On a property of the expected value of a determinant

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Abstract: Suppose that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$ in $\mathbb{R}^p$ are i.i.d. with some underlying distribution μ, and consider the matrix $\mathbf{M}_N = (1/N)\sum_{i=1}^N \mathbf{x}_i \mathbf{x}_i^T$. We show that, with constant $C(N,p) = N!/(N^p(N-p)!)$ not depending on μ, the expected value of the determinant is of $\mathbf{M}_N$ is proportional to the determinant of the expected value of $\mathbf{M}_N$. This implies in particular that in a linear regression problem with regressors independently generated with a measure μ, the measure maximizing the expected value of the determinant of the information matrix is D-optimal, whatever the number of observations.

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1 A theorem for expected determinants

Suppose that N vectors are independently identically distributed in a set $\mathcal{X} \subseteq \mathbb{R}^p$ with a probability measure μ. We shall denote $\mathbf{M}(\mathbf{x}_1^N)$ the matrix,

$$\mathbf{M}(\mathbf{x}_1^N) = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^T \,,$$

with $\mathbf{x}_1^N = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$. The expected value $E\{\mathbf{M}(\mathbf{x}_1^N)\}$, with $E\{\cdot\}$ the expectation for the measure μ, will be denoted $\bar{\mathbf{M}}(\mu)$, that is:

$$\bar{\mathbf{M}}(\mu) = \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^T \mu(d\mathbf{x}) \,.$$