Generalizing Quantification in Fuzzy Description Logics

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Summary. In this paper we introduce $\mathcal{ALCQ}_F^+$, a fuzzy description logic with extended qualified quantification. The proposed language allows for the definition of fuzzy quantifiers of the absolute and relative kind by means of piecewise linear functions on $\mathbb{N}$ and $\mathbb{Q} \cap [0, 1]$ respectively. These quantifiers extends the usual (qualified) $\exists$, $\forall$ and number restriction. The semantics of quantified expressions is defined by using method GD [4], that is based on recently developed measures of the cardinality of fuzzy sets.

1 Introduction

Description logics (DL) [1] are a family of logic-based knowledge-representation formalisms emerging from the classical AI tradition of semantic networks and frame-based systems. DLs are well-suited for the representation of and reasoning about terminological knowledge, configurations, ontologies, database schemata, etc.

The need of expressing and reasoning with imprecise knowledge and the difficulties arising in classifying individuals with respect to an existing terminology is motivating research on nonclassical DL semantics, suited to these purposes. To cope with this problem, fuzzy description logics have been proposed that allow for imprecise concept description by using fuzzy sets and fuzzy relations. However, these approaches have paid little attention to the quantification issue (only the semantics of $\exists$ and $\forall$ have been extended to the fuzzy case [8]).

This is an important lack by several reasons. On the one hand, number restriction is a kind of quantification that arises very frequently in concept description, so it is necessary to extend it to the fuzzy case. But another important reason is that not only concepts, but also quantifiers are imprecise in many cases (e.g. "around two", "most").

For example, suppose you are the marketing director of a supermarket chain. You are about to launch a new line of low-calorie products. In order to set up your budget, you need to project the sales of this new line of products. This can be done either by means of an expensive market research, or by means of some kind of inference based on your knowledge of customer habits. For instance, you could expect prospective buyers of this new line of products to be essentially faithful customers who mostly buy foods with low energy value. We have here all the ingredients of imprecise knowledge: a "faithful customer" is a fuzzy concept; "low" energy value is a linguistic value, which might be modeled as a fuzzy number; to "mostly" buy a given kind of product is equivalent to a quantified statement of the form "most of the bought products are of this kind", where "most" is an imprecise quantifier.

Zadeh [10] showed that imprecise quantifiers can be defined by using fuzzy sets, and by incorporating them into the language and providing the tools to define their semantics we can provide a very powerful knowledge representation tool, with greater expressive power, and closer to the humans' way of thinking. This is the objective of our work.

The paper is organized as follows: section 2 introduces briefly existing developments on fuzzy description logics. Section 3 is devoted to fuzzy quantifiers and the strongly linked issue of cardinality of fuzzy sets. Our proposal of fuzzy description logic with extended fuzzy quantification is described in section 4. Finally, section 5 contains our conclusions.

2 Fuzzy Description Logics

The idea of fuzzifying description logics to deal with imprecision is not new. Recently, a quite general fuzzy extension of description logics has been proposed, with complete algorithms for solving the entailment problem, the subsumption problem, as well as the best truth-value bound problem [8].

2.1 Fuzzy $\mathcal{ALC}$

The $\mathcal{ALC}$ description language is a basic yet significant representative of DLs. The syntax of the $\mathcal{ALC}$ language is very simple: a *concept* is built out of primitive (or *atomic*) concepts according to the grammar

$$
\begin{aligned}
\langle \text{concept\_description} \rangle ::= \; & \langle \text{atomic\_concept} \rangle \mid \\
& \top \mid \bot \mid \neg\langle \text{concept\_description} \rangle \mid \\
& \langle \text{concept\_description} \rangle \sqcap \langle \text{concept\_description} \rangle \mid \\
& \langle \text{concept\_description} \rangle \sqcup \langle \text{concept\_description} \rangle \mid \\
& \langle \text{quantification} \rangle \\
\langle \text{quantification} \rangle ::= \; & \langle \text{value\_restriction} \rangle \mid
\end{aligned}
$$

$\langle$existential_quantification$\rangle$
$\langle$value_restriction$\rangle$ ::= $\forall$ $\langle$role$\rangle$.$\langle$concept_description$\rangle$
$\langle$existential_quantification$\rangle$ ::= $\exists$ $\langle$role$\rangle$.$\langle$concept_description$\rangle$

When necessary to avoid ambiguities, parentheses should be used. From a logical point of view, concepts can be seen as unary predicates, whereas roles can be interpreted as binary predicates linking individuals to their attributes.

