WEIGHTED CLOSED-FORM ESTIMATORS FOR BLIND SOURCE SEPARATION

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Abstract

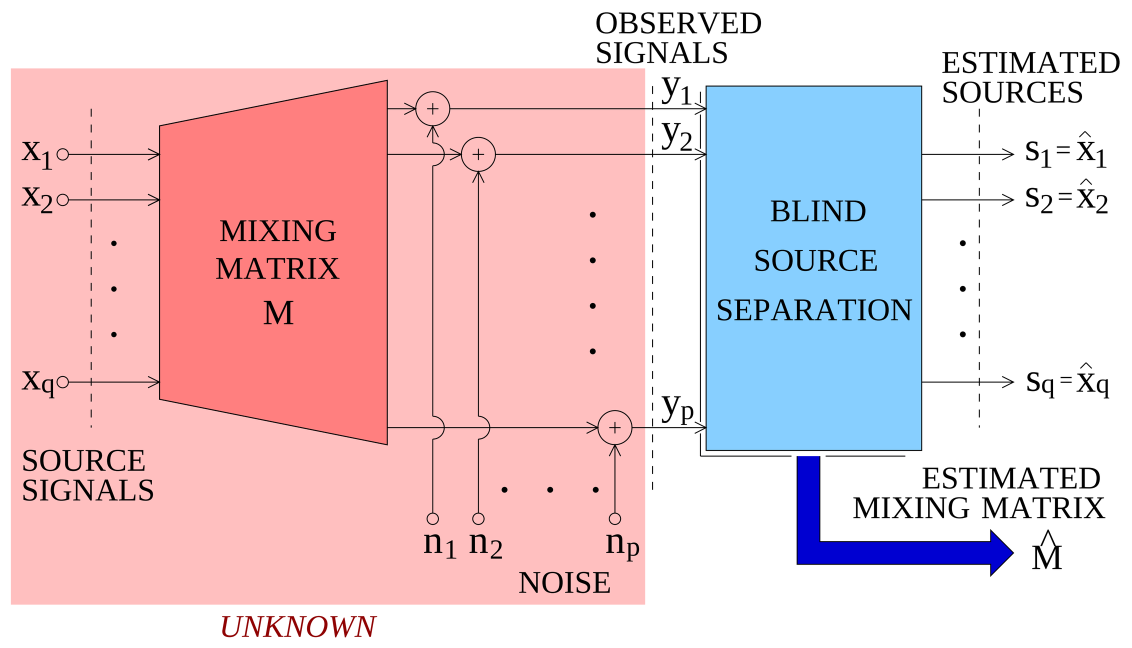
This paper investigates a novel closed-form estimation class, so-called weighted estimator (WE), for blind source separation in the basic two-signal problem. Proper combination of previously proposed estimators yields consistent estimates of the separation parameters under general conditions. In the real-mixture case, we determine analytic expressions for the WE asymptotic (large-sample) variance and the source-dependent weight value of the most efficient estimator in the class. By means of the bicomplex-number formalism, the WE is extended to the complex-mixture scenario, for which Cramér-Rao bounds are also derived. Simulations compare the WE with other methods, demonstrating its potential.

PROBLEM

- **Blind source separation (BSS)** of instantaneous linear mixtures:

