Some Complexity Results on Fuzzy Description Logics

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Abstract. We present and discuss some novel and somewhat surprising complexity results for a basic but significant fuzzy description logic (DL) which extends the classical $\mathcal{ALC}$ language. In particular we show that checking the consistency of a concept or a KB in fuzzy DLs has a complexity which jumps from linear-time to EXPTIME-complete, while the subsumption problem is always (at least) as hard as in crisp DLs.

1 Introduction

Description logics (DL) [1] are a family of logic-based knowledge-representation formalisms emerging from the classical AI tradition of semantic networks and frame-based systems. DLs are well-suited for the representation of and reasoning about terminological knowledge, configurations, ontologies, database schemata, etc.

The need of expressing and reasoning with imprecise knowledge and the difficulties arising in classifying individuals with respect to an existing terminology is motivating research on nonclassical DL semantics, suited to these purposes. Recently, a quite general fuzzy extension of description logics has been proposed, with complete algorithms for solving the entailment problem, the subsumption problem, as well as the best truth-value bound problem [4]. All these algorithms have been shown to be PSPACE-complete, i.e., fuzzification of description logics has no impact from a computational complexity point of view.

The properties of the fuzzy extensions of DLs proposed so far are not yet completely understood. In this paper we note that there is no unique way of generalizing concept consistency in a fuzzy setting, and prove complexity results for this and other reasoning tasks of interest in the fuzzy extension of the DL $\mathcal{ALC}$. We obtain rather surprising results: the complexity of concept and knowledge base consistency checking in fuzzy DLs may range from linear-time to EXPTIME-complete, while the subsumption problem is always (at least) as hard as in crisp DLs.

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2 Fuzzy $\mathcal{ALC}$

$\mathcal{ALC}$ is a basic yet significant representative of DLs. The syntax of the $\mathcal{ALC}$ language is very simple: a *concept* (denoted by C or D) is built out of primitive (or *atomic*) concepts according to the grammar

$$\begin{aligned} C, D \rightarrow\ & C \sqcap D \mid \\ & C \sqcup D \mid \\ & \neg C \mid \\ & \forall R.C \mid \\ & \exists R.C \mid \\ & \top \mid \bot \mid A \end{aligned}$$

where A denotes an atomic concept, and R denotes an atomic role. From a logical point of view, concepts can be seen as unary predicates, whereas roles can be interpreted as binary predicates linking individuals to their attributes.

Fuzzy $\mathcal{ALC}$ retains the same syntax, only semantics changes. A fuzzy interpretation $\mathcal{I}$ consists of a non-empty domain $\Delta^{\mathcal{I}}$ (the universe of discourse), and an assignment $\cdot^{\mathcal{I}}$, which maps every atomic concept A onto a fuzzy subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, every atomic role R onto a fuzzy binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and every individual name a onto an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The special atomic concepts $\top$ and $\bot$ are mapped respectively onto $\Delta^{\mathcal{I}}$ (the function that maps every individual onto 1) and the empty set (the function that maps every individual onto 0).

The semantics of compound concepts is defined as follows: for all $x \in \Delta^{\mathcal{I}}$,

$$(C \sqcap D)^{\mathcal{I}}(x) = \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}; \tag{1}$$

$$(C \sqcup D)^{\mathcal{I}}(x) = \max\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}; \tag{2}$$

$$(\neg C)^{\mathcal{I}}(x) = 1 - C^{\mathcal{I}}(x); \tag{3}$$

$$(\exists R.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} \min\{R^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y)\}; \tag{4}$$

$$(\forall R.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y)\}. \tag{5}$$

Axioms and queries can be of two kinds: inclusions and assertions. An inclusion is a statement of the form $C \sqsubseteq D$, which is true (i.e., 1) in $\mathcal{I}$ if for all $x \in \Delta^{\mathcal{I}}$,

$$C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x), \tag{6}$$

and false (i.e., 0) otherwise. By $C \equiv D$ we abbreviate the pair of assertions $\{C \sqsubseteq D, D \sqsubseteq C\}$. If $C \sqsubseteq D$ is valid (true in every interpretation), then we say that D *subsumes* C. An assertion can be either of the form $C(x) \leq \alpha$, or $C(x) \geq \alpha$, where C is a concept, x is an individual constant and α is a rational number. The two kinds of assertions are true in $\mathcal{I}$ if $C^{\mathcal{I}}(x^{\mathcal{I}}) \leq \alpha$ (resp. $C^{\mathcal{I}}(x^{\mathcal{I}}) \leq \alpha$), and false otherwise.