A usual extension of the $\mathcal{ALC}$ language, called $\mathcal{ALCN}$, is obtained by allowing number restrictions of the form $\leq nR$ and $\geq nR$. It is highly relevant to this work since number restriction is a form of quantification. The syntax of $\mathcal{ALCN}$ can be described by the following additional production rules:

$\langle$quantification$\rangle$ ::= $\langle$value_restriction$\rangle$ |
$\langle$existential_quantification$\rangle$
$\langle$number_restriction$\rangle$
$\langle$number_restriction$\rangle$ ::= $\langle$comparison_operator$\rangle\langle$natural_number$\rangle\langle$role$\rangle$
$\langle$comparison_operator$\rangle$ ::= $\leq$ | $=$ | $\geq$

Fuzzy $\mathcal{ALC}$ [8] retains the same syntax as its crisp counterpart, only semantics changes. Semantics for fuzzy $\mathcal{ALCN}$ are part of the contribution of this paper and will be discussed in next sections.

2.2 Fuzzy Interpretation

A fuzzy interpretation $\mathcal{I}$ consists of a non-empty domain $U^{\mathcal{I}}$ (the universe of discourse), and an assignment $\cdot^{\mathcal{I}}$, which maps every atomic concept A onto a fuzzy subset $A^{\mathcal{I}}$ of $U^{\mathcal{I}}$, every atomic role R onto a fuzzy binary relation $R^{\mathcal{I}} \subseteq U^{\mathcal{I}} \times U^{\mathcal{I}}$, and every individual name a onto an element $a^{\mathcal{I}} \in U^{\mathcal{I}}$. The special atomic concepts $\top$ and $\bot$ are mapped respectively onto $U^{\mathcal{I}}$ (the function that maps every individual onto 1) and the empty set (the function that maps every individual onto 0).

The semantics of the intersection, disjunction, and negation of concepts is defined as follows: for all $a \in U^{\mathcal{I}}$,

$$(C \sqcap D)^{\mathcal{I}}(a) = \min\{C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\}; \tag{1}$$

$$(C \sqcup D)^{\mathcal{I}}(a) = \max\{C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\}; \tag{2}$$

$$(\neg C)^{\mathcal{I}}(a) = 1 - C^{\mathcal{I}}(a). \tag{3}$$

For the existential quantification, there is only one possible semantics that can be given in terms of fuzzy set theory, namely

$$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in U^{\mathcal{I}}} \min\{R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b)\}. \tag{4}$$

The value restriction construct $\forall R.C$ of FDLs is interpreted in [9, 8] by translating the implication of the crisp interpretation into the *classical* fuzzy implication which directly maps the $P \supset Q \equiv \neg P \vee Q$ logical axiom:

$$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in U^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b)\}. \tag{5}$$

2.3 Fuzzy Assertions

An assertion can be either of the form $C(a) \leq \alpha$ (respectively $R(a,b) \leq \alpha$), or $C(a) \geq \alpha$ (resp. $R(a,b) \geq \alpha$), where C is a concept, R is a role, a and b are individual constants and $\alpha \in \mathbb{Q} \cap [0,1]$ is a truth degree. The two kinds of assertions are true in $\mathcal{I}$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq \alpha$ (resp. $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$), and false otherwise.

2.4 Fuzzy Terminological Axioms

Axioms and queries can be of two kinds: specializations and definitions.

A fuzzy concept specialization is a statement of the form $C \sqsubseteq D$, where C and D are concepts. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy concept specialization $C \sqsubseteq D$ if, for all $a \in U^{\mathcal{I}}$, $C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$.

A fuzzy concept definition is a statement of the form $C \equiv D$, which can be understood as an abbreviation of the pair of assertions $\{C \sqsubseteq D, D \sqsubseteq C\}$.

If $C \sqsubseteq D$ is valid (true in every interpretation), then we say that D *subsumes* C.

2.5 Fuzzy Knowledge Bases

A fuzzy knowledge base comprises two components, just like its crisp counterpart: a TBox and an ABox. While the TBox of a fuzzy knowledge base has formally nothing fuzzy with it, for the syntax of the terminological part of the fuzzy $\mathcal{ALC}$ language as defined in [8] is identical to the crisp $\mathcal{ALC}$ language, the ABox can contain fuzzy assertions.

