A Possibilistic Framework for Asset Allocation

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Abstract. This paper is a first step in laying down a theoretical foundation for asset allocation under uncertainty, based on possibility theory. First, fuzzy financial knowledge representation is treated. Then, some issues relevant to financial decision making are approached from the standpoint of possibility theory.

1 Introduction

Asset allocation has become a very complicated problem, as regulatory constraints, individual investor's requirements, non-trivial indices of risk and subjective quality measures are taken into account, together with multiple investment horizons, cash-flow planning and forecasting techniques based on refined econometric models.

This problem has been approached in a probabilistic framework by using sampling techniques to build a tree of possible scenarios for the future from a set of predictions (in the form of joint probability distributions), and an evolutionary algorithm was used to optimize an investment plan against the desired criteria and the possible scenarios [2].

Even though such an approach gave interesting results, we felt that an approach based on probabilistic predictions only is not fully satisfactory, for reasons illustrated below, and that a framework grounded on possibility theory would provide a much more reliable foundation for asset allocation. An approach using fuzzy logic for portfolio selection has been proposed by Ramaswamy in [11]. This work is structured in two parts: Section 2 discusses how uncertainty in financial knowledge can be modeled using fuzzy logic; Section 3 deals with how to use uncertain knowledge to evaluate decisions. Section 4 suggests future directions of research.

2 Uncertainty Representation in Financial Knowledge

2.1 Vagueness, Uncertainty and Imprecision in Finance

Knowledge in Finance is constituted essentially of the views the experts have on the behavior of the market. These views like “index A has good prospects for growth"” are described by the economists as being uncertain, vague, or imprecise in terms of both the direction and the size of market moves.

In the most known method of portfolio management due to Markowitz [12], the *risk* of an investment is used as a synonym for *uncertainty*, and it is measured as the *variance* of a (normally distributed) random variable. The intuitive sense of this hypothesis is that high fluctuations mean greater unpredictability and therefore greater risk. This choice might be explained by the fact that main-stream economists are well acquainted with probability theory, and this has been their main (if not their only) tool to work with uncertainty.

To begin with, it has been shown that variance does not capture the essence of risk for an investor [9]. Besides, some economists make a technical distinction between risk, as a situation in which there are well defined, unique, and generally accepted objective probabilities, and uncertainty [7]. We believe that the scope of uncertainty in this sense is much broader in finance than it is generally acknowledged.

We must draw a clear distinction between uncertainty, vagueness, and imprecision. Uncertainty has to do with telling which of a set of possible events will happen; in the case of vagueness and imprecision, a variable cannot be characterized by a single value of the domain: a vague value is a linguistic label with a possibility distribution, while an imprecise value is an interval of the domain.

These considerations point out that uncertainty differs from imprecision and vagueness, even though it may sometimes result from them [4]. For example, if an analyst says: “There are good chances that the quarterly revenues of company *X* will show a very slight acceleration”, we have uncertainty, in that it is not certain that the event “the quarterly revenues of company *X* will show a slight acceleration” will happen, but this event is assigned a “probability” through the phrase “there are good chances”; the event itself vague.

States and Scenarios We may define a *state s* as an assignment of given values to all the variables describing the system of interest, in our case a market. We will denote by *S* the set of all possible states of the market. We suppose that there exists a state in *S* that is completely possible, i.e., whose possibility is one. In the case of total ignorance, all states are completely possible.

Although the concept of state is fully appropriate to describe market conditions in the past or in the present, which are known exactly, when reasoning about the future it is convenient to deal with more macroscopic entities, like scenarios.

The word *scenario* is widely used by economists and market analysts to describe their predictions or hypotheses about market behavior. Here, we want to assign to this word a precise meaning, namely, a set of conditions that delimit a conceivable set of states of affairs in the future. For example: "inflation will grow and GNP will increase by less than 3%". As we can see, the formulation of a scenario concentrates on a few variables of interest, and it does not say anything about other variables that are not relevant to the point the analyst wants to make.

The Case for Fuzzy Logic Although economists normally state their arguments in probabilistic terms, in many cases it makes no real sense to speak of probability, for the repeatability of events, which is typical of the frequentist approach, is lacking. The most commonly used theory of decision-making under uncertainty is subjective expected utility theory, which implies that an individual has beliefs over uncertain events which have the form of probabilities. These are referred to as *subjective probabilities*.

However, it does not seem plausible that people are always able to assign probabilities to relevant events when they are faced with a problem involving uncertainty, especially if this uncertainty stems from vague or imprecise information. As a matter of fact, there is evidence that people do not have subjective probabilities in some situations involving uncertainty [1]. Moreover, this approach, which would appear more plausible at first sight, is dangerous, for those probabilities are employed in exact calculations and may give the (false) impression that the results thus obtained are exact.

Alternative theories have been proposed by economists to explain uncertainty-averse behavior, for instance complete ignorance, convex sets of probabilities, and non-additive probabilities [7, 8]. These theories are anyway grounded on classical probability theory.

