A Fuzzy Frame-Based Knowledge Representation Formalism

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Abstract. This paper describes a formalism for representing imprecise knowledge which combines traditional frame-based formalisms with fuzzy logic and fuzzy IF-THEN rules. Inference in this formalism is based on unification and the calculus of fuzzy IF-THEN rules, and lends itself to an efficient implementation.

1 Introduction

This paper describes a frame-based formalism for representing imprecise knowledge, developed within a large research project on knowledge management.

The formalism is frame-based, but frames are considerably simplified to achieve an elegant algebraic description, which is heavily inspired by the unification-based grammar formalisms [4] developed in the last two decades by the natural language processing community. Unification, indeed, plays a central role in this formalism as the main inference rule, which implements multiple inheritance.

The frame-based formalism is extended to accommodate uncertainty and imprecision by combining it with fuzzy logic [5], in a way analogous to other proposals whose aim was to combine frame-based knowledge representation formalisms with Bayesian networks [2]. Furthermore, the formalism incorporates procedural information in the form of fuzzy IF-THEN rules.

According to this formalism, knowledge consists of three basic types of objects:

– knowledge *elements*, which can be either atomic (*atoms*) or complex (*frames*);
– fuzzy sets, or *linguistic values*;
– *relations*, which can be fuzzy rules or subsumption relations.

2 Knowledge Elements

A knowledge element captures the intuitive notion of a *concept*. The set of all knowledge elements will be denoted by $\mathcal{E}$. An element can be either atomic or complex.

2.1 Atomic Elements

An atomic element or, simply, an *atom*, is a concept which cannot be (or is not) analyzed as an aggregate of simpler components. In an application which takes

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for granted the nature of numbers, this might be the case of numerical values like 1, 4/5, 193,439,499, or π; another general example of atomic elements might be the two opposites *yes* : *no*, or *present* : *absent*, or $+ : -$, used to specify whether a given feature is possessed or not by a concept.

This is what is sometimes called an *individual* in description logics [1]. The set of all atomic elements will be denoted by $\mathcal{C}$.

2.2 Complex Elements

A complex element, on the contrary, is a concept which can (and actually is) broken down into more basic components, or features. A complex element can be thought of as a logic type or sort, or, using the terminology of object-oriented systems, a *class*.

We represent such an element as a *frame* [3], i.e., an aggregate of *slots*, where a slot is an attribute-value pair. However, we depart from conventional frames in dispensing with *facets*, and in treating type, value restriction, and values uniformly. The set of all complex elements will be denoted by $\mathcal{F}$.

Each slot predicates a given feature of the element, identified by its attribute, from the set $\mathcal{A}$. This predication is obtained by making a restriction on the values (other elements) that attribute may take up. In order to capture the (possibly) uncertain nature of knowledge, this value restriction is regarded as a *possibility distribution* over $\mathcal{E}$. In other words, the set of values of an attribute may take up is fuzzy: classical sets are thus provided for as a special case[1].

Therefore, a slot is an association between an attribute and a fuzzy set of knowledge elements, which must be regarded as its possible (or admissible) values.

Given a frame $x \in \mathcal{F}$, the (possibly fuzzy) value of its slot identified by attribute $a \in \mathcal{A}$ will be denoted by $x.a$. Therefore, $x.a$ is the fuzzy set of possible values (a possibility distribution) for attribute a that an individual of class x might have. So, in a sense, we can regard $x.a$ both as a restriction on the values of attribute a, or as an assignment of a set of values to attribute a. The most specific case is when $x.a$ is the singleton set $\{e\}$, with $e \in \mathcal{E}$, which we may read "the a of x is e", as in "the *elevation* of *Mount Everest* is *8,850 m*".

We say attribute $a \in \mathcal{A}$ in frame $x \in \mathcal{F}$ is unrestricted if and only if $x.a = \mathcal{E}$. The frame for which all attributes are unrestricted is a representation of the most general element, the one that subsumes all other knowledge elements, atomic or complex: we call it the *top* element, and denote it by the symbol $\top$.

