Observer design for invariant systems with homogeneous observations

C. Lageman[*], J. Trumpf[†] and R. Mahony[‡]

October 4, 2008

Abstract

This paper considers the design of nonlinear observers for invariant systems posed on finite-dimensional connected Lie groups with measurements generated by a transitive group action on an associated homogeneous space. We consider the case where the group action has the opposite invariance to the system invariance and show that the group kinematics project to a minimal realisation of the systems observable dynamics on the homogeneous output space. The observer design problem is approached by designing an observer for the projected output dynamics and then lifting to the Lie-group. A structural decomposition theorem for observers of the projected system is provided along with characterisation of the invariance properties of the associated observer error dynamics. We propose an observer design based on a gradient-like construction that leads to strong (almost) global convergence properties of canonical error dynamics on the homogeneous output space. The observer dynamics are lifted to the group in a natural manner and the resulting gradient-like error dynamics of the observer on the Lie-group converge almost globally to the unobservable subgroup of the system, the stabiliser of the group action.

[*]C. Lageman is with the Department of Electrical Engineering and Computer Science, Université de Liège, B-4000 Liège Sart-Tilman, Belgium, `christian.lageman@montefiore.ulg.ac.be`. His work was supported by the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.

[†]J. Trumpf is with the Department of Information Engineering, The Australian National University, Canberra ACT 0200, Australia, `Jochen.Trumpf@anu.edu.au`.

[‡]R. Mahony is with the Department of Engineering, The Australian National University, Canberra ACT 0200, Australia, `Robert.Mahony@anu.edu.au`.

1 Introduction

There is a growing body of work concerned with the design of nonlinear observers for invariant systems on Lie groups. Recent results are partly motivated by the need for highly robust and computationally simple state estimation algorithms for robotic vehicles. The classical approach to state estimation for such applications is based on nonlinear filtering techniques such as extended Kalman filters [1] or particle filters [12]. Non-linear observers offer less information than a nonlinear filter (state-estimates rather than full *posterior* distributions for the state), however, it is often possible to prove strong stability results with large (almost global) basins of attraction and provide computationally simple implementations of the observers. Historically, work on nonlinear observers for invariant systems was divided into a body of applied work, closely linked to specific examples, and a separate body of theoretical work concerned with fundamental systems theory for invariant systems. One of the earliest applied results concerned the design of a nonlinear observer for attitude estimation of a rigid-body using the unit quaternion representation of the special-orthogonal Lie group [30]. This work is seminal to a series of papers undertaken over the last fifteen years that develop nonlinear attitude observers for rigid-body dynamics [27, 36, 22, 26, 19, 4, 25, 20, 35, 23, 39], exploiting either the unit quaternion group structure or the rotation matrix Lie-group structure of $SO(3)$. Recent observer designs have comparable performance to state-of-the-art nonlinear filtering techniques [11], generally have much stronger global stability and robustness properties [20], and are simple to implement. The full pose estimation problem has also attracted recent attention [40, 29, 3, 38] in which case the underlying state space is the special Euclidean group $SE(3)$ comprising both attitude and translation of a rigid-body. Another promising body of applied work involves development of heading reference systems for UAV systems [23, 24]. The theoretical investigation of observability and controllability of invariant systems on Lie-groups was first considered in the seventies [8, 16, 15] in the context of the formulation of systems theory on Lie-groups and coset spaces. Parallel work by Sussmann investigated quotient structure of realisations of nonlinear systems and associated observability and controllability properties [31, 32]. There have been only occasional extensions of this material since the early work, for example [10, 2]. A characteristic of all the work mentioned above is that authors consider only matched invariance assumptions on the system. That is, all the invariances in the system are of the same handedness; left invariant kinematics, left invariant outputs, left invariant metrics, etc. Recent work on understanding the generic structure of observers for left invariant systems on Lie-groups [7, 14, 20, 5] has challenged this formulation and lead to an understanding of the importance of invariance properties of observer error dynamics and the underlying properties of the observer design problem [17, 6, 18]. The present paper contains further results in this direction.