A more promising approach is made possible by fuzzy logic [14], which allows us to model not only uncertainty (to which probability could be suited), but also vagueness and imprecision. Fuzzy sets lead to a gradual theory of uncertainty which differs in its purpose from the one of frequentist probability. This is possibility theory, where gradual notions of possibility and necessity are present [6].

2.2 Modeling

Two basic types of knowledge are of interest for asset allocation:

1. knowledge about the states of the world at any given time;
2. knowledge about the laws that describe either how the state of the world can change in time, or what relations are there among variables, or both.

Knowledge about the States This type of knowledge involves in the first place the definition of a number of scenarios. The vagueness and imprecision in the definition of a scenario is modeled in the most natural way by using fuzzy sets. In particular, a scenario will be defined by assigning a linguistic value to some of the relevant (*linguistic*) variables, much like *frames* [10] are defined in terms of *slots*.

In the second place, uncertainty about which scenario will happen in the future can be modeled using possibility theory i.e., as a possibility distribution over the scenarios. Probability distributions can also be used whenever it make sense to use them, considering the remarks made in Section 2.1.

A *fuzzy scenario C* is an object

$$C = \begin{bmatrix} x_1 & is & A_1 \\ & \vdots & \\ x_n & is & A_n \end{bmatrix}, \tag{1}$$

where $x_1, \ldots, x_n$ are n linguistic variables defining the scenario, and $A_1, \ldots, A_n$ are the fuzzy sets of possible values for $x_1, \ldots, x_n$ (i.e., *their linguistic values*). C then represents the fuzzy set $A_1 \times A_2 \times \ldots \times A_n \subseteq S$. For convenience of notation, we will not make a difference between a scenario and the fuzzy set of states it represents.

We will refer to variable $x_i$of scenario C as $C.x_i$; therefore, Equation 1 will be equivalent to the fuzzy proposition

$$C.x_1 \text{ is } A_1 \text{ AND} \ldots \text{ AND } C.x_n \text{ is } A_n. \tag{2}$$

Let $(x_1, \ldots, x_n)$ be a possible state of the market. Then, its degree of membership in scenario C is given by

$$\mu_C(s) = \min_{i \in \{1,\cdots,n\}} \mu_{A_i}(x_i). \tag{3}$$

We will denote by $\Pi(C)$ the possibility of scenario C, by $Pr(C)$ the probability of the same scenario, if applicable, and by $\pi(s)$ the possibility of state $s \in S$. Thus,

$$\pi(s) = \min_{C \subseteq S} \min(\Pi(C), \mu_C(s)), \tag{4}$$

(we are assuming that $\Pi(C)$ is part of the given knowledge) and

$$\Pr(C) = \sum_{s \in S} \mu_C(s) \Pr(s). \tag{5}$$

Knowledge about the Laws The laws governing financial phenomena can be synchronic (e.g., “all telecommunications stocks tend to behave more or less in the same way”) or diachronic (e.g., “index Y is mean-reverting”).

Both types of laws can be formally modeled as relations between financial linguistic variables, and can be expressed as fuzzy relations. A convenient way of representing fuzzy relations is by means of fuzzy IF-THEN rules, for which there are well-established inference methods [15].

3 Evaluation of Financial Strategies under Uncertainty

As a pre-requisite for the evaluation, we need to determine, on the basis of the current state of the world and of the available knowledge, which are the most plausible scenarios for the next period. This can be done by generating all conceivable scenarios one by one and calculating their possibility. This will yield a ranking of scenarios, of which, for efficiency reasons, only the first m will be considered.

Strategy evaluation will take place against those m most plausible scenarios. Of course, the greater m, the more accurate will be our evaluation, but the more time-consuming it will be.

3.1 Investment Strategies

For the purposes of this work, we can equate an investment strategy with a *portfolio*, i.e, an allocation of an investment among N asset classes. An *asset class* is a form of investment. For the sake of generality, we assume that one of the N asset classes is *cash* (which, in fact, one would rather define as "no investment"). A portfolio may be represented as a vector w in the standard simplex W, satisfying the constraints, for all $i = 1, \ldots, N$,

$$w_i \geq 0, \tag{6}$$

$$\sum_{i=1}^{N} w_i = 1 \cdot \tag{7}$$

We assume a one-period investment with fixed horizon, whereby an amount of money is invested in the current period in a portfolio, and in the next period the portfolio is transformed back into money, by selling all non-cash assets at market price.

3.2 Fuzzy Utility

The outcome of an investment strategy can be evaluated by means of a utility function, which maps the return of the investment at the final period to a real number, according to some investment objective.