[1] In general, one would expect frames defining types with a high level of abstraction to have crisp sets as the values of their slots, and low level class frames and instance frames to show more and more fuzziness. The intuitive motivation for this is simple: we may have a very clear abstract idea of what *white* and *yellow* are: these are our conceptual categories, our ontology. Things become more complicated when we want to describe what we know about the color of a real, actual object, say an old sheet of paper. Then we might find its color to fit into the white type only to a degree, and into the yellow type to another degree. This agrees very well with our everyday experience: abstractions are clean-cut and simple (because they abstract away from unimportant details), whereas reality is fuzzy and complex.

Dual to this element is the *bottom* element $\perp$, subsumed by all other elements, which, in intuitive terms, is the equivalence class of those frames that cannot correspond to any "actual" concept, just because at least one of their slots has the empty set as the set of admissible values. We might also refer to this element as the *inadmissible* element. The $\perp$ element is something we would never want to have in our knowledge base, since it would be the index of a logical contradiction or inconsistency.

It is clear that the definition of a complex element is recursive, since we do not restrict the elements in the fuzzy set of admissible values of attributes to be atomic. In principle, there is no limit to the level of nesting of complex elements.

3 Fuzzy Sets of Elements

We will use min as the t-norm and max as the t-conorm, and represent a fuzzy set S of knowledge elements as a mapping $S: \mathcal{E} \to [0, 1]$, thus writing $S(e)$, for all $e \in \mathcal{E}$, to denote the membership degree of e in S. We will also adopt Zimmermann's [7] notation of fuzzy sets as formal summations (or integrals).

Accordingly, given two fuzzy sets of elements S and T:

- $S \subseteq T$ if and only if, for all $e \in \mathcal{E}$, $S(e) \leq T(e)$;
- $[S \cup T](e) = \max\{S(e), T(e)\}$ for all $e \in \mathcal{E}$;
- $[S \cap T](e) = \min\{S(e), T(e)\}$ for all $e \in \mathcal{E}$;
- $\bar{S}(e) = 1 - S(e)$ for all $e \in \mathcal{E}$.

Given a fuzzy set S, its *support*, denoted by $\mathrm{supp}(S)$, is the crisp set of all $e \in \mathcal{E}$ such that $S(e) > 0$.

A slot mapping attribute $a \in \mathcal{A}$ into a fuzzy set of elements S will be denoted as $a: S$, and a frame will be denoted as a column vector of slots.

Actually, a complex knowledge element is a bipartite graph, whose nodes are of two types: knowledge elements and fuzzy sets; arcs from an element to a set are labeled by an attribute, whereas arcs from a set to an element are labeled by a membership degree. To represent such structures on paper, when more than one attribute share the same set of values or the same frame is a member of more than one set, a label (the name of a variable) enclosed in parentheses will be put at the left of the first mention of the referred object and will be used to stand for the same object, like in

$$\begin{bmatrix} \text{grandparent: (x)} \dfrac{1}{\begin{bmatrix} \text{name: } \frac{1}{\text{John}} \end{bmatrix}} \\ \text{parent: } \dfrac{1}{\begin{bmatrix} \text{parent: (x)} \\ \text{name: } \frac{1}{\text{Peter}} \end{bmatrix}} \\ \text{name: } \dfrac{1}{\text{George}} \end{bmatrix}. \tag{1}$$

4 Interpretation

The semantics of this formalism is given by a fuzzy interpretation, consisting of a non-empty domain $\mathcal{U}$, the universe of discourse, and an assignment $\cdot^{\mathcal{I}}$, which maps every knowledge element e into a fuzzy subset $e^{\mathcal{I}}$ of $\mathcal{U}$. We extend this assignment to map fuzzy sets of knowledge elements into fuzzy subsets of $\mathcal{U}$. In particular, it is useful to define

$$\mathcal{E}^{\mathcal{I}} \equiv \bigcup_{e \in \mathcal{E}} e^{\mathcal{I}}, \quad \mathcal{C}^{\mathcal{I}} \equiv \bigcup_{c \in \mathcal{C}} c^{\mathcal{I}}, \text{ and } \quad \mathcal{F}^{\mathcal{I}} \equiv \bigcup_{x \in \mathcal{F}} x^{\mathcal{I}}.$$

In general, given a fuzzy set S of knowledge elements, for all $u \in \mathcal{U}$,

$$S^{\mathcal{I}}(u) = \max_{e \in \mathcal{E}} \min\{S(e), e^{\mathcal{I}}(u)\}.$$

This is the fuzzy equivalent of saying that $u \in S^{\mathcal{I}}$ iff there exists a knowledge element e such that $e \in S$ and $u \in e^{\mathcal{I}}$.