For example, the knowledge base describing the business of running a supermarket chain could contain the following terminological axioms:

$$\begin{aligned} \mathsf{FaithfulCustomer} &\sqsubseteq \mathsf{Customer} \sqsubseteq \top \\ \mathsf{FoodProduct} &\sqsubseteq \mathsf{Product} \sqsubseteq \top \\ \mathsf{LowCalorie} &\sqsubseteq \mathsf{EnergyMeasure} \sqsubseteq \top \\ \mathsf{LowCalorieFood} &\equiv \mathsf{FoodProduct} \sqcap \forall \mathsf{energyValue}.\mathsf{LowCalorie} \end{aligned}$$

The ABox describing facts about your supermarket chain might contain fuzzy assertions which we might summarize as follows:

- given an individual customer c and a product p, $\mathsf{buys}(c,p)$ might be defined as
$$\mathsf{buys}(c,p) = f(\text{weeklyrevenue}(c,p)),$$
where $f : \mathbb{R} \to [0,1]$ is nondecreasing, and $\text{weeklyrevenue}(c,p) : \mathsf{Customer}^{\mathcal{I}} \times \mathsf{Product}^{\mathcal{I}} \to \mathbb{R}$ returns the result of a database query which calculates the average revenue generated by product p on customer c in all the stores operated by the chain;

- given an individual customer c, $\mathsf{FaithfulCustomer}(c)$ might be defined as

$$\mathsf{FaithfulCustomer}(c) = g(\text{weeklyrevenue}(c)),$$

where $g : \mathbb{R} \to [0, 1]$ is nondecreasing, and weeklyrevenue(c) : $\mathsf{Customer}^{\mathcal{I}} \to \mathbb{R}$ returns the result of a database query which calculates the average revenue generated by customer c in all the stores operated by the chain;
- finally, $\mathsf{LowCalorie}(x)$, where x is an average energy value per 100 g of product measured in kJ, could be defined as

$$\mathsf{LowCalorie}(x) = \begin{cases} 1 & x < 1000, \\ \frac{2000-x}{1000} & 1000 \leq x \leq 2000, \\ 0 & x > 2000. \end{cases}$$

By using this knowledge base, you would be able, for example, to deduce the degree to which a given food product would be a low-calorie food, and other useful knowledge implied in the TBox and ABox.

However, it would be impossible to even express the notion of a "faithful customer who mostly buys low-calorie food", let alone using that concept in deductions!

3 Cardinality and Fuzzy Quantification

Crisp quantification is strongly linked to crisp cardinality since a crisp quantifier Q represents a crisp subset of absolute (values in $\mathbb{N}$) or relative (values in $\mathbb{Q} \cap [0, 1]$) cardinalities, we call $S(Q)$. For example, $\exists$ represents the set of absolute cardinalities $\mathbb{N}\backslash\{0\}$ (equivalently the set of relative cardinalities $S(\exists) = \mathbb{Q} \cap (0, 1])$[3]. In the same way, $S(\forall) = \{1\} \subseteq (\mathbb{Q} \cap (0, 1])$ and $S(\geq n) = \mathbb{N}\backslash\{0, \ldots, n-1\}$.

Hence, cardinality plays a crucial role in the assessment of crisp quantified statements. For example, let $D \equiv QR.C$ be a concept definition and let $\mathcal{I}$ be a crisp interpretation. Let $R^{\mathcal{I}}_a$ be the projection of relation $R^{\mathcal{I}}$ on individual a: for all $b \in U^{\mathcal{I}}$,

$$b \in R^{\mathcal{I}}_a \Leftrightarrow (a, b) \in R^{\mathcal{I}}.$$

Then $D(a)$ is true iff $\left|C^{\mathcal{I}} \cap R^{\mathcal{I}}_a\right| / \left|R^{\mathcal{I}}_a\right| \in S(Q)$ (when Q is relative), or $\left|C^{\mathcal{I}} \cap R^{\mathcal{I}}_a\right| \in S(Q)$ (when Q is absolute).

In summary, the evaluation of the truth degree of a quantified sentence consists in calculating the compatibility between cardinality and quantifier. Hence, in order to extend quantification to the fuzzy case, we must discuss first about cardinality of fuzzy sets.

[3]This is the only quantifier that can be represented both ways in the general case, i.e., when the cardinality of the referential is not known.

3.1 Fuzzy cardinality

We consider two different kinds of cardinalities: absolute and relative. "Absolute cardinality", or simply "cardinality", measures the amount of elements in a set, while "relative cardinality" measures the percentage of objects of one set that are in another set.

Absolute cardinality

The most widely used definition of fuzzy set cardinality introduced in [7] is the following: given a fuzzy set F,

$$|F| = \sum_{x \in U} F(x), \tag{6}$$

which is, in general, a real number, and not an integer as it is the case with classical set cardinality.

However, this definition lends itself to many objections and leads to paradoxes. On the one hand, it is well known that the aggregation of many small values can yield a value of cardinality that does not correspond to the amount of elements in the set. One usual solution is to add only values over a certain threshold, but this is not satisfactory.