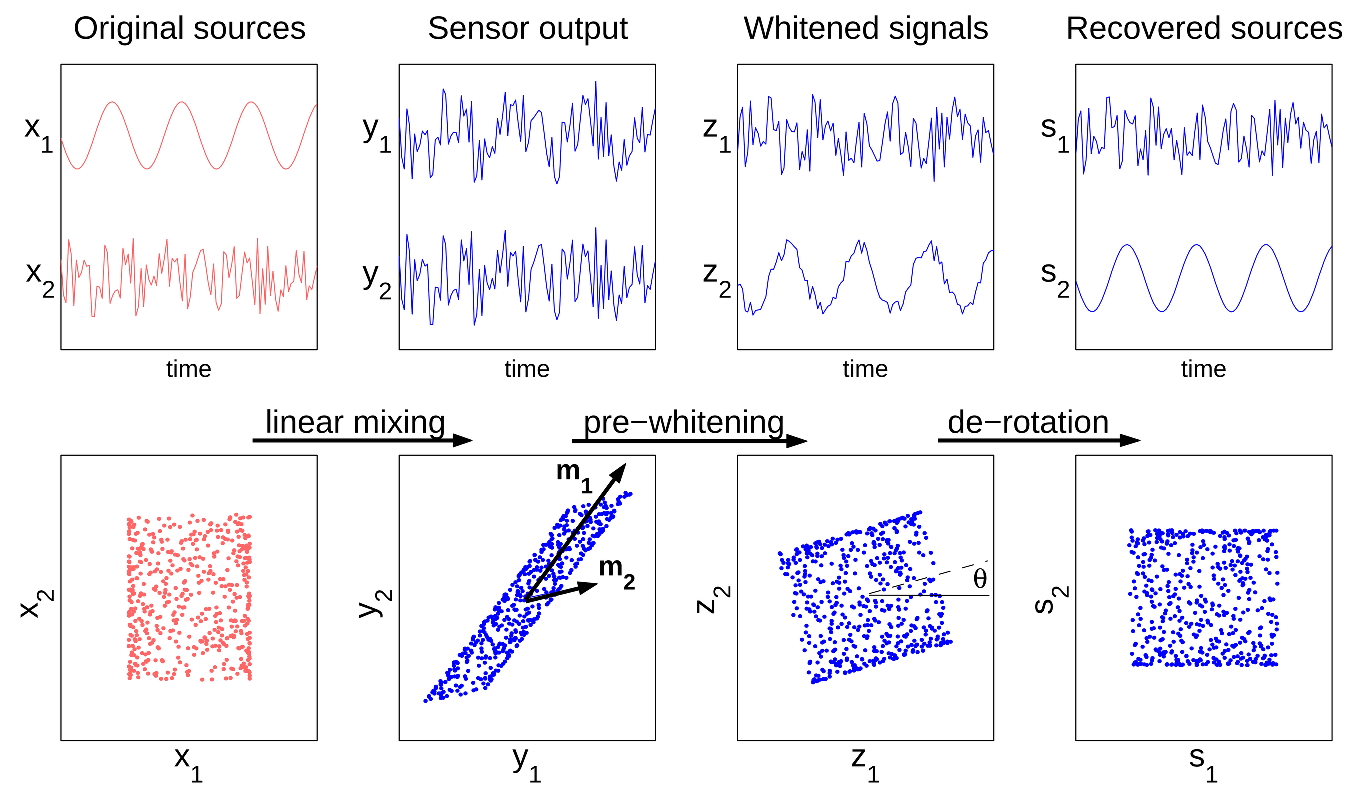
$$\mathbf{y} = M\mathbf{x}, \quad \begin{cases} \mathbf{y} \in \mathbb{C}^p : & \text{sensor array output} \\ \mathbf{x} \in \mathbb{C}^q : & \text{independent source signals} \\ M \in \mathbb{C}^{p\times q} : & \text{mixing matrix.} \end{cases}$$

- **Objective**: from only knowledge of $\mathbf{y} \rightarrow$ estimate $\mathbf{x}$ and M.



- After 2nd-order spatial whitening: $\mathbf{z} = Q\mathbf{x}$, $Q \in \mathbb{C}^{q\times q}$ unitary.
- Fundamental two-signal case ($p = q = 2$):

$$Q = \begin{bmatrix} \cos\theta & -e^{-j\alpha}\sin\theta \\ e^{j\alpha}\sin\theta & \cos\theta \end{bmatrix} \quad \Rightarrow \quad \text{estimation of } \theta,\, \alpha \in \mathbb{R}.$$



REAL-MIXTURE CASE

Fourth-Order Weighted Estimator

- **Idea**: complex linear combinations (*centroids*) of whitened-sensor HOS $\rightarrow$ explicit expressions for estimation of θ.
- **EML**: $\xi_4 = (\kappa^z_{40} + \kappa^z_{04} - 6\kappa^z_{22}) + j4(\kappa^z_{31} - \kappa^z_{13}) = \gamma e^{j4\theta} \Rightarrow \boxed{\hat{\theta}_{\text{EML}} = \frac{1}{4}\angle(\hat{\gamma}\hat{\xi}_4)}$
- **AEML**: $\xi_2 = (\kappa^z_{40} - \kappa^z_{04}) + j2(\kappa^z_{31} + \kappa^z_{13}) = \eta e^{j2\theta} \Rightarrow \boxed{\hat{\theta}_{\text{AEML}} = \frac{1}{2}\angle\hat{\xi}_2}$
- **AML, MaSSFOC**: combination of EML and AEML.
- **WE**: $\xi_{\text{WE}} = w\gamma\xi_4 + (1-w)\xi_2^2 = \big(w\gamma^2 + (1-w)\eta^2\big)e^{j4\theta} \Rightarrow \boxed{\hat{\theta}_{\text{WE}} = \frac{1}{4}\angle\hat{\xi}_{\text{WE}}}$
  - *Consistent for any source distribution,* $0 < w < 1$.
  - *Optimal choice of* w*?*

Optimal Finite-Sample Performance

- **Asymptotic (large-sample) variance of WE** (T samples):

$$\sigma^2_{\hat{\theta}_{\text{WE}}} = \frac{\mathrm{E}\Big\{\big[w\gamma(x_1^3x_2 - x_1x_2^3) + (1-w)\eta(x_1^3x_2 + x_1x_2^3)\big]^2\Big\}}{T\big[w\gamma^2 + (1-w)\eta^2\big]^2}.$$