Individual objects of the application domain (individuals for short) are the elements of $\mathcal{U}$; they will be denoted by u, v, etc.

Therefore, for all $e \in \mathcal{E}$ and $a \in \mathcal{A}$, and for all individuals $u, v \in \mathcal{U}$,

1. $0 \leq e^{\mathcal{I}}(u) \leq 1$;
2. $0 \leq a^{\mathcal{I}}(u, v) \leq 1$;
3. $\top^{\mathcal{I}}(u) = \mathcal{E}^{\mathcal{I}}(u) = 1$;
4. $\bot^{\mathcal{I}}(u) = 0$.

A fuzzy set S of knowledge elements might be equivalent, in terms of the interpretation, to a single knowledge element x: we use precisely this property to define equality between a fuzzy set of knowledge elements and knowledge elements, as in the case of $\top$ and $\mathcal{E}$. We define $x = S$ if and only if, for all $u \in \mathcal{U}$, $x^{\mathcal{I}}(u) = S^{\mathcal{I}}(u)$.

Furthermore, the interpretation maps every attribute a into a fuzzy binary relation $a^{\mathcal{I}} \subseteq \mathcal{F}^{\mathcal{I}} \times \mathcal{U}$, whereby, for all $u, v \in \mathcal{U}$,

$$\begin{aligned} a^{\mathcal{I}}(u, v) &= \max_{x \in \mathcal{F}} \min\{x^{\mathcal{I}}(u), (x.a)^{\mathcal{I}}(v)\} \\ &= \max_{x \in \mathcal{F}} \min\{x^{\mathcal{I}}(u), \max_{e \in \mathcal{E}} \min\{x.a(e), e^{\mathcal{I}}(v)\}\} \end{aligned}$$

as it is intuitive, i.e., for all the frames x by which individual u is described (to a certain degree), we say that the possibility that v is a value of attribute a for u is in fact the possibility that $u \in x^{\mathcal{I}}$ *and* $v \in (x.a)^{\mathcal{I}}$; of all the possible frames describing u, we take the one that yields the maximum degree of membership.

Atomic knowledge elements are all mutually disjoint, and, as a whole, they are disjoint from complex knowledge elements[2].

[2] Therefore, one might assume that atomic knowledge elements represent individual elements in the universe of discourse, and nothing would change. As a matter of fact, if the interpretation of an atom contains more than one element, those elements are virtually indistinguishable for the formalism.

5 Subsumption

Given $x, y \in \mathcal{F}$, x subsumes y ($x \sqsupseteq y$), iff, for all $u \in \mathcal{U}$, $y^{\mathcal{I}}(u) \leq x^{\mathcal{I}}(u)$. Subsumption can also be extended to fuzzy sets of elements: given $S, T \subseteq \mathcal{E}$, $S \sqsupseteq T$ iff $(S \cap \mathcal{C}) \supseteq (T \cap \mathcal{C})$ and, for all $a \in \mathcal{A}$,

$$\bigcup_{x \in S \cap \mathcal{F}} x.a \sqsupseteq \bigcup_{y \in T \cap \mathcal{F}} y.a. \tag{2}$$

By their very nature, atomic elements cannot subsume each other or any other element than $\bot$, and be subsumed by any other element than $\top$.

Frames, on the other hand, can subsume and be subsumed by other frames. However, in order to precisely define subsumption over $\mathcal{F}$, we need to extend the definition of subsumption to the fuzzy sets of elements.