Even with high membership degrees some unintuitive results can arise. Though this is true in general for fuzzy sets, let us illustrate it with an example in terms of FDLs: consider the fuzzy concept Blonde with the interpretation $1/\mathsf{MIKE} + 0.5/\mathsf{JOHN} + 0.5/\mathsf{TONY}$. According to Equation 6, there would be exactly two instances of Blonde in this set. But, who are they? Obviously, MIKE is Blonde (i.e., $\mathsf{Blonde}(\mathsf{MIKE}) = 1$), so the other one should be JOHN or TONY. But if we consider JOHN is Blonde, then we must accept TONY also is, because $\mathsf{Blonde}(\mathsf{JOHN}) = \mathsf{Blonde}(\mathsf{TONY})$. In other words, the cardinality of this set could never be two.

These problems have motivated research over the last twenty years, where several alternative definitions have been proposed. There is a wide agreement that the absolute cardinality of a fuzzy set should be a fuzzy subset of $\mathbb{N}$. In particular, this approach is employed in the definition of the fuzzy cardinality measure ED introduced in [3] as follows:

Definition 1 ([3]). *The fuzzy cardinality of a set G is the set $ED(G)$ defined for each $0 \le k \le |\mathrm{supp}(G)|$ as*

$$ED(G)(k) = \begin{cases} \alpha_i - \alpha_{i+1} & \alpha_i \in \Lambda(G) \ \textit{and} \ |G_{\alpha_i}| = k \\ 0 & \textit{otherwise} \end{cases} \tag{7}$$

with $\Lambda(G) = \{\alpha_1, \dots, \alpha_p\} \cup \{1\}$ the level set of G, and $\alpha_i > \alpha_{i+1}$ for every $i \in \{1, \dots, p\}$, and $\alpha_{p+1} = 0$.

It is easy to see that $ED(\mathsf{Blonde}^{\mathcal{I}})(2) = 0$ since no α-cut has cardinality 2.

Relative cardinality

In the crisp case it is easy to obtain relative cardinalities from absolute ones, i.e., the relative cardinality of set G with respect to set F (i.e. the percentage of elements of G that are in F) is

$$\mathrm{RelCard}(G/F) = \frac{|G \cap F|}{|F|} \tag{8}$$

However, performing this quotient between fuzzy cardinalities is not easy and can even lead to misleading results, so it has been historically preferred to calculate it directly from the definitions of G and F.

The scalar approach based on Equation 6 poses the same problems commented before. To cope with this, a fuzzy cardinality measure called ER, that extends ED, was also introduced in [3] as follows:

Definition 2 ([3]). *The fuzzy relative cardinality of a set G with respect to a set F is the set $ER(G/F)$ defined for each $0 \leq q \leq 1$ as*

$$ER(G/F)(q) = \sum_{\alpha_i \mid C(G/F, \alpha_i) = q} (\alpha_i - \alpha_{i+1}) \tag{9}$$

with $\Lambda(F) \cup \Lambda(G \cap F) = \{\alpha_1, \ldots, \alpha_p\}$ and $\alpha_i > \alpha_{i+1}$ for every $i \in \{1, \ldots, p\}$, and $\alpha_0 = 1$, $\alpha_{p+1} = 0$.

ER is an extension of ED since, if F is crisp and $|F| = n$, then $ER(G/F)(k/n) = ED(G)(k)$ [3].

3.2 Fuzzy quantifiers

The concept of fuzzy linguistic quantifier is due to L. A. Zadeh [10]. Fuzzy quantifiers are linguistic labels representing imprecise quantities or percentages. It is usual to distinguish two basic types of fuzzy quantifiers:

- *Absolute quantifiers* express vague quantities (e.g., "Around 2") or quantity intervals (i.e., "Approximately between 1 and 3"). They are represented as fuzzy subsets of $\mathbb{N}$. For example, we could define

$$\begin{aligned} \text{“Around 2”} &= 0.5/1 + 1/2 + 0.5/3 \\ \text{“Approx. 1–3”} &= 0.5/0 + 1/1 + 1/2 + 1/3 + 0.5/4 \end{aligned}$$

- *Relative quantifiers* express fuzzy percentages and they are represented by fuzzy subsets of the real unit interval, although in practice only rational values make sense. In this category belong the standard predicate-logic quantifiers $\exists$ and $\forall$, that can be defined as

$$\exists(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 1 \end{cases} \qquad \forall(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

Some other examples are

$$\text{(Approx. Half)}(x) = \begin{cases} 2x & x \leq 0.5 \\ 2(1-x) & x \geq 0.5 \end{cases}$$

$$\text{(Approx. Half or more)}(x) = \begin{cases} 2x & x \leq 0.5 \\ 1 & x \geq 0.5 \end{cases}$$

3.3 Evaluation of quantified sentences

Quantified sentences are natural language sentences involving fuzzy linguistic quantifiers, and therefore they express claims about the (fuzzy) quantity or percentage of elements of a (possibly fuzzy) set that verify a certain imprecise property.

