Reduction of Discrete Dynamical Systems & Linear Network Coding

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(arXiv: 1512.5310)

(I) <u>Introduction</u>

(II) Approximation of $\max(G, \rho)$

(III) Reduction

(IV) Application to linear network coding

(V) Conclusion

A finite dynamical system with n components is a function

$$f : A^n \to A^n \qquad x = (x_1 \dots x_n) \mapsto f(x) = (f_1(x) \dots f_n(x))$$

where A is a finite set. Here, $A = \{0, 1, \dots, p-1\} = [p]$ for some $p \geqslant 2$.

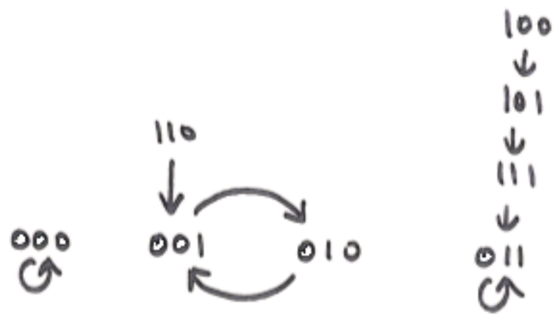
The interaction graph of f is the directed graph $G(f)$ on $\{1 \dots n\}$ defined by

$$j \to i \in G(f) \iff f_i \text{ depends on } x_j$$

Example with $n=3$ and $A=\{0,1\}=[2]$

x	$f(x)$
000	000
001	010
010	001
011	011
100	101
101	111
110	001
111	011

$$\begin{cases} f_1(x) = x_1 \wedge \overline{x_2} \\ f_2(x) = x_3 \\ f_3(x) = x_1 \vee x_2 \end{cases}$$



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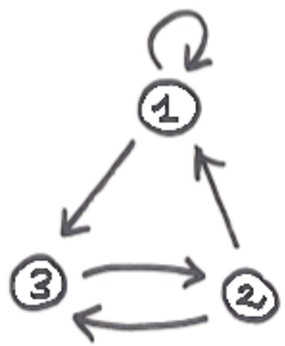

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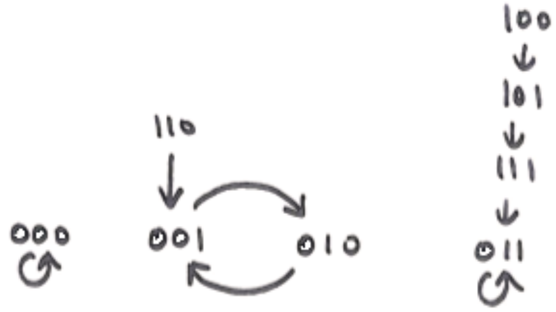

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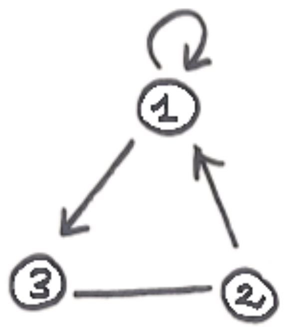

Many Aplications :

- Neural networks (Mc Culloch & Pitts 1943)
- Gene networks (Kauffman 1969, Thomas 1973)
- Network coding (Ahlswede et al, 2000)

In the context of gene network :

- First reliable information are often on the interaction graph $G(f)$
- Fixed points of f have often a biological meaning

What can be said on the fixed points of f according to $G(f)$?

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In the context of gene network :

- First reliable information are often on the interaction graph $G(f)$
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What can be said on the fixed points of f according to $G(f)$?

$\max(G,p)$ = maximum number of fixed points among all the systems $f : [p]^n \to [p]^n$ with $G(f) \subseteq G$

I. Introduction

II. **Approximation of $\max(G,\rho)$**

III. Reduction

IV. Application to linear network coding

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$\nu(G)$ = maximum number of vertex-disjoint cycles in G

= packing number of G

$\tau(G)$ = minimum size of a set of vertices I such that $G - I$ is acyclic

= transversal number of G

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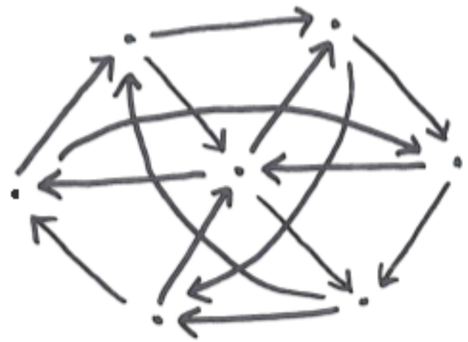

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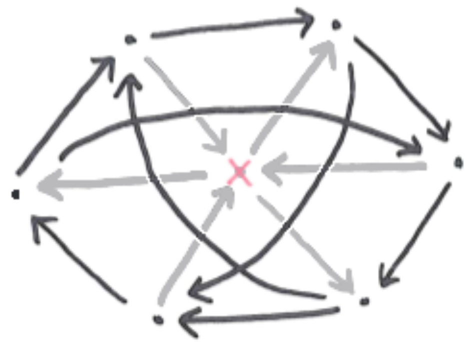

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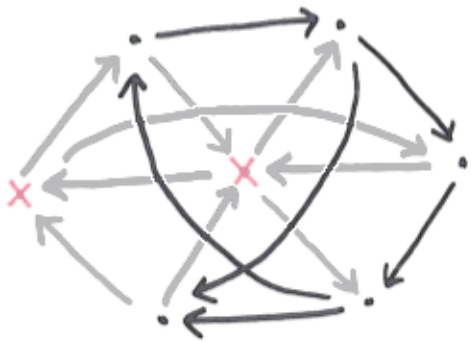

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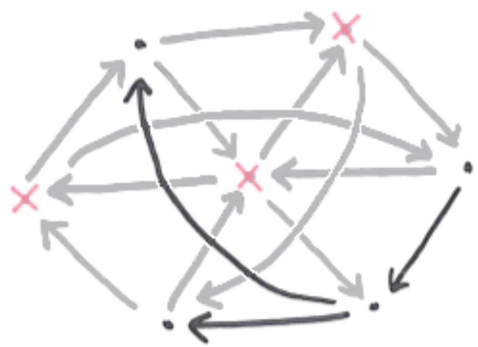

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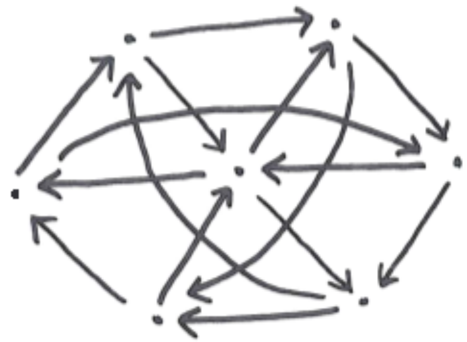

$\nu(G) = 1$

$\tau(G) = 3$

$\nu(G) = 1 \Rightarrow \tau(G) \leq 3$ (McCuaig 1993)

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$\nu(G) \leq \tau(G) \leq h(\nu(G))$ (Reed et al 1996)

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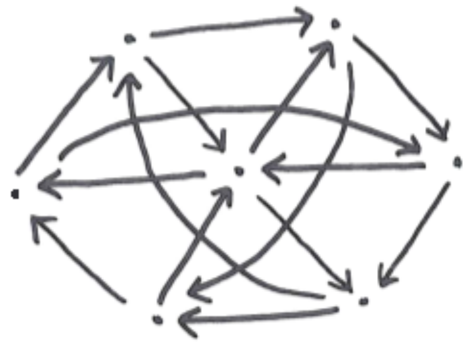