In this paper, we consider the design of nonlinear observers for state space systems where the state evolves on a finite dimensional, connected Lie group and the measurements are generated by a transitive group action acting on an associated homogeneous space. We consider *complementary* output invariance; that is, for a left invariant system we consider a right invariant output group action, a structure that is suggested by a body of recent work [5, 20, 6, 18]. We show that a system with complementary outputs projects to a quotient system on the homogeneous output space corresponding to the minimal observable realisation of the full system, and consequently, that the stabiliser of the output group action is the unobservable subgroup of the system. Since the projected system is a minimal realisation of the full system it is natural to study the observer design problem for this system. We introduce the concept of synchrony of observer and plant [18] for projected systems and use this to define a "template" for observer design as a combination of an internal model and an innovation term. The invariance properties of the innovation term that ensure the error dynamics of the observer/system couple are autonomous are also fully characterised. The second part of the paper introduces a constructive process for observer design based on the design of gradient-like innovation terms. We consider cost functions defined on the homogeneous output space and design a gradient-like innovation for the projected system. This design can be lifted to an observer on the Lie-group state space of the full system. Under suitable assumptions on the cost function we obtain (almost) global asymptotic stability of the observer on the homogeneous output space and corresponding asymptotic stability to the unobservable subgroup on the Lie-group state-space. An example on $SO(3)$ is provided to demonstrate the applications of results provided in the body of the paper.

After the introduction, Section 2 provides an overview of the systems considered and notation used. Section 3 studies the structure of invariant systems with complementary invariance in the outputs. Section 4 studies the properties of observers for the projected system on the homogeneous output space. Section 5 considers the question of observer synthesis. First we consider synthesis of observers for the projected system (Subsection 5.1) and then the lift of these observers onto the Lie-group state-space (Subsection 5.2). Section 6 provides an example based on the attitude observers that have been studied recently [7, 19, 14, 20, 5]. A short paragraph of conclusions is also provided in Section 7.

2 Invariant systems with homogeneous output space

In this section, we introduce the notation used throughout the paper and discuss some of the intuition in the models considered.

The symbol G denotes a connected Lie group with Lie algebra $\mathfrak{g}$. We consider left invariant systems on G of the form

$$\dot{X} = Xu. \tag{1}$$

An input $u\colon \mathbb{R} \to \mathfrak{g}$ is called admissible if the resulting time-variant differential equation on the Lie group has unique solutions for all initial values. We assume that the velocity u is measured and can be used in the observer construction. The system (1) is termed left invariant since it is invariant under application of left multiplication on the group $T_X L_Y \dot{X} = (YX)u$ where $L_Y : X \mapsto YX$ and T_X denotes the tangent map evaluated at X. An equivalent definition is used for right invariance with respect to right multiplication $R_Y : X \mapsto XY$ on the group.

A smooth, connected manifold is denoted M and a Lie-group action on M is denoted $h\colon G \times M \to M$, where the h notation anticipates that the group action will be acting as an output map. A group action on a manifold is called a right action if for all $X, Y \in G$, $y \in M$: $h(X, h(Y, y)) = h(YX, y)$ and a left action if $h(X, h(Y, y)) = h(XY, y)$. All actions considered in this paper are transitive, that is, for any $x, y \in M$ there exists $X \in G$ such that $h(X, x) = y$.

We choose a fixed a point $y_0 \in M$ to be the 'reference' output. We consider outputs for the system (1) generated by the group action $y = h(X, y_0)$, $y, y_0 \in M$. In particular, for the fixed reference y_0 we define $h_{y_0} : G \to M$, $y_0 \in M$, and the output

$$y = h_{y_0}(X) = h(X, y_0). \tag{2}$$

We use the terminology **homogeneous output space** to refer to a homogeneous manifold M associated with the scenario described above. In general the group kinematics considered may be either left or right invariant and the group action can be independently either a left or right action. We term the case where the outputs are generated by a group action of opposite invariance to the system invariance as **complementary observations**. Conversely, we term the case where both the system and outputs have the same invariance as **matched observations**. That is, right invariant outputs are complementary to left invariant group kinematics and matched with right invariant group kinematics, and *vice versa* for left invariant outputs.