A natural extension in terms of the sementics defined in Section 4 is the following: given $S, T \subseteq \mathcal{E}$, $S \sqsupseteq T$ if and only if

$$(S \cap \mathcal{C}) \supseteq (T \cap \mathcal{C}). \tag{3}$$

In other words, Equation 3 requires the set of atoms in fuzzy set S to include the set of atoms in fuzzy set T, whereas Equation 2 requires the union of the values of attribute a for all frames in S to subsume the union of the values of the same attribute for all the frames in T.

Given this definition, subsumption between frames can be defined as follows: Given $x, y \in \mathcal{F}$, $x \sqsupseteq y$ if and only if, for all $a \in \mathcal{A}$, $x.a \sqsupseteq y.a$. This definition is recursive just like frames are recursive; however, recursion ends as soon as atomic values are reached, and Equation 3 is applied, or an unrestricted attribute appears on the left-hand side of Equation 2, whose value $\mathcal{E} = \top$ subsumes everything.

Subsumption defines a partial ordering of elements, and it is easy to verify that $\mathcal{E}$ forms a complete lattice with respect to the subsumption ordering relationship.

6 Unification

The *meet* operation on this lattice is called *unification*: the unification of two elements $x, y \in \mathcal{E}$ is the element $z = x \sqcap y$ such that, for all $\hat{z} \in \mathcal{E}$, $x \sqsupseteq \hat{z}$ and $y \sqsupseteq \hat{z}$ implies $z \sqsupseteq \hat{z}$, i.e., the most general element subsumed by both x and y.

It is easy to derive a more operational definition, leading to a unification algorithm whose time complexity is $O(nm)$, where n and m are the "size" of the two frames or fuzzy sets that are to be unified: for all $x \in \mathcal{E}$, $c, d \in \mathcal{C}$,

- $x \sqcap x = x$;
- $\top \sqcap x = x \sqcap \top = x$;
- $\bot \sqcap x = x \sqcap \bot = \bot$.
- $c \neq d$ implies $c \sqcap d = \bot$.
- $x \in \mathcal{F}$, $\top \neq x \neq \bot$, implies $c \sqcap x = x \sqcap c = \bot$.

For all $x, y \in \mathcal{F}$, and for all $a \in \mathcal{A}$,

$$(x \sqcap y).a = x.a \sqcap y.a; \tag{4}$$

furthermore, for all fuzzy sets of elements $S, T \subseteq \mathcal{E}$,

$$S \sqcap T = \sum_{e \in \mathcal{E} \setminus \perp} \frac{\min_{x \in S, y \in T : x \sqcap y = e} \{S(x), T(y)\}}{e}, \tag{5}$$

or, equivalently, for all $x \in S$ and $y \in T$,

$$[S \sqcap T](e) = \begin{cases} \min_{x \in S, y \in T : x \sqcap y = e} \{S(x), T(y)\}, & \text{if } e \neq \perp; \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Translated into a simple algorithm, Equation 6 says that, in order to unify two fuzzy sets S and T, one must:

1. try to unify every member of S with every member of T, obtaining a collection of elements;
2. discard all $\perp$ results;
3. associate with each resulting element a degree of membership that is the minimum between those of the two operands in their respective set;
4. merge multiple copies of the resulting elements while taking as their degree of membership the minimum;
5. build the unified set $S \sqcap T$ from all the remaining results.

It is clear that the number of intermediate elements $x \sqcap y$ to be calculated is $\|\text{supp}(S)\| \cdot \|\text{supp}(T)\|$; therefore, if n is the typical cardinality of the supports of the fuzzy sets involved in unification operations, the computational complexity of calculating unification between sets is $O(n^2)$ frame unifications. Frame unification, in turn, is linear in the number of slots. Since some of the unifications $x \sqcap y$, with $x \in S$ and $y \in T$ might fail, $\|\text{supp}(S \sqcap T)\| \leq \|\text{supp}(S)\| \cdot \|\text{supp}(T)\|$.