For example, suppose that an investor wants to achieve a return of θ or more from the investment; the investor associates a constant marginal utility to increased return; also, the investor is risk-averse and associates a disutility with underperformance which grows with the square of the shortfall. Then, the investor's utility function will be something like

$$U(R(w);\theta) = R(w) - \lambda[R(w) < \theta](\theta - R(w))^2, \tag{8}$$

where $R(w)$ is the return of portfolio w at the investment horizon, and λ is a trade-off coefficient between risk and return. $R(w)$ depends on the vector r of returns of asset classes at the moment they are sold, which in turn depends on the state of the world s (as defined in Section 2.1) in the period in which the investment ends:

$$R(w) = w \cdot (1 + r(s)). \tag{9}$$

Therefore, utility depends on the investment strategy chosen w, on the state of the market s at the time the investment ends, and is parameterized by any objective θ:

$$U(R(w);\theta) \equiv U(w, s; \theta). \tag{10}$$

Now, in general, the knowledge possessed at the time an investment strategy is to be chosen will result in a possibility distribution over the states s$\in$ S. Therefore, instead of knowing in advance the exact utility of every strategy, we will have to be satisfied with a possibility distribution over the utilities, which we can conveniently define *fuzzy utility*. By applying the extension principle, we can write

$$\Pi(u) = \sup_{s:U(w,s;\theta)} \pi(s). \tag{11}$$

Substituting Equation 4 in Equation 11 yields

$$\Pi(u) = \sup_{s:U(w,s;\theta)} \min_{C\subseteq S} \min(\Pi(C), \mu_C(s)). \tag{12}$$

which relates the possibility distribution of utility for a given strategy to the possibilities of the defined scenarios.

3.3 Optimal Investment Strategies

The considerations made in the previous subsection allow us to evaluate the fuzzy utility for any given investment strategy in the light of the knowledge available at the time an investment strategy has to be chosen. Now we want to find the best investment strategies in term of their fuzzy utilities. Basically, we must choose, among all $w\in W$, a portfolio w^*, such that its fuzzy utility $\Pi_{w^*}(u)$ is "better", or "preferable" to the fuzzy utility $\Pi_w(u)$ of any other w. In order to do this, we need to be able to compare possibility distributions over the reals (i.e., fuzzy sets of real numbers) with one another.

Various methods to compare fuzzy numbers have been proposed in the literature [5]. Fuzzy utilities, however, are not always fuzzy numbers. Besides, a comparison method makes sense only with respect to a specific application [3].

In general, we have to admit the possibility that some fuzzy utilities are not comparable, so that we can establish only a partial ordering of fuzzy utilities. Therefore, we have to restate our objective by saying that we must find the set of all non-dominated investment strategies, whose fuzzy utilities are not comparable with one another, but are not "worse" than the fuzzy utility of any other investment strategy.

3.4 Ranking Fuzzy Utilities

Here, we propose three methods to compare fuzzy utilities, which are designed specifically for asset allocation.

Weighted Pessimistic Ranking This method relies on the definition of a relative *merit* of a fuzzy utility U^1 with respect to another fuzzy utility U^2, as a weighted comparison of the infima of all their α-cuts:

$$\tau(U^1 > U^2) = 2\int_0^1 \alpha[\inf(U^1_\alpha) > \inf(U^2_\alpha)]d\alpha. \tag{13}$$

This merit $\tau(U^1 > U^2)$ may be interpreted as the truth degree of proposition " U^1 is better than U^2". According to this ranking, a fuzzy utility U^1 will be preferable to another fuzzy utility U^2 if its worst utility (weighted by its possibility) is higher than the worst utility compatible with U^2.

Weighted Optimistic Ranking This method is dual to the weighted pessimistic ranking: the relative merit is defined as

$$\tau(U^1 > U^2) = 2\int_0^1 \alpha[\sup(U_\alpha^1) > \sup(U_\alpha^2)]d\alpha. \tag{14}$$

3.5 Weighted Center of Gravity Ranking

This methods is based on the center of gravity (cf. the first ranking function proposed by Yager [13]).

$$G = \frac{\int_{-\infty}^{\infty} u\Pi(u)du}{\int_{-\infty}^{\infty} \Pi(u)du}. \tag{15}$$

The idea is to prefer the fuzzy utility whose center of gravity, weighted by its height (i.e., maximum possibility), is greater:

$$U^1 > U^2 \equiv h(U^1)G(U^1) > h(U^2)G(U^2). \tag{16}$$

3.6 Combined Method

The three ranking methods proposed above can be combined at least in two ways:

– hierarchically: apply weighted pessimistic ranking first, then, in case of tie, apply weighted center of gravity ranking, then, if there is still a tie, apply weighted optimistic ranking;
– according to a majority rule: apply all three methods in parallel, then take the advice suggested by the majority.

4 Conclusions and Future Work

The work presented above should be considered a starting point for a series of theoretical and practical investigations in the field of asset allocation and finance. A number of extensions are possible (or rather necessary) in order to apply this possibilistic framework to problems of practical relevance:

– we have dealt with the simplest case of asset allocation, namely one-period, fixed-horizon asset allocation; the same ideas could be extended to the case of multi-period, variable-horizon asset allocation;

– we have considered the case in which only possibilities can be assigned to scenarios; an interesting extension would consist in investigating the case in which probabilities are known for some scenarios, and possibilities for other scenarios;
– we have not treated the algorithmic aspects of the framework, i.e., how do we select the most plausible scenarios? How do we use them to find the non-dominated set of investment strategies? How do we do that efficiently?

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