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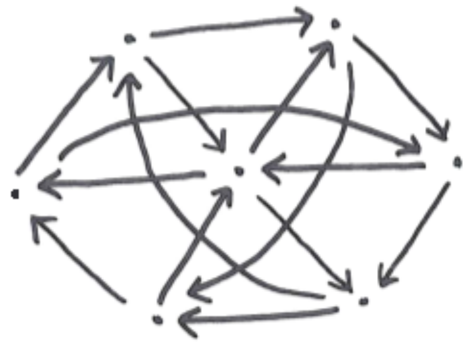

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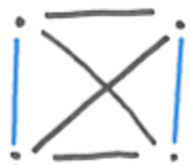

$\nu(G) = 2$
$\tau(G) = 3$

$\nu(G) = 1 \quad \Rightarrow \quad \tau(G) \leq 3 \quad$ (McCuaig 1993)

Theorem (Aracena 2004, Riis 2007)

$$p^{\nu(G)} \leqslant \max(G, p) \leqslant p^{\tau(G)}$$

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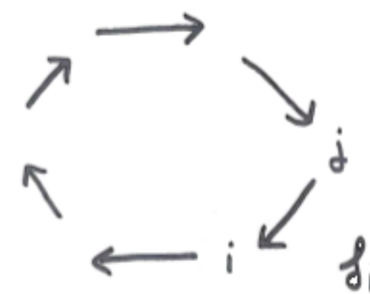
Lower bound

$H =$ 

$f_i(x) = x_j$

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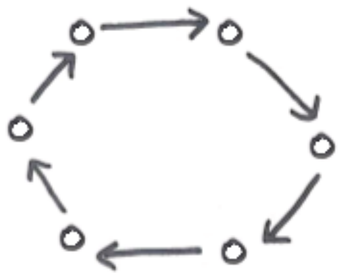
$H =$ 

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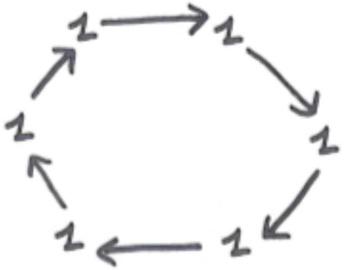
$H =$ 

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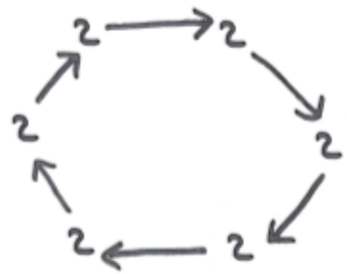
H = 

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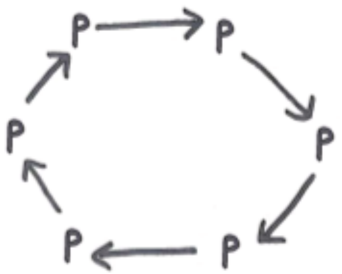
H = 

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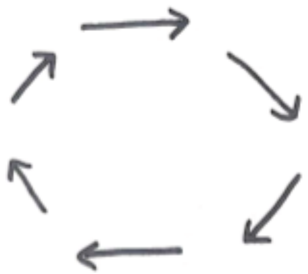
$H =$ 

f has p fixed points

$$\boxed{p \leq \max(H, p)}$$

Theorem (Aracena 2004, Riis 2007)

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$H =$ 

k disjoint cycles

$$\boxed{p^{k} \leqslant \max(H,p)}$$

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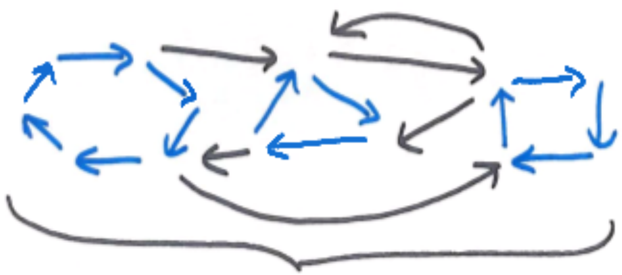

$\nu(G)$ disjoint cycles

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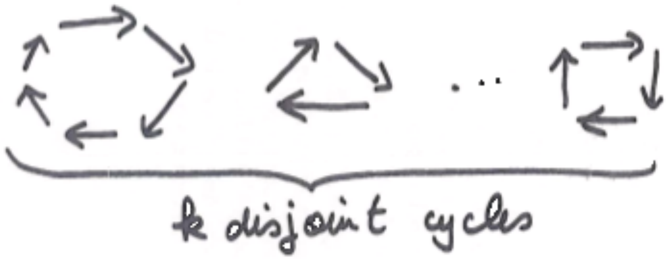

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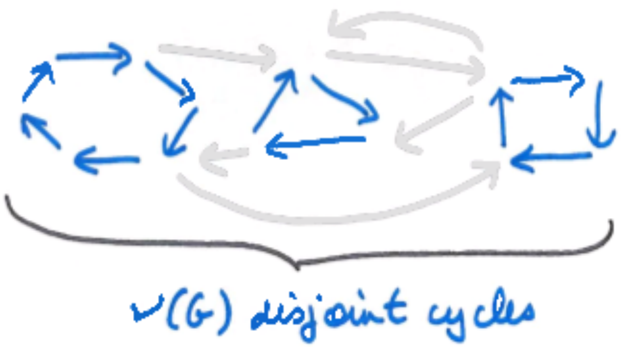

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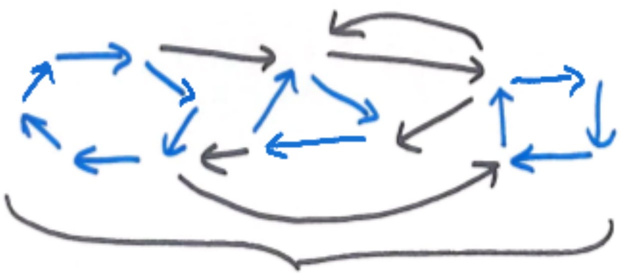
G = 

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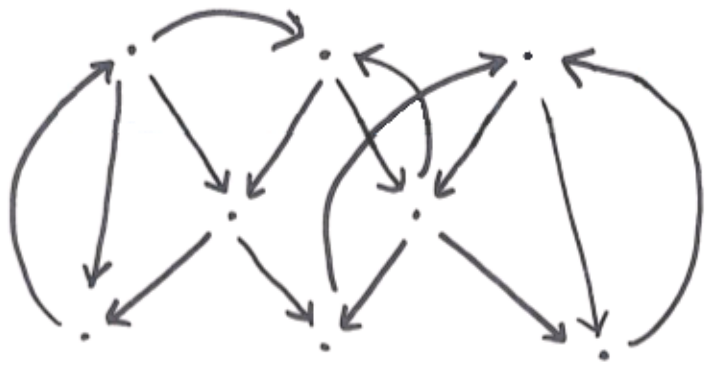
$G =$ 

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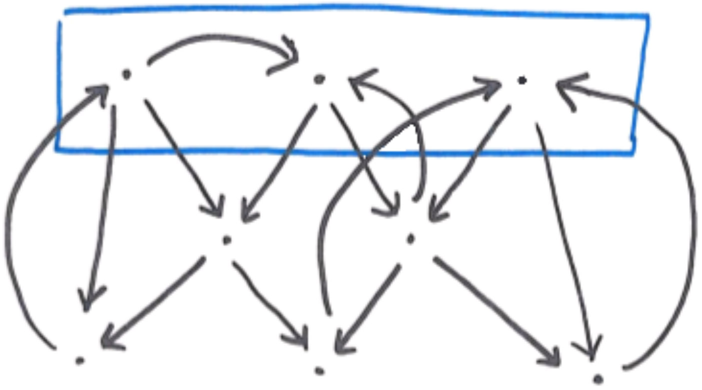
$G =$ 