- **Minimum-variance WE** (if $|\kappa^x_{40}| \neq |\kappa^x_{04}|$):

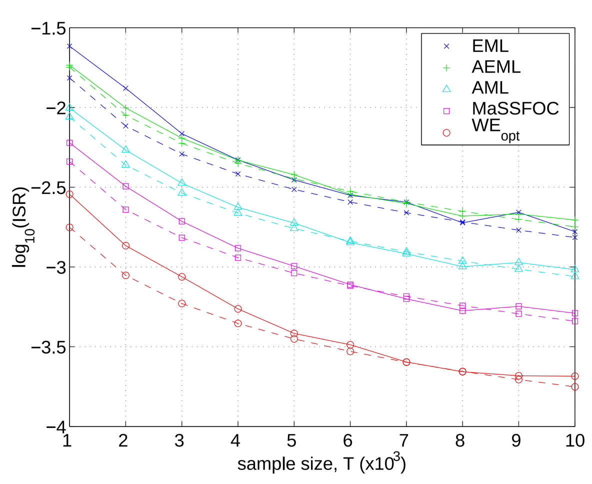
$$w_{\text{opt}} = \frac{1}{2} + \frac{\mu^x_{40}\mu^x_{04}\big[(\kappa^x_{40})^2 - (\kappa^x_{04})^2\big] + \kappa^x_{40}\kappa^x_{04}(\mu^x_{60} - \mu^x_{06})}{2\big[(\kappa^x_{40})^2\mu^x_{06} - (\kappa^x_{04})^2\mu^x_{60}\big]}.$$

Simulation Results

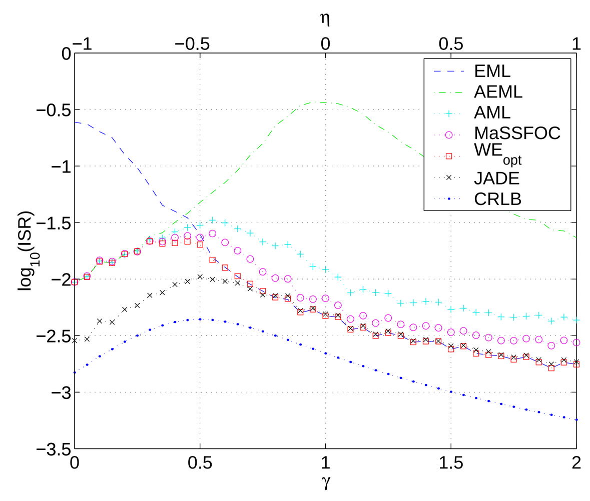
- **Performance index**:

$$\mathrm{ISR} = \mathop{\mathrm{E}}_{i\neq j}\left\{\frac{|(\widehat{Q}^{\mathrm{T}}Q)_{ij}|^2}{|(\widehat{Q}^{\mathrm{T}}Q)_{ii}|^2}\right\}$$

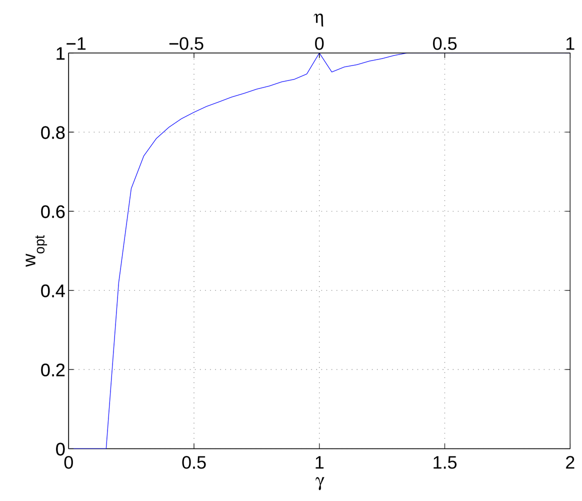
$\widehat{Q}$: rotation of angle $\hat{\theta}$.



ISR vs. sample size. Uniform–Rayleigh sources, $\theta = 15^{\mathrm{o}}$, ν independent Monte Carlo runs, with $\nu T = 5\times 10^6$. Solid lines: average empirical values. Dashed lines: asymptotic variances.



ISR vs. sks γ and skd η. GGD sources, $\kappa^x_{04} = 0.5$, $\theta = 15^{\mathrm{o}}$, $T = 5\times 10^3$ samples, 10^3 Monte Carlo iterations.



Optimal value of the WE weight parameter.