Following Zadeh [10] there are two main types of quantified sentences, whose general structure is the following:

Type I sentences: Q of X are G
Type II sentences: Q of F are G

where Q is a linguistic quantifier, X is a crisp finite set, and F and G are two fuzzy subsets of X that represent imprecise properties. Examples are

Type I: *Around 30 students are young*
Type II: *Most of the efficient students are young*

Type I sentences are suitable for both absolute and relative quantifiers, whilst type II sentences only make sense for relative quantifiers (i.e. a sentence like "Around 2 F are G" is in fact a type I sentence "Around 2 X are $F \cap G$").

The evaluation of a quantified sentence is the process of calculating its fuzzy accomplishment degree. There are several methods available in the literature (some methods are discussed in [4], recent developments are [6, 5]).

In order to extend fuzzy DLs with absolute and relative quantifiers, we shall employ method GD [4]. This method calculates the accomplishment degree of a quantified sentence "Q of F are G" as the compatibility degree between the fuzzy relative cardinality measure $ER(G/F)$ [3] and Q (if Q is relative) or between its absolute counterpart $ED(G \cap F)$ [3] and Q (if Q is absolute). A convenient formulation of GD is the following [4]:

Definition 3. *The method GD obtains the evaluation of a quantified sentence "Q of F are G" as*

$$GD_Q(G/F) = \sum_{\alpha_i \in \Lambda(G/F)} (\alpha_i - \alpha_{i+1})\, Q\left(\frac{|(G \cap F)_{\alpha_i}|}{|F_{\alpha_i}|}\right) \tag{10}$$

for relative quantifiers, and

$$GD_Q(G/F) = \sum_{\alpha_i \in \Lambda(G/F)} (\alpha_i - \alpha_{i+1})\, Q\left(|(F \cap G)_{\alpha_i}|\right) \tag{11}$$

for absolute ones, where $F \cap G$ *is computed using the minimum, and* $\Lambda(G/F) = \Lambda(G \cap F) \cup \Lambda(F)$. *We label these values as* $\Lambda(G/F) = \{\alpha_1, \ldots, \alpha_p\}$ *with* $\alpha_i > \alpha_{i+1}$ *for every* $i \in \{1, \ldots, p\}$ *and* $\alpha_{p+1} = 0$.

As discussed in [4], $GD_Q(G/F)$ is undefined for type II sentences when F is not normalized since $F_\alpha = \emptyset$ for at least one $\alpha \in [0,1]$ (in particular for $\alpha = 1$). Possible solutions to this problem are to normalize F (applying the same factor to $F \cap G$ after) or to define the value of any relative quantifier when the relative cardinality is undefined (we note this value as $Q(u)$, by considering $|\emptyset|/|\emptyset| = u$). In this work we shall employ this last option. In particular, $\exists(u) = 0$ and $\forall(u) = 1$.

4 Fuzzy DLs with Fuzzy Quantification

Quantified expressions in description logics are expressions like $QR.C$, where Q is a quantifier, R is a role and C is a concept, called sometimes a *qualifier*. These expressions are useful for defining new concepts, like $D \equiv QR.C$, meaning that an individual a is an instance of concept D if Q of the individuals which fill its role R are instances of C.

In crisp description logics, quantification is limited to the classical (relative) quantifiers $\exists$ and $\forall$, as well as to the so-called number restriction, $\leq n$ and $\geq n$ (crisp absolute quantifiers). In this section we propose a fuzzy description logic able to represent and reason with general absolute and relative quantifiers. The goal is to be able to express such fuzzy definitions as, e.g., $D \equiv QR.C$, where Q is now a fuzzy quantifier, meaning that an individual a is an instance of concept D *to the extent that* Q of the individuals which fill its role R are instances of C.