3 Concept Satisfiability

It is proven in [4] that given acyclic inclusion axioms plus assertions like $C(x) \leq \alpha$ and $C(x) \geq \alpha$, fuzzy inference is PSPACE-complete; however, the discussion of concept consistency provided there is incomplete, because in fuzzy $\mathcal{ALC}$, concept consistency can be formulated in two ways, namely

- checking whether $C \not\sqsubseteq \bot$, and
- checking whether C is satisfiable to a given degree α.

As a matter of fact, the complexity of checking whether $C \not\sqsubseteq \bot$ or, equivalently, $C \not\equiv \bot$, which is PSPACE-complete in the classical case, turns out to be linear-time in fuzzy $\mathcal{ALC}$. However, unlike in the classical case, this version of consistency checking cannot be taken as the basis for other reasoning problems, i.e., it is less complex but also less expressive.

The proof goes as follows: given the definition of compound concept C, rewrite it by using the following equivalence-preserving rewriting system:

1. $\neg\top \to \bot$;
2. $\neg\bot \to \top$;
3. $\top \sqcup C \to \top$;
4. $C \sqcup \top \to \top$;
5. $\bot \sqcup C \to C$;
6. $C \sqcup \bot \to C$;
7. $\exists R.\bot \to \bot$.[1]

Repeatedly applying the above rewrite rules yields a concept in what we might call *Top-Bottom Normal Form* (TBNF), i.e., $\top$, $\bot$, or a $\bot$-free concept where $\top$ occurs only in subconcepts of the form $\exists R.\top$ (*nontrivial* TBNF).

By a simple structural induction on the concept C we obtain:

Lemma 1. *Let $\mathcal{I}_{0.5}$ be any interpretation that maps every atomic concept (resp. role) onto the constant function $\lambda x.0.5$ (resp. $\lambda x \lambda y.0.5$). For all nontrivial TBNF concepts C, $C^{\mathcal{I}_{0.5}} = \lambda x.0.5$.*

In other words, the interpretation in which all membership degrees are one half, renders all nontrivial TBNF concepts "half-satisfiable". A direct corollary of this property is that

Lemma 2. *$C \equiv \bot$ iff $C \to^* \bot$.*

Furthermore, the rewriting process is confluent and eliminates one connective at each step; therefore, we derive the following result.

Theorem 1. *In fuzzy $\mathcal{ALC}$, the inclusion $C \sqsubseteq \bot$ can be checked in linear time.*

The second version of consistency, called α-consistency (cf. [2, 3]), is defined as follows: a compound concept C is α-consistent, iff there exist an interpretation $\mathcal{I}$ and an individual $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) \geq \alpha$.

[1] Note that there is no corresponding rewrite rule for $\exists R.\top$.

Theorem 2. *If $\alpha \leq 0.5$ then α-consistency can be checked in linear time.*

Proof. If $\alpha = 0$ then every concept is α-satisfiable. If $0 < \alpha \leq 0.5$ then note that C is 0.5-satisfiable (hence α-satisfiable) iff its TBNF is not $\bot$ (by the above lemmata). The latter test takes linear time, hence the theorem holds. $\Box$

If, however, $0.5 < \alpha \leq 1$, then α-consistency turns out to be PSPACE-complete, like in the case of crisp $\mathcal{ALC}$. This is an unusually abrupt complexity jump.

The PSPACE-completeness proof requires some preliminaries:

Definition 1. *For all interpretations $\mathcal{I}$, let $\sharp\mathcal{I}$ be the crisp interpretation such that $\Delta^{\sharp\mathcal{I}} = \Delta^{\mathcal{I}}$ and for all atomic concepts A and all $x \in \Delta^{\mathcal{I}}$*

$$A^{\sharp\mathcal{I}}(x) = \begin{cases} 1 \text{ if } A^{\mathcal{I}}(x) > 0.5 \\ 0 \text{ otherwise} \end{cases}$$

and similarly for (atomic) roles.