The stabiliser of an element $y \in G$ is given by

$$\mathrm{stab}(y) = \{X \in G \mid h(X, y) = y\},$$

and is a subgroup of G. We will mostly be interested in $\mathrm{stab}(y_0)$ and assume that it is a closed subgroup of G. We use the notation $h_X : M \to M$ for the symmetry map

$$h_X(y) := h(X, y).$$

While we have chosen to work explicitly with left invariant system dynamics on the group (1), the case of right invariant dynamics is analogous and all the results in the paper will carry through with the obvious changes. The choice of complementary or matched output invariance, however, is a key assumption. The main results of the paper require that the output observations are complementary and the results do not hold for matched observations. It is instructive to consider an example to provide some physical intuition for this choice.

Example 1 *We use the attidue estimation problem as an illustration [4, 5, 6, 7, 11, 14, 17, 18, 19, 20, 22, 21, 24, 23, 25, 26, 30, 35, 34, 36, 39, 40]. The attitude of a rigid body, with body-fixed-frame* $\{B\}$*, measured with respect to an inertial frame* $\{A\}$*, can be identified with an element* R *of* $SO(3)$*. The left invariant dynamics*

$$\dot{R} = R\Omega_\times \tag{3}$$

on $SO(3)$ *correspond to the natural body-fixed-frame kinematics of the system. Here* $\Omega = (\Omega_1, \Omega_2, \Omega_3)^\top \in \{B\}$ *is the angular velocity of the rigid-body expressed in the body-fixed-frame and*

$$\Omega_\times = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

Consider the case where only partial information on the rigid-body attitude is measured. The most common situation is when an inertial direction, such as the magnetic field, or gravitational field, is measured by sensor systems such as magnetometers or accelerometers on board a vehicle. The inertial direction $y_0 \in \{A\}$ *of the measured direction is known* a-priori*, for example, the gravitational direction lies in the* z*-axis of the inertial frame. The measured direction in the body-fixed-frame,*

$$y = R^\top y_0 \in \{B\},$$

is obtained by using the attitude matrix to transform the inertial direction y_0 *from the inertial frame into the body fixed frame. The associated output group action that we consider is*

$$h : G \times S^2 \to S^2, \quad h(R, y_0) = R^\top y_0,$$

a right-invariant group action. This example demonstrates the case of complementary outputs, typical for applications involving robotic vehicles where measurements are made by embarked sensor systems.

An alternatively scenario involves a fixed sensor system that observes an autonomous vehicle in motion. An example would be a computer vision system that extracts a principle axis of symmetry of an observed vehicle from a video sequence of images. In this case the known data is a direction $\bar{y}_0 \in \{B\}$ *that denotes the orientation of the symmetry of the vehicle in the body-fixed-frame. The measurement is obtained by transforming* y_0 *into the inertial frame*

$$\bar{y} = Ry_0 \in \{A\}.$$

The associated group action would be

$$\bar{h} : G \times S^2 \to S^2, \quad h(R, \bar{y}_0) = R\bar{y}_0,$$

a left-invariant group action. This second example demonstrates the case of matched observations. There are fewer applications where matched observations are the natural model.

The complementary case of a left invariant system (1) with right output action (2) is the case that has attracted interest recently [5, 17, 6, 18] and is most important for applications in robotic vehicles. It turns out, perhaps somewhat counter-intuitively, that the structure of invariant systems with complementary outputs is relatively straightforward compared to the case of matched outputs. A number of recent works have shown that the case of complementary outputs yields autonomous error dynamics when the observer is properly designed [17, 6, 5, 18] and this work has lead to some promising applied results [23, 24]. Our work can be viewed as providing further perspective and extension of this research direction. The case of matched observations was studied in some detail in the seventies [8, 16, 15] although observer design was not a focus of this early work.