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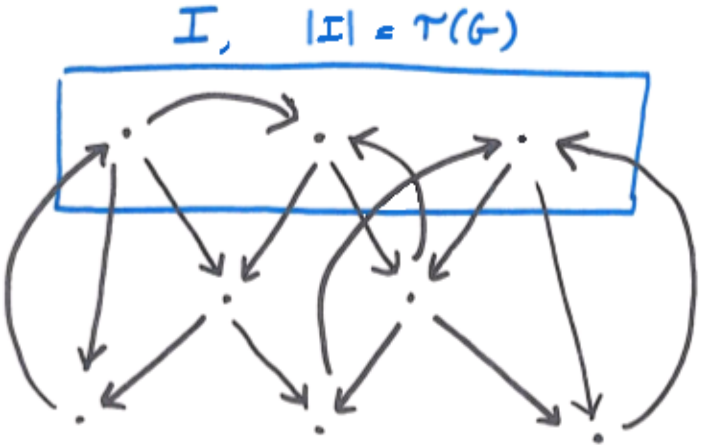
Let $f : [p]^n \to [p]^n$

with $G(f) \subseteq G =$



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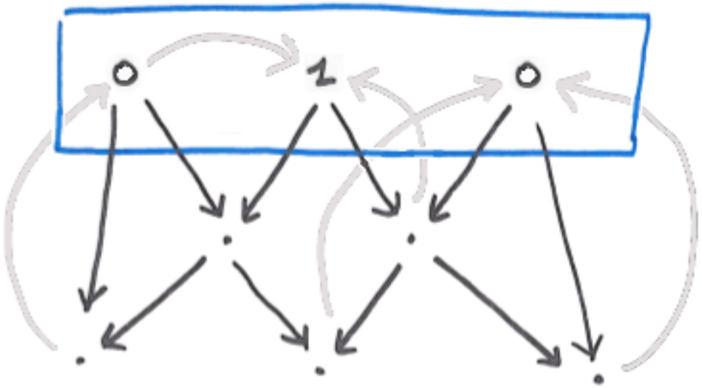

$x \in [p]^I$

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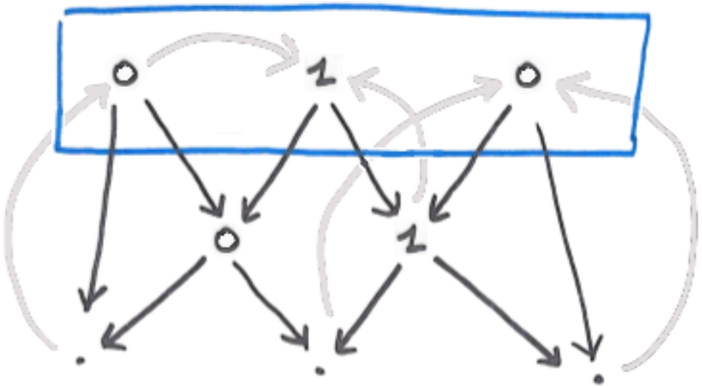

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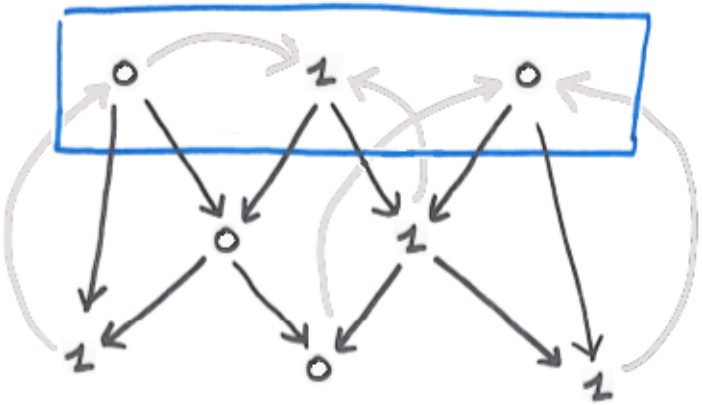

$$x \in [p]^I$$

$$\Big\downarrow \ \text{Diffusion (depends on } f\text{)}$$

$$x^* \in [p]^n$$

Theorem (Aracena 2004, Riis 2007)

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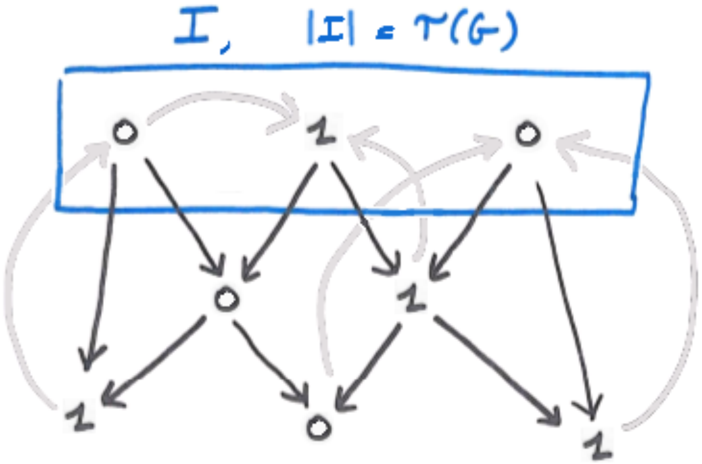

$$x \in [p]^I \xrightarrow{\text{Diffusion (depends on } f\text{)}} x^* \in [p]^n$$

All the fixed points of f are in $X = \{ x^* \mid x \in [p]^I \}$

Theorem (Aracena 2004, Riis 2007)

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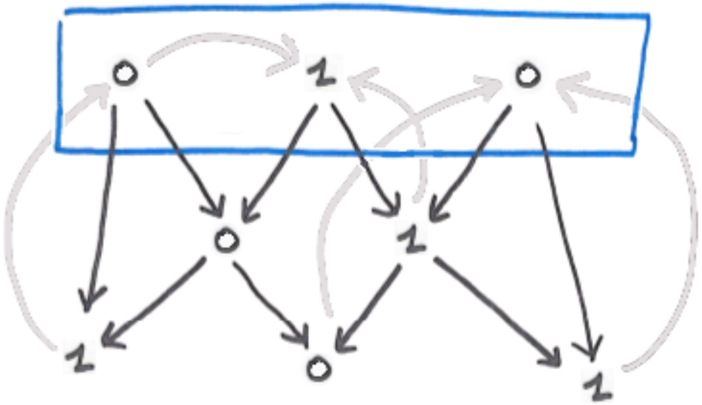

$x \in [p]^I$

$\big\downarrow$ Diffusion (depends on f)

$x^* \in [p]^n$

All the fixed points of f are in $X = \{x^* \mid x \in [p]^I\}$

$$\boxed{f \text{ has at most } |X| \leqslant p^{|I|} = p^{\tau(G)} \text{ fixed points}}$$

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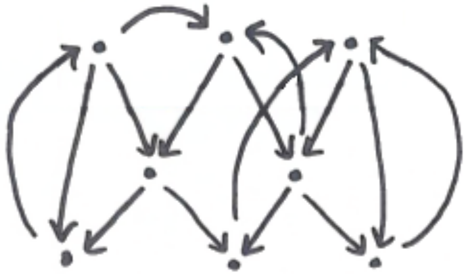
$G(f) =$ 