4.1 The $\mathcal{ALCQ}_F^+$ Language

The extended language we introduce, for which we propose the name $\mathcal{ALCQ}_F^+$, in keeping with DL naming conventions[4], has the following syntax:

$$
\begin{aligned}
\langle\text{concept\_description}\rangle ::= {} & \langle\text{atomic\_concept}\rangle \mid \\
& \top \mid \bot \mid \neg\langle\text{concept\_description}\rangle \mid \\
& \langle\text{concept\_description}\rangle \sqcap \langle\text{concept\_description}\rangle \mid \\
& \langle\text{concept\_description}\rangle \sqcup \langle\text{concept\_description}\rangle \mid \\
& \langle\text{quantification}\rangle \\
\langle\text{quantification}\rangle ::= {} & \langle\text{quantifier}\rangle\langle\text{atomic\_role}\rangle.\langle\text{concept\_description}\rangle
\end{aligned}
$$

[4]The superscript plus is to suggest that, in addition to **q**ualified number restrictions available in the description logic $\mathcal{ALCQ}$ introduced by De Giacomo and Lenzerini [2], we provide also more general fuzzy linguistic **q**uantifiers. The subscript F means that the language deals with infinitely many truth-values, as in the language $\mathcal{ALC}_{F_M}$ of Tresp and Molitor [9].

$\langle$quantifier$\rangle$ ::= "(" $\langle$absolute_quantifier$\rangle$ ")" | "(" $\langle$relative_quantifier$\rangle$ ")" | $\exists$ | $\forall$
$\langle$absolute_quantifier$\rangle$ ::= $\langle$abs_point$\rangle$ | $\langle$abs_point$\rangle$ + $\langle$absolute_quantifier$\rangle$
$\langle$relative_quantifier$\rangle$ ::= $\langle$fuzzy_degree$\rangle$/u | $\langle$fuzzy_degree$\rangle$/u + $\langle$piecewise_fn$\rangle$
$\langle$piecewise_fn$\rangle$ ::= $\langle$rel_point$\rangle$ | $\langle$rel_point$\rangle$ + $\langle$piecewise_fn$\rangle$
$\langle$abs_point$\rangle$ ::= $\langle$val$\rangle$/$\langle$natural_number$\rangle$
$\langle$rel_point$\rangle$ ::= $\langle$val$\rangle$/$\langle$[0,1]-value$\rangle$
$\langle$val$\rangle$ ::= [$\langle$fuzzy_degree$\rangle$ $\lhd$]$\langle$fuzzy_degree$\rangle$[$\rhd$ $\langle$fuzzy_degree$\rangle$]

In this extension the semantics of quantifiers is defined by means of piecewise-linear membership functions. In the case of absolute quantifiers, the quantifier is obtained by restricting the membership function to the naturals.

The piecewise-linear functions are defined by means of a sequence of points. These points are expressed as $\alpha \lhd \beta \rhd \gamma/x$, where x is the cardinality value, β is the membership degree of x, and α and γ are the limit when the membership function tends to x at the left and at the right, respectively. When the function is continuous, this can be summarized as β/x (since $\alpha = \beta = \gamma$), whereas discontinuities on the left ($\alpha \neq \beta = \gamma$) or right ($\alpha = \beta \neq \gamma$) can be summarized as $\alpha \lhd \beta/x$ and $\beta \rhd \gamma/x$, respectively.

Obviously, for a membership function definition of the form $val_1/x_1 + val_2/x_2 + \cdots + val_p/x_p$ it is required $x_i \neq x_j$ $\forall i < j$.

For relative quantifiers, as pointed out in the previous section, we should take into account the definition of a membership degree for the case "undefined" that arises when the referential set with respect to which the relative cardinality is calculated is empty[5]. This value could depend on the subjective view of the user or the application at hand, though it is well known and fixed for some quantifiers like $\exists$ and $\forall$, as we have seen.

Unless a different definition is provided explicitly, we shall assume that

- the points 0/0 and 1/1 are part of the definition of any relative quantifier, and
- the point 0/0 is part of the definition of any absolute quantifier. Also, let x_l be the greatest natural value in the definition of an absolute quantifier and let $\alpha_l \lhd \beta_l \rhd \gamma_l$ be the values for x_l. Then, for any $x > x_l$ we assume γ_l/x.

The following is a set of absolute quantifiers and their corresponding (subjective) expressions using the proposed notation:

[5]Furthermore, the presence of the definition of a membership degree for the case "undefined" univocally identifies a quantifier as relative, even in those rare cases in which a doubt might arise.

(Around 2)	$(1/2 + 0/4)$
(Approx. between 1 and 3)	$(0.5/0 + 1/1 + 1/3 + 0/5)$
(Exactly 3)	$(0 \lhd 1 \rhd 0/3)$
> 5	$(0 \rhd 1/5)$
< 8	$(1/0 + 1 \lhd 0/8)$
(Around 7)	$(0/5 + 1/7 + 0/9)$

Some examples of relative quantifiers are the following, where $Q_M(x) = x$ is a quantifier sometimes called "Most"[6], and α_i represent user-defined values for the case "undefined":