Lemma 3. *For all C, $\mathcal{I}$, and $x \in \Delta^{\mathcal{I}}$,*

$$C^{\mathcal{I}}(x) > 0.5 \text{ iff } C^{\sharp\mathcal{I}}(x) = 1,$$
$$C^{\mathcal{I}}(x) < 0.5 \text{ iff } C^{\sharp\mathcal{I}}(x) = 0.$$

We are thus able to reduce α-satisfiability, for $\alpha > 0.5$, to classical satisfiability: for all $\alpha > 0.5$, C is α-consistent iff C is classically consistent. As a consequence (by standard complexity results) we have:

Theorem 3. *If $0.5 < \alpha \leq 1$ then checking α-consistency is PSPACE-complete.*

4 KB Satisfiability

We now address the problem of checking whether a general, terminological knowledge base (KB) is satisfiable, i.e., whether there exists an interpretation satisfying simultaneously a given set of inclusions.

First of all, we can use the special interpretation $\mathcal{I}_{0.5}$ and Lemma 1 to prove that:

Corollary 1. *If all compound concepts C_{ij} are nontrivial TBNF concepts then the KB $\{C_{i1} \sqsubseteq C_{i2} \mid 1 \leq i \leq n\}$ is satisfiable.*

Furthermore, it is possible to prove that the general problem is much harder.

Theorem 4. *Checking whether an arbitrary set of inclusions is α-satisfiable is EXPTIME-complete.*

The proof sketch is as follows: after normalization, all cases are trivial but $\top \sqsubseteq C$ and $C \sqsubseteq \bot$. Such inclusions are satisfiable iff they are classically satisfiable (use the crisp interpretation $\sharp\mathcal{I}$ and the results from the previous section). The latter problem is known to be EXPTIME-complete (for arbitrary inclusions).

5 Subsumption

In crisp $\mathcal{ALC}$, checking whether $C \sqsubseteq D$ is valid (i.e., D subsumes C) can be reduced to a concept consistency check. However, adding fuzziness to $\mathcal{ALC}$ destroys this equivalence, and it can be shown that checking whether $C \sqsubseteq D$ is valid is at least as hard as in the classical case, even under the assumption which makes satisfiability checking of $C \sqsubseteq D$ a linear-time problem.

Theorem 5. *In fuzzy $\mathcal{ALC}$ the subsumption problem is PSPACE-hard even if the concepts involved are nontrivial TBNF concepts.*

The proof of this theorem consists in reducing crisp unsatisfiability of a concept C (PSPACE-complete) to the fuzzy subsumption problem $C \sqsubseteq \neg C$ using the crisp interpretation $\sharp\mathcal{I}$ and the results of the previous section. If C is not in nontrivial TBNF, it can be first reduced to this form by replacing $\top$ and $\bot$ with $A \sqcup \neg A$ and $A \sqcap \neg A$, respectively, where A is any atomic concept.

6 Conclusions and Future Work

Careful investigation of the computational complexity of inference in fuzzy DLs unveils unusual complexity results, some of which are reminiscent of properties of disjunctive logic programs, namely:

- concept satisfiability of the first kind is easy (linear-time);
- the complexity of concept satisfiability of the second kind, i.e., α-satisfiability, depends on α, and jumps from linear-time to PSPACE-complete as $\alpha > 0.5$;
- KB satisfiability is linear-time if all concepts are in nontrivial TBNF, and EXPTIME-complete otherwise;
- subsumption is (at least) as difficult as subsumption in crisp DLs, even in those cases where satisfiability is easy.

The proofs of these result made clear that there exist simple and strong relationships between the semantics of fuzzy DLs (the fuzzy interpretations $\mathcal{I}$) and the semantics of crisp DLs (the corresponding "defuzzified" interpretations $\sharp\mathcal{I}$).

Several important issues remain still open. Having proved that subsumption in fuzzy DLs is at least as hard as in crisp DLs, a precise complexity characterization for it will have to be the subject of further investigation. Another interesting problem is evaluating lower and upper bounds for the membership value of a given individual to a given concept.

Fuzzy $\mathcal{ALC}$ could be extended in a variety of ways to make it a usable tool for a vast range of real-world application domains: cardinality-based quantifiers and compound roles could be added, alternative t-norms and co-norms could be used to define the basic connectives, and other types of assertions about individuals could be allowed.

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