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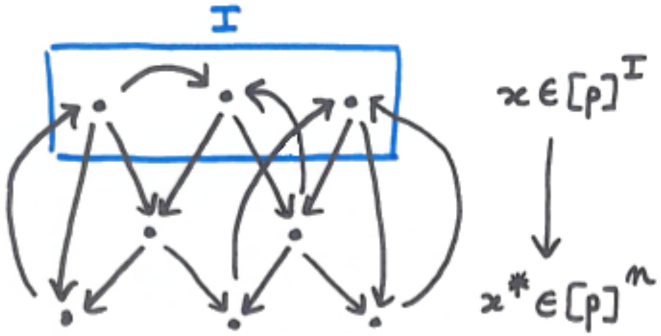
$G(f) =$ 

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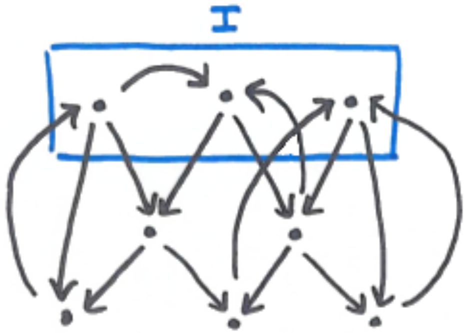


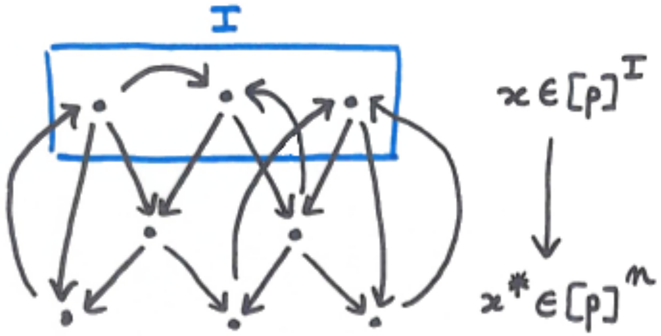
$$x \in [p]^I$$
$$\downarrow$$
$$x^* \in [p]^n$$

The reduced system $f^I : [p]^I \to [p]^I$ is defined by

$$f^I(x) = f(x^*)_I$$

Let $f : [p]^n \to [p]^n$ with

$G(f) =$



The reduced system $f^I : [p]^I \to [p]^I$ is defined by

$$\forall i \in I \quad f^I_i(x) = f_i(x^*)$$

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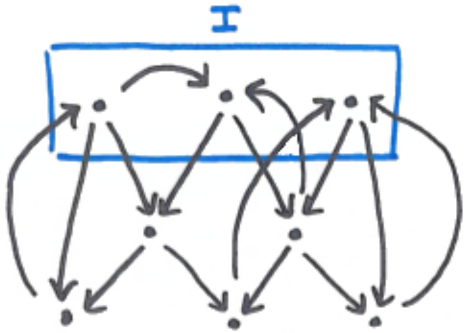
$G(f) =$ 

$$\begin{array}{c} x \in [p]^I \\ \downarrow \\ x^* \in [p]^n \end{array}$$

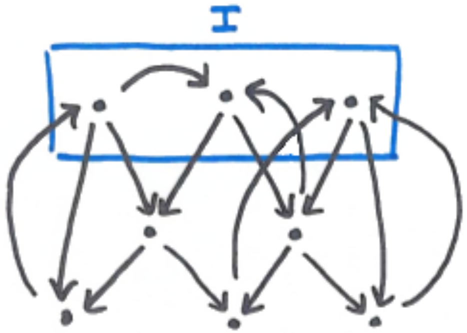
The reduced system $f^I : [p]^I \to [p]^I$ is defined by

$$\forall i \in I \quad f^I_i(x) = f_i(x^*)$$

f and f^I have the same number of fixed points

$$f^I(x) = x \iff f(x^*) = x^*$$

Let $f : [p]^n \to [p]^n$ with

$G(f) =$ 

$$x \in [p]^I \\ \downarrow \\ x^* \in [p]^n$$

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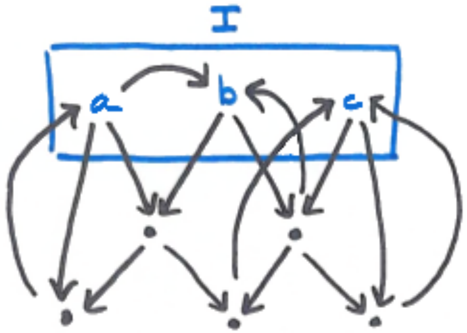
What can be said on $G(f^I)$?

Let $f : [p]^n \to [p]^n$ with

$G(f) =$ 

$$x \in [p]^I \;\big\downarrow\; x^* \in [p]^n$$

$G(f)^I =$ 

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f and f^I have the same number of fixed points

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The reduced interaction graph $G(f)^I$ is defined by

- The vertex set is I
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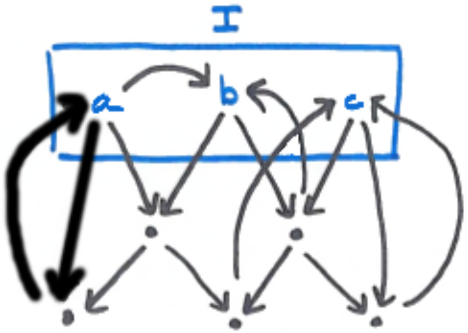
$G(f) =$ 

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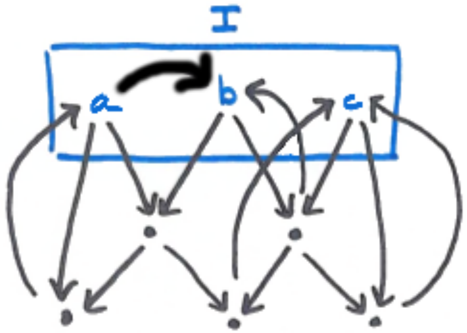

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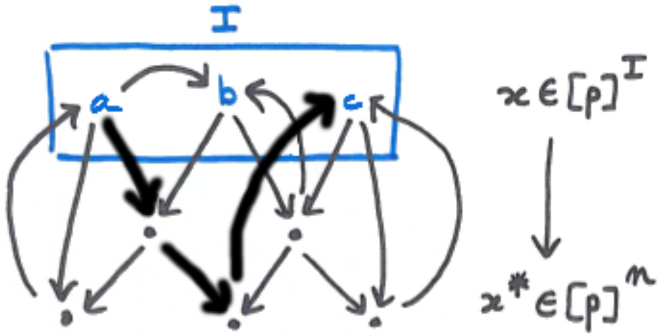
$G(f) =$ 

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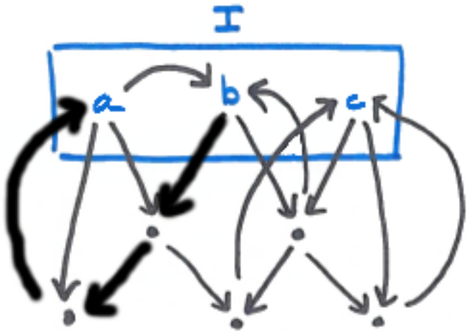
$G(f) =$ 

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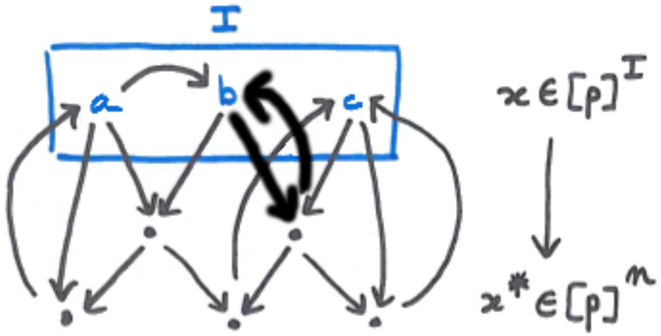
$G(f) =$ 