$\forall$	$(1/u + 0 \lhd 1/1)$
$\exists$	$(0/u + 0 \rhd 1/0)$
(Approx. half)	$(\alpha_1/u + 1/0.5 + 0/1)$
(Approx. half or more)	$(\alpha_2/u + 1/0.5)$
Q_M	(α_3/u) or $(\alpha_3/u + 0/0 + 1/1)$
(Around 75%)	$(\alpha_4/u + 0/0.25 + 1/0.75 + 0.5/1)$

The definition of quantifiers we have introduced generalizes the classical quantifiers $\exists$ and $\forall$ (particular cases of relative quantifiers), as we have just seen, so the symbols $\exists$ and $\forall$ are included in the language only by historical reasons and to preserve backward compatibility. In addition, the language employed to define quantifiers generalizes number restriction (particular cases of absolute quantifiers), since $\leq n$ translates to $(1/0 + 1 \rhd 0/n)$ and $\geq n$ translates to $(0 \lhd 1/n)$.

Finally, in order to name quantifiers we shall employ the same notation used to name concepts, for example

(Around 2) $\equiv (1/2 + 0/4)$.

As for the classical quantifiers $\exists$ and $\forall$, by including them in the $\mathcal{ALCQ}_F^+$ language we are assuming implicitly the definitions:

$\forall \equiv (1/u + 0 \lhd 1/1)$;
$\exists \equiv (0/u + 0 \rhd 1/0)$.

When a quantifier is defined which is denoted by a single mathematical symbol (possibly followed by a single number), the parentheses around the quantifier name might be dropped without risk of confusion. For example,

$$\simeq 2 \equiv (1/2 + 0/4);$$
$$\widetilde{<}2 \equiv (1/0 + 1/2 + 0/4).$$

[6] Q_M could be called "Cardinal", because the membership degree is exactly the cardinality in the crisp case.

4.2 Semantics

Given a fuzzy interpretation $\mathcal{I}$, the semantics of the intersection, disjunction, and negation of concepts in our language keep being those introduced in section 2. For the general quantification, we can translate a general expression like $D \equiv (QR.C)$ into the usual notation of quantified sentences introduced in the previous chapter as follows: if Q is an absolute quantifier, the degree to which an individual a verifies concept D is the result of the evaluation of the quantified sentence

Q of $U^{\mathcal{I}}$ are $R_a^{\mathcal{I}} \cap C^{\mathcal{I}}$,

whilst for relative quantifiers, it is the result of the evaluation of the sentence

Q of $R_a^{\mathcal{I}}$ are $C^{\mathcal{I}}$,

where $R_a^{\mathcal{I}}$ is the projection of fuzzy relation $R^{\mathcal{I}}$ on individual a: for all $b \in U^{\mathcal{I}}$,

$$R_a^{\mathcal{I}}(b) = R^{\mathcal{I}}(a, b).$$

Hence, we consider two cases depending on whether the quantifier is absolute (12) or relative (13):

$$(Q_{\text{abs}}R.C)^{\mathcal{I}}(a) = GD_{Q_{\text{abs}}}((R_a^{\mathcal{I}} \cap C^{\mathcal{I}})/U^{\mathcal{I}}) \tag{12}$$

$$(Q_{\text{rel}}R.C)^{\mathcal{I}}(a) = GD_{Q_{\text{rel}}}(C^{\mathcal{I}}/R_a^{\mathcal{I}}) \tag{13}$$

where GD is the evaluation method introduced in definition 3. In particular the semantics for the quantifiers $\exists$ and $\forall$ is:

$$(\exists R.C)^{\mathcal{I}}(a) = \sum_{(R_a^{\mathcal{I}})_{\alpha_i} \cap (C^{\mathcal{I}})_{\alpha_i} \neq \emptyset} (\alpha_i - \alpha_{i+1}) \tag{14}$$

$$(\forall R.C)^{\mathcal{I}}(a) = \sum_{(R_a^{\mathcal{I}})_{\alpha_i} \subseteq (C^{\mathcal{I}})_{\alpha_i}} (\alpha_i - \alpha_{i+1}) \tag{15}$$

with $\alpha_i \in \Lambda(G/F)$. Following the properties of GD [4], the semantics of the existential quantification keep being as usual, i.e., Equation 14 is equivalent to

$$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in U^{\mathcal{I}}} \min\{R_a^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\}.$$

As a particular case, when the referential F is crisp, then GD verifies De Morgan's laws, i.e.,

$$1 - GD_{\forall}(G/F) = GD_{\exists}(\neg G/F)$$
$$1 - GD_{\exists}(G/F) = GD_{\forall}(\neg G/F)$$

Hence, if $R_a^{\mathcal{I}}$ is crisp then Equation 15 is equivalent to

$$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in U^{\mathcal{I}}} \max\{1 - R_a^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\}.$$