COMPLEX-MIXTURE CASE

Bicomplex Numbers

- Unitary matrix $U = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$, $a, b \in \mathbb{C} \mapsto$ *bicomplex number* $\bar{x} = a + \jmath b$, $\jmath^2 = 1$, $\jmath \neq j$.
- Product: $\bar{x}_1 = a_1 + \jmath b_1$, $\bar{x}_2 = a_2 + \jmath b_2 \Rightarrow \bar{x}_1\bar{x}_2 = (a_1a_2 - b_1^*b_2) + \jmath(b_1a_2 + a_1^*b_2)$.
- Unitary matrices under matrix product and bicomplex numbers under above product operation: isomorphic.
- Matrix $Q \rightarrow$ bicomplex exponential: $e_\alpha^{\jmath\theta} = \cos\theta + \jmath e^{j\alpha}\sin\theta$.

Fourth-Order Weighted Estimator

- **Bicomplex centroids**.
- **CEML**: $\bar{\xi}_4 = (\kappa^z_{40} + \kappa^z_{04} - 6\kappa^z_{22}) + \jmath 4(\kappa^z_{31} - \kappa^z_{13}) = \gamma e_\alpha^{\jmath 4\theta}$.
- **CAEML**: $\bar{\xi}_2 = (\kappa^z_{40} - \kappa^z_{04}) + \jmath 2(\kappa^z_{31} + \kappa^z_{13}) = \eta e_\alpha^{\jmath 2\theta}$.
- **CWE**: $\bar{\xi}_{\text{CWE}} = w\gamma\bar{\xi}_4 + (1-w)\bar{\xi}_2^2 = \big(w\gamma^2 + (1-w)\eta^2\big)e_\alpha^{\jmath 4\theta}$.

Cramér-Rao Lower Bounds

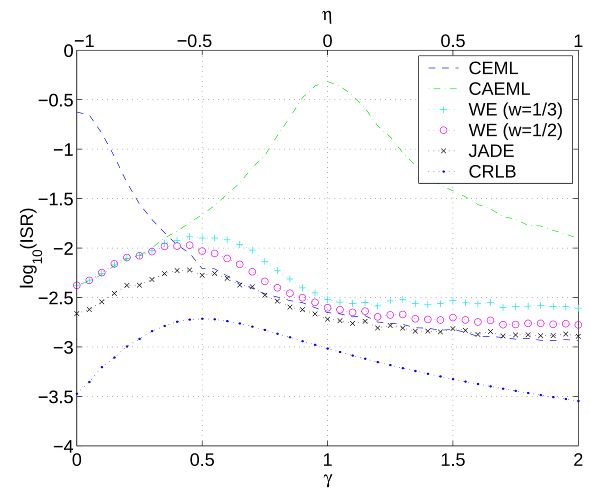
- Circularly distributed source signals, T samples: $\mathrm{FIM}_{(\theta,\alpha)} = T\begin{bmatrix} I & 0 \\ 0 & \frac{1}{4}I\sin^2 2\theta \end{bmatrix}$,

$$I = I_1 + I_2 - 4,\; I_k = \frac{1}{2}\iint_{D_k}\frac{1}{p_k}\left[\left(\frac{\partial p_k}{\partial u}\right)^2 + \left(\frac{\partial p_k}{\partial v}\right)^2\right]\mathrm{d}u\,\mathrm{d}v,$$

$p_k(u, v)$: pdf of the kth source signal $x_k = u_k + jv_k$, $u_k, v_k \in \mathbb{R}$, $k = 1, 2$.

- CRLBs of θ and α decoupled $\Rightarrow \begin{cases} \mathrm{CRLB}_\theta = (TI)^{-1} \\ \mathrm{CRLB}_\alpha = 4(TI\sin^2 2\theta)^{-1} \end{cases}$.
- Two Gaussian sources $\rightarrow$ FIM = 0.

Simulation Results



ISR vs. sks γ and skd η. CGGD sources, $\kappa^x_{04} = 0.5$, $\theta = 15^{\mathrm{o}}$, $\alpha = 65^{\mathrm{o}}$, $T = 5\times 10^3$ samples, 10^3 independent Monte Carlo iterations.

CONCLUSIONS AND OUTLOOK

- Linear combination of 4th-order centroids for closed-form BSS $\rightarrow$ consistent estimates of separation parameter for any source distribution $\Rightarrow$ weighted estimator (WE).
- Prior knowledge of source statistics $\rightarrow$ WE with optimal finite-sample performance (minimum asymptotic variance).
- Bicomplex numbers $\rightarrow$ extension to complex mixtures. CRLB derived for circular sources. WE performance follows closely CRLB trend.
- **Further work**:
  - w_{opt} as a function of array-output statistics
  - behaviour in additive noise and impulsive interference
  - aymptotic performance analysis of complex WE.

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