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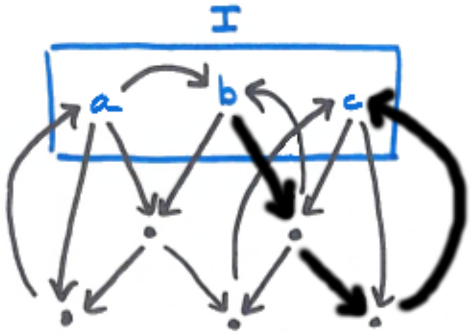
$G(f) =$ 

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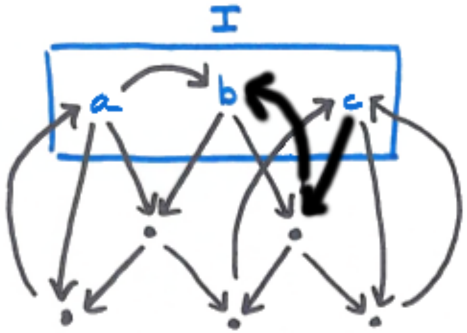

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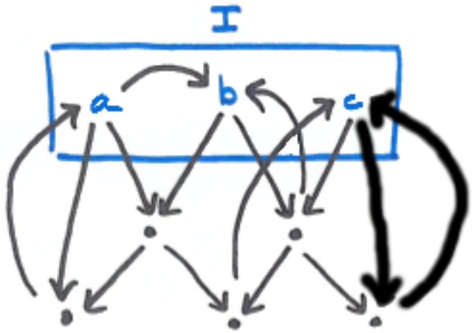
$G(f) =$ 

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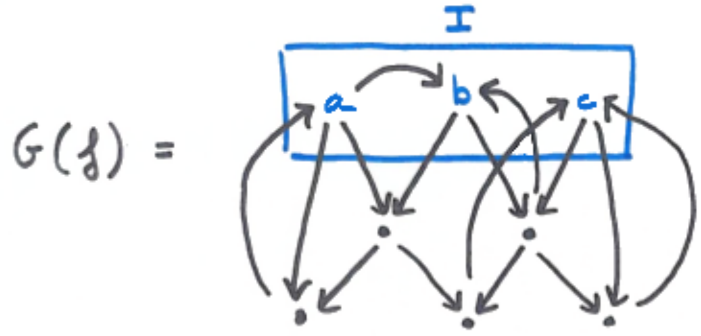

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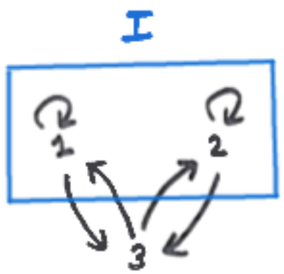
$G =$ 

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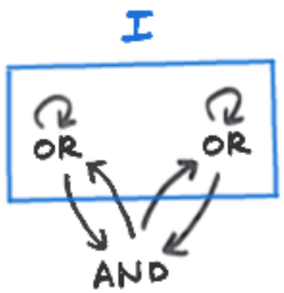

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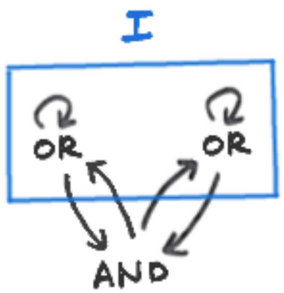

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x	x^*
0 0	0 0 0
0 1	0 1 0
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1 1	1 1 1

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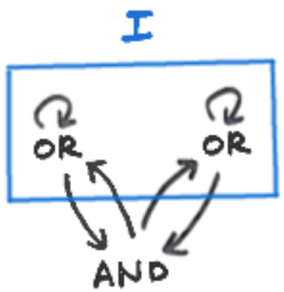

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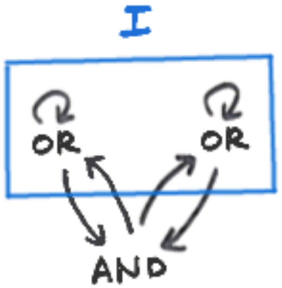

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\end{array}
$$

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01	010	010	01
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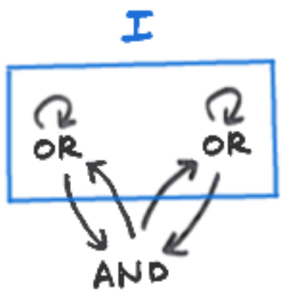

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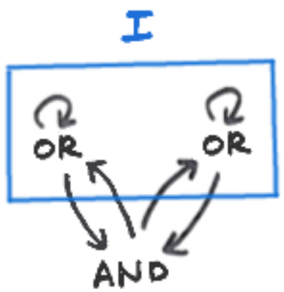

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$G(f^I) = \circlearrowleft 1 \quad \circlearrowleft 2$

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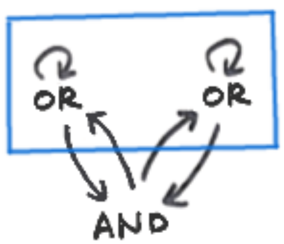

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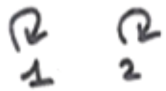
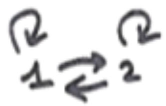
$G(f^I) =$ 

$G(f)^I =$ 

I Introduction

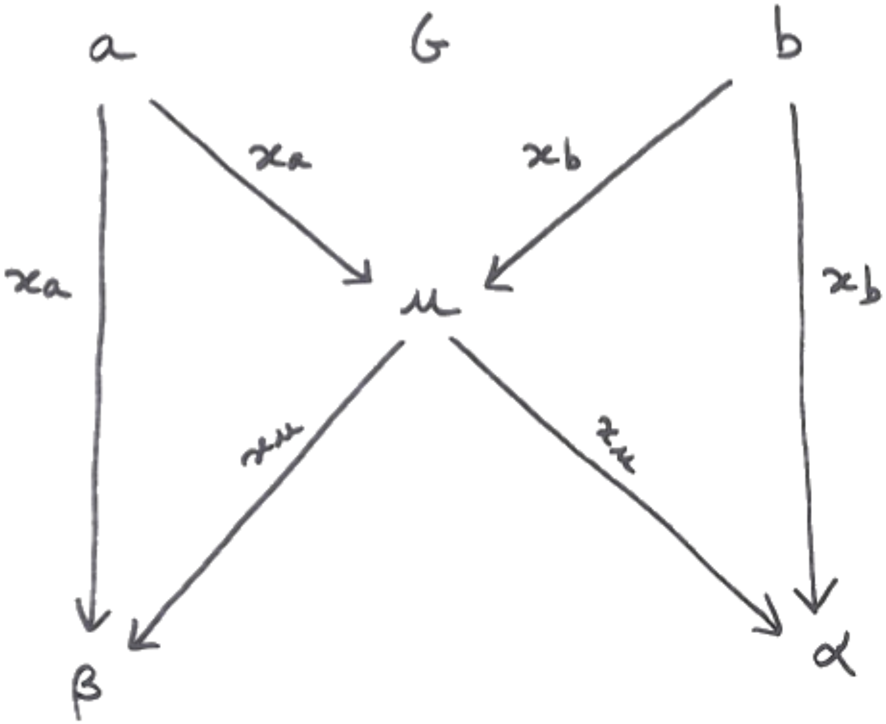
II Approximation of max (G, p)