In the special case where $x.a \subseteq \mathcal{C}$ and $y.a \subseteq \mathcal{C}$, i.e, when attribute a can take only atomic values, Equations 4 and 6 combine to give

$$(x \sqcap y).a = x.a \cap y.a. \tag{7}$$

7 Fuzzy Rules

Whereas frames and elements provide the descriptive devices of the representation formalism, relations, in the form of subsumption relations and fuzzy rules, provide the main mechanisms for inference.

A fuzzy IF-THEN rule has the form

$$\text{IF } S_1 \text{ is } T_1 \text{ AND } \ldots \text{ AND } S_{n-1} \text{ is } T_{n-1} \text{ THEN } S_n \text{ is } T_n, \tag{8}$$

where $S_i, T_i \subseteq \mathcal{E}$, $i = 1, \ldots, n$.

The degree of truth of an antecedent clause "S_i is T_i" is given by the maximum degree of membership of the members of $S_i \sqcap T_i$, or

$$\tau(S_i \text{ is } T_i) = \sup_{e \in \mathcal{E}} \{(S_i \sqcap T_i)(e)\}. \tag{9}$$

The degree of truth of the consequent clause equals the smallest degree of truth of its antecedents:

$$\tau(S_n \text{ is } T_n) = \min_{i=1,\ldots,n-1} \{\tau(S_i \text{ is } T_i)\}. \tag{10}$$

8 Inference

When a knowledge engineer models a domain, she constructs an ontology, say $\mathcal{O}$, by defining knowledge elements and connecting them by means of

1. subsumption axioms, of the form $x \sqsupseteq y$, where $x, y \in \mathcal{F}$,
2. slot-value axioms of the form $x.a(e) = \alpha$, where $x \in \mathcal{F}$, $e \in \mathcal{E}$, and $0 \leq \alpha \leq 1$ is a membership degree of e in $x.a$,
3. fuzzy IF-THEN rules, defined in Section 7.

During this process, it is important to find out whether a newly defined knowledge element makes sense or whether it is contradictory. From a logical point of view, a knowledge element makes sense if there is some interpretation that satisfies the axioms of $\mathcal{O}$ (that is, a model of $\mathcal{O}$) such that the concept denotes a nonempty fuzzy set in that interpretation. A knowledge element with this property is said to be *satisfiable* with respect to $\mathcal{O}$ and *unsatisfiable* otherwise.

Afterwards, when the knowledge contained in $\mathcal{O}$ is used, it is important to be able to calculate logical consequences of the above three types of axioms. Both tasks require a system to perform some kind of inference.

Two basic mechanisms for inference are provided for within this knowledge representation formalism:

- *inheritance* according to the subsumption relation;
- *mapping*, i.e., functional dependence, according to the calculus of the fuzzy IF-THEN rules [6].

The two mechanisms are strictly combined and operate at the same time, in the sense that the value of every slot is the unification of all the slots subsuming it (inheritance) and of all the values of the consequent clauses referring to it.

Inheritance and mapping combine as follows:

$$x.a = \left(\coprod_{r} T_r \cap \tau(x.a \text{ is } T_r) \right) \sqcap \prod_{y \sqsupseteq x} y.a. \tag{11}$$

where the T_r's are the values predicated by the consequents of the rules

$$\text{IF} \ldots \text{THEN } x.a \text{ is } T_r.$$

It should be noted that in Equation 11 the inherited value of an attribute is given by the unification of all its values in the subsuming frames, *including* the frame itself. Therefore, the semantics of the explicitly defined value of a slot is to be understood "modulo" any more restrictive definition inherited from superclasses. Furthermore, the contribution of fuzzy rules, as dictated by the calculus of fuzzy rules, is disjunctive. This allows us to exploit the interpolative behavior of fuzzy rules.

9 Conclusions

A knowledge representation formalism has been described which is based on the combination of three concepts:

– frame-based knowledge representation formalisms;
– unification, as found in unification-based grammar formalisms;
– fuzzy set theory.

The resulting formalism has been specially developed for and used in an innovative technological framework for knowledge management, whose main feature is the ability to semantically index documents by representing the (uncertain) knowledge about their content and relating it to an ontology. That framework is currently being validated by means of three vertical applications in the domains of banking regulations, insurance, and IT service outsourcing.

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