However, this is not true in general. In fact, in order to verify De Morgan's laws it is necessary that

$$R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}} \Leftrightarrow R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} = \emptyset$$

but this equivalence holds only when $R_a^{\mathcal{I}}$ (or $C^{\mathcal{I}}$) is crisp. Otherwise we could have a situation where both sets are fuzzy and $R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ (and hence the evaluation using $\forall$ is expected to yield 1), but $R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} \neq \emptyset$ (and hence the evaluation using $\exists$ is expected to yield a value greater than 0). For example, suppose

$$\begin{aligned} R_a^{\mathcal{I}} &= 1/b_1 + 0.6/b_2 + 0.4/b_3 \\ C^{\mathcal{I}} &= 1/b_1 + 0.9/b_2 + 0.5/b_3 \end{aligned}$$

then $R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ and $R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} = 0.1/b_2 + 0.4/b_3 \neq \emptyset$, consequently $GD_\forall(C^{\mathcal{I}}/R_a^{\mathcal{I}}) = 1$ and $GD_\exists((\neg C)^{\mathcal{I}}/R_a^{\mathcal{I}}) = 0.4 \neq 0$.

4.3 An Example

Let us go back to our low-calorie product line example. By using the $\mathcal{ALCQ}_F^+$ language, it is now possible to express the notion of a faithful customer who mostly buys food with low energy value as

$$C \equiv \mathsf{FaithfulCustomer} \sqcap (\mathsf{Most})\mathsf{buys.LowCalorieFood},$$

where $(\mathsf{Most}) \equiv (0/u + 0/0.5 + 1/0.75)$.

A useful deduction this new axiom allows you to make is, for instance, calculating the extent to which a given individual customer or, more precisely, a fidelity card, say CARD0400009324198, is a C. For instance, you could know that

$$\mathsf{FaithfulCustomer}(\mathsf{CARD0400009324198}) = 0.8,$$

and, by querying the sales database, you might get all the degrees to which that customer buys each product. For sake of example, we give a small subset of those degrees of truth in Table 1, along with the energy values of the relevant products.

According to the semantics of $\mathcal{ALCQ}_F^+$,

$$C(\mathsf{CARD0400009324198}) \approx 0.742$$

i.e., the degree to which most of the items purchased by this customer are low-calorie is around 0.742. This seems to be in accordance with the data in table 1, where we can see that four products (those products p in rows 2, 4, 5, and 6) verify

$$\mathsf{buys}(\mathsf{CARD0400009324198}, p) \leq \mathsf{LowCalorieFood}(p)$$

Product	Energy [kJ/hg]	$\mathsf{LowCalorieFood}(\cdot)$	$\mathsf{buys}(\mathsf{CARD}\ldots,\cdot)$
GTIN8001350010239	1680	0.320	0.510
GTIN8007290330987	1475	0.525	0.050
GTIN8076809518581	1975	0.025	0.572
GTIN8000113004003	1523	0.477	0.210
GTIN8002330006969	498	1.000	1.000
GTIN8005410002110	199	1.000	1.000
GTIN017600081636	1967	0.033	0.184

Table 1. The energy value, membership in the LowCalorieFood, and the degree to which customer CARD0400009324198 buys them for a small sample of products.

while for the products in rows 1 and 7 the difference between being purchased and being low-calorie food is not so high. Only the item in row 3 seems to be a clear case of item purchased but not low-calorie.

As another justification of why this result appears in agreement with the data, in Table 2 we show the percentage of purchased items that are low-calorie at α-cuts of the same level. At any other level, the percentage obtained is one of those shown in Table 2.

Level	Percentage
1.000	$1.000 = 2/2$
0.572	$0.667 = 2/3$
0.510	$0.500 = 2/4$
0.320	$0.750 = 3/4$
0.210	$0.800 = 4/5$
0.184	$0.667 = 4/6$
0.050	$0.714 = 5/7$
0.033	$0.857 = 6/7$

Table 2. Percentage of purchased items that are low-calorie at significant levels.

At many levels the percentage is above 0.75, therefore fitting the concept of Most as we have defined it. At level 0.050 the percentage is almost 0.75. The only level that clearly doesn't fit Most is 0.510, but at the next level (0.320) we have again 0.75 and $\mathsf{Most}(0.75) = 1$.

5 Conclusions

$\mathcal{ALCQ}_F^+$ allows for concept descriptions involving fuzzy linguistic quantifiers of the absolute and relative kind, and using qualifiers. It provides also semantics for crisp quantifiers like $\forall$, $\exists$, and number restriction in those cases where the roles and/or qualifiers employed are fuzzy.

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