III Reduction

IV Application to linear network coding

V Conclusion

Network Coding



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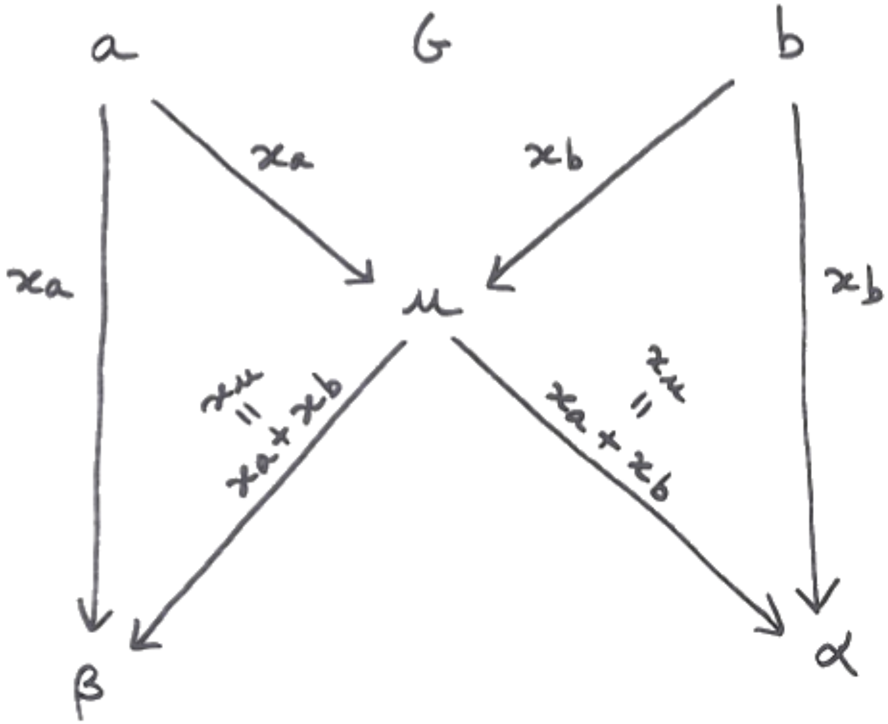

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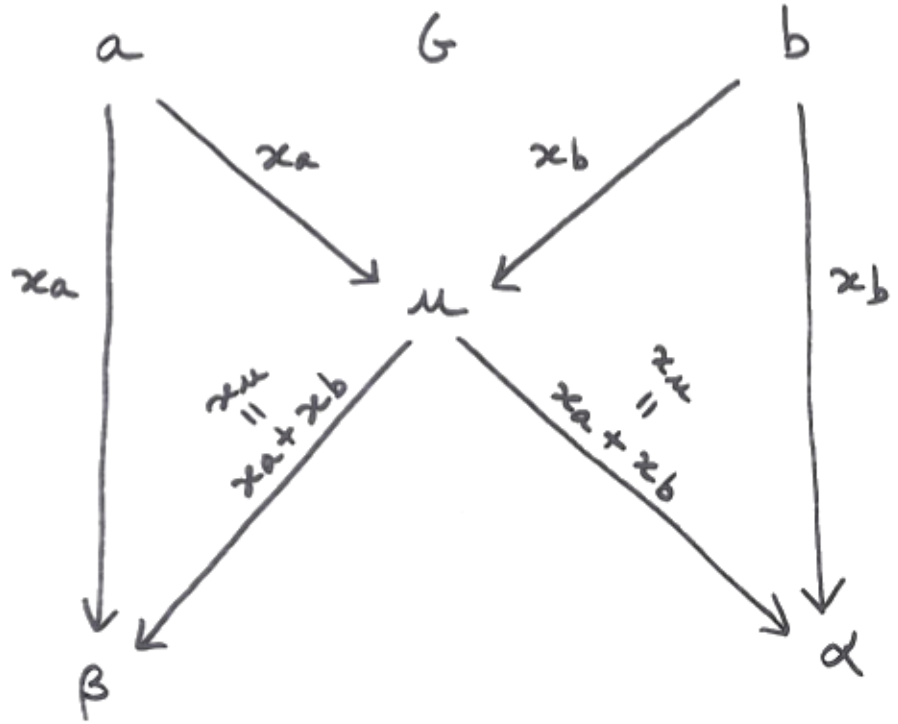

$x_\beta = x_a + x_u$

$x_\alpha = x_u + x_b$

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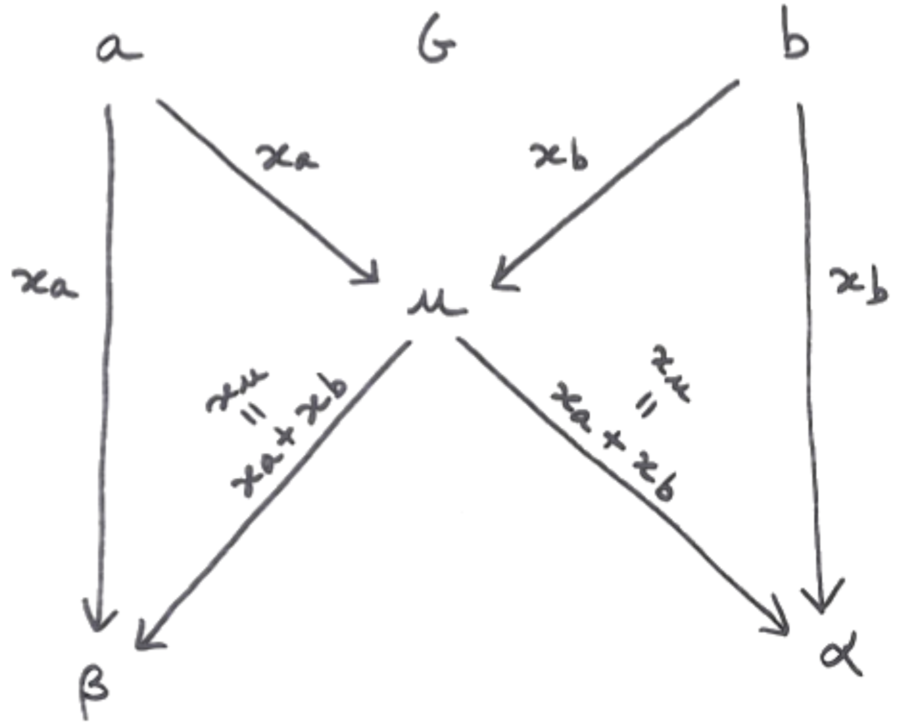

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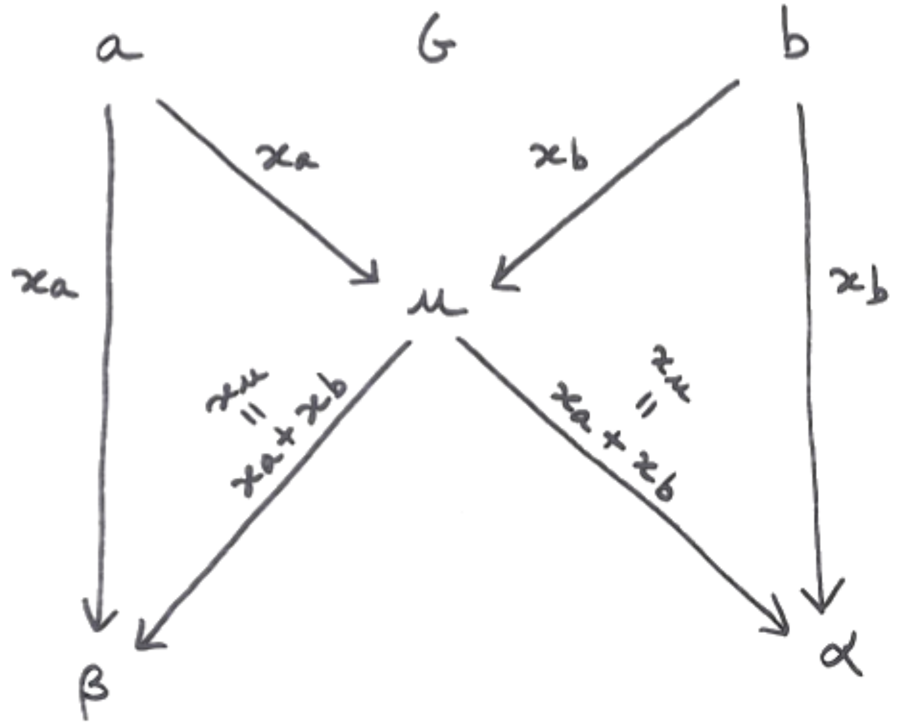

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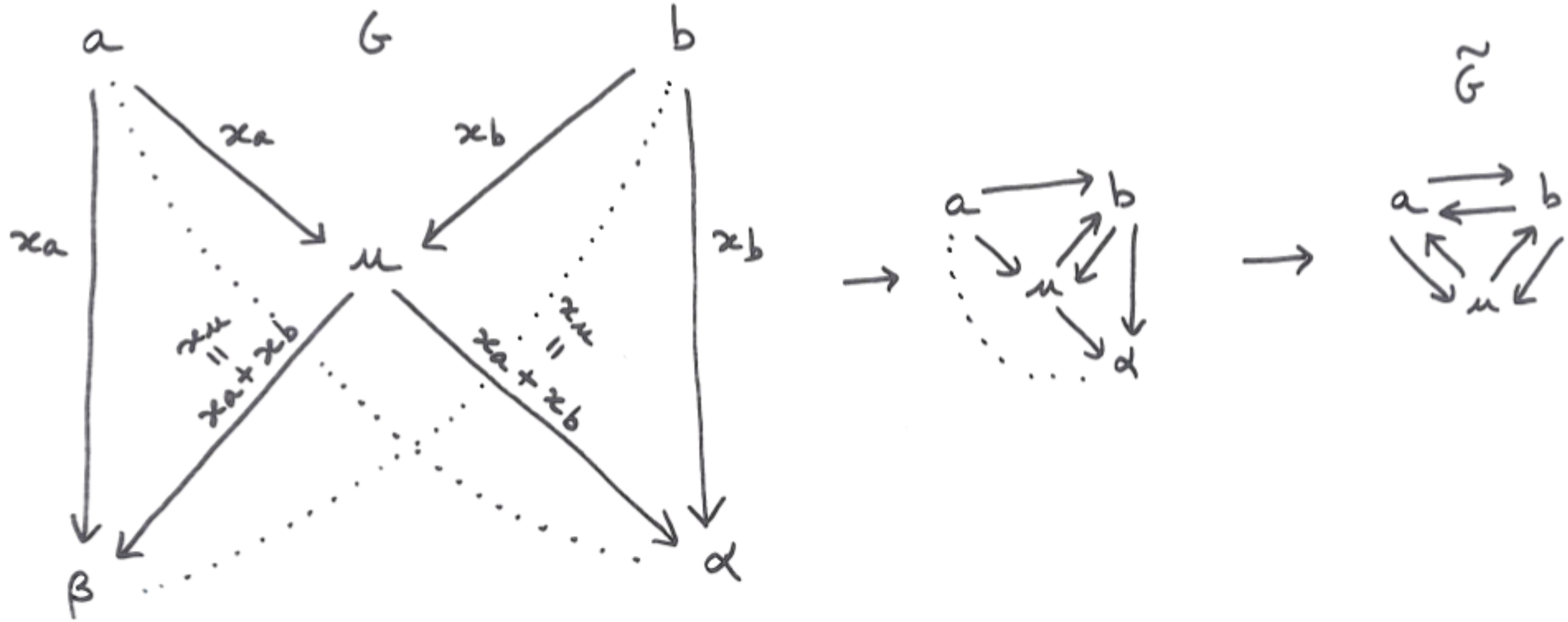

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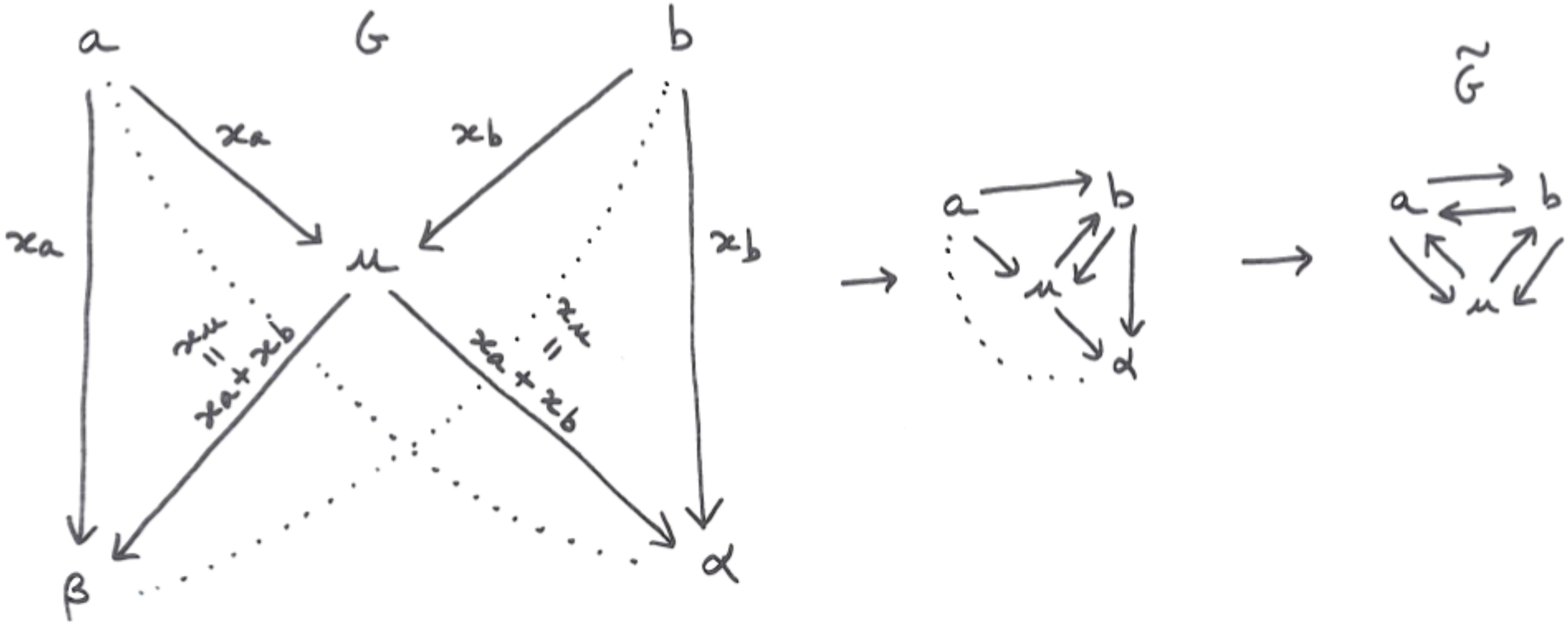

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G is solvable $\iff$ $\max(\tilde{G}, p) = p^{\tau(\tilde{G})}$ for some p

Central question in network coding

Which are the interaction graphs G such that

$$\max(G,p) = p^{\tau(G)} \quad \text{for some } p \qquad (\text{solvable for some } p)$$

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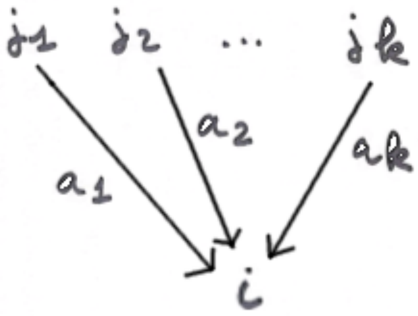

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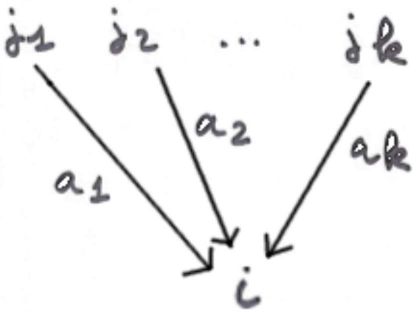

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$$f : [p]^n \to [p]^n \quad \text{with } G(f) \subseteq G$$

Central questions in network coding

Which are the interaction graphs G such that

- $\max(G,p) = p^{\tau(G)}$ for some p (solvable for some p)
- $\max L(G,p) = p^{\tau(G)}$ for some p (linearly solvable for some p)

The number of fixed points is often maximized by linear systems

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$f:[p]^n \to [p]^n$ with $G(f) \subseteq G$

Theorem (Fanchon, Gadouleau, Richard 2014)

Let G be an undirected triangle-free graph. Then the following are equivalent:

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$\boxed{C_5 \text{ is not linearly solvable}}$

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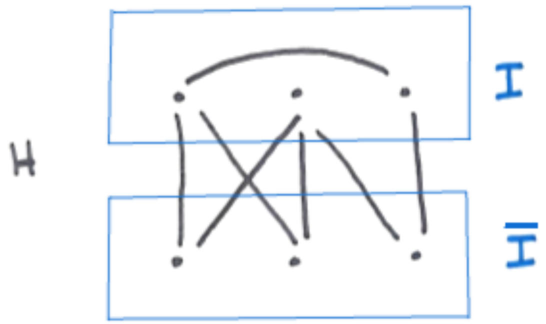

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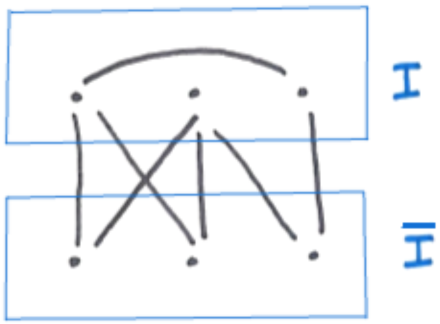

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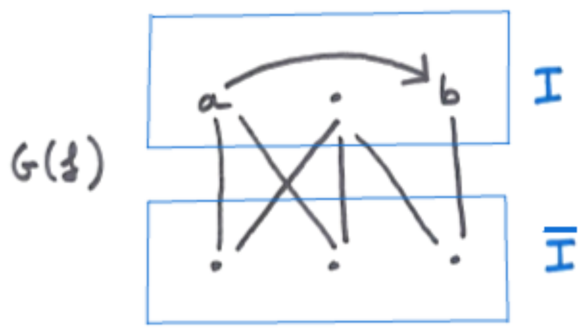

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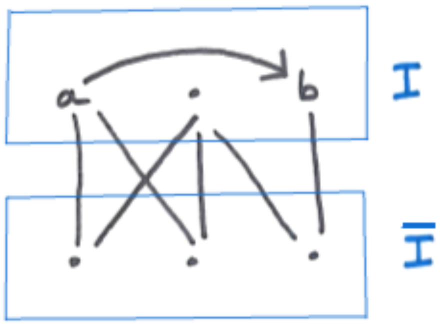
$G(f)$ 

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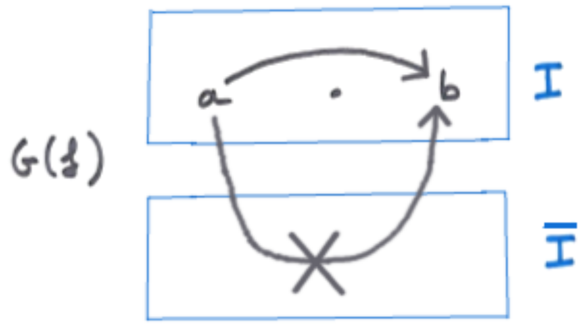

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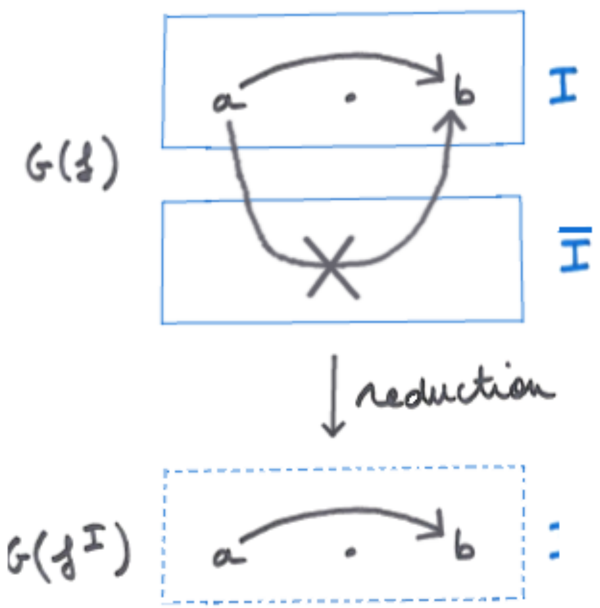

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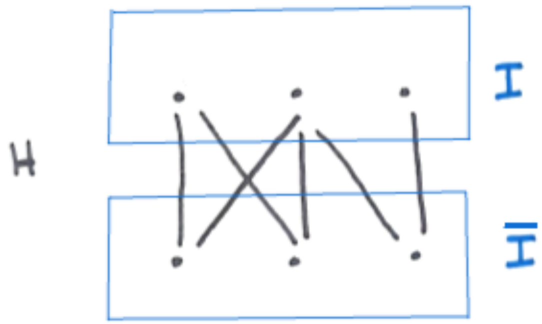

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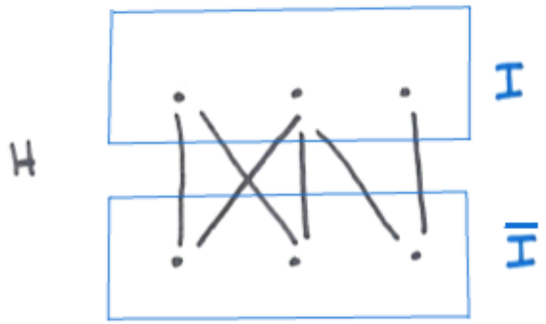

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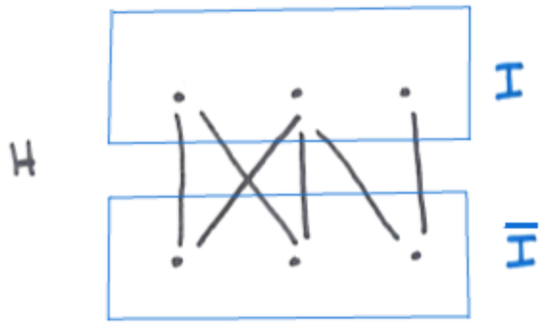

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I Introduction

II Approximation of $\max(G, \rho)$

III Reduction

IV Application to linear network coding

V Conclusion

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- Good tool for theoretical studies

    - Network Coding

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      Ex: Characterization of the interaction graphs G such that there exists a monotonous boolean network f with

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    ↳ Naldi's talk tomorrow!