Remark 2 *In the system* (1) *and* (2)*, the map* $u \mapsto Xu$ *is surjective and the system is fully controllable. In fact, the controllability of the system is not of particular interest in the main body of the paper except in the context of the realisation results we present. The results will hold equally well for systems with restricted inputs (corresponding to limited degrees of control actuation) as long as system is controllable. If the system is not controllable then the results presented in the paper apply to the system restricted to the maximal controllable subgroup. In this case the homogeneous output space considered must be restricted to the orbit of the maximal controllable subgroup.*

3 Structure of left invariant systems with complementary observations

In this section, we consider the structure of left invariant systems with complementary observations. The main result shows that there is a natural projection of the full system dynamics onto a system on the homogeneous output space that is a minimal realisation of the input-output behaviour of the full system, and that this in turn characterises the unobservable subspace on G as $\mathrm{stab}(y_0)$.

In the remainder of this paper, we consider left-invariant system kinematics (1) and complementary right invariant 'output' group action h (2). Any system on a manifold N given globally by $\dot{x} = f(x, u)$ can be considered as a bundle map $f\colon B \to TN$ from a trivial smooth fiber bundle B over N to the tangent bundle TN. In general any bundle map from a fiber bundle B over N to TN can be considered as a realization of a control system. We will use the notion of a system with symmetry [37, 13, 28]. A system $f\colon B \to TN$ with output map $h\colon N \to M$ has a symmetry H (where H is a Lie group) if there are left actions $S^B\colon H \times B \to B$, $S^N\colon H \times N \to N$ and a right action $S^M : H \times M \to M$, such that for all $X \in H$, $v \in B$, $x \in M$

$$\begin{aligned} f(S^B(X,v)) &= TS^N_X f(v), \\ h(S^N(X,x)) &= S^M(X, h(x)). \end{aligned}$$

Here S^N_X denotes the map $v \mapsto S^N(X,v)$.

Remark 3 *Note that the systems with symmetries considered in [13, 28] do not have specific output maps. The work [37] considers symmetries for systems with outputs, however, the groups considered were commutative and the difference between left and right symmetries was lost. The definition we use is extended in the natural manner to conform to the structure we consider. The factorisation results for system with symmetries [13, 28, 37], as we will use here, have also motivated recent work on general quotient systems [33].*

Proposition 4 *The stabiliser* $\mathrm{stab}(y_0)$ *is a symmetry of system* (1) *with complementary observations* (2)*.*

Proof: The system (1) and (2) can be regarded as a bundle map $f : G \times \mathfrak{g} \to TG$ via $f(X,u) = Xu$. Define the left actions $S^B(Y,(X,u)) = (YX,u)$ and $S^N(Y,Xu) = YXu$ of $\mathrm{stab}(y_0)$ on $G \times \mathfrak{g}$ and TG. Note additionally that $\mathrm{stab}(y_0)$ acts on M via $S^M(Y,y) = h(Y,y) = y$. The symmetry condition on f follows trivially from the definition, while the symmetry condition on h follows from the definition of the stabiliser. $\square$

The main structural property of the system (1) with complementary observations (2) is that it *projects to a system on* M; that is, the system on G maps via the group action to a system on M with trivial output map and this projected system is strongly equivalent to (1) and (2) in the sense of [32]. In other words, the system on G can be projected to a system on M with the same input-output behaviour.

Theorem 5 *The system* (1) *with complementary observations* (2) *projects to the system*

$$\dot{y} = T_X h_{y_0}(Xu), \quad X \in h^{-1}_{y_0}(y) \tag{4}$$

on M*.*

Proof: Since the control system $f\colon G\times\mathfrak{g}\to TG$, $f(X,u)=Xu$ has the symmetry $\operatorname{stab}(y_0)$ which acts freely and properly, it can be projected to a control system $\tilde f$ on the corresponding orbit spaces [28]. As $\operatorname{stab}(y_0)$ acts on $G\times\mathfrak{g}$ and TG by the corresponding left actions, we can identify the orbit spaces $\operatorname{stab}(y_0)\diagdown G\times\mathfrak{g}$ and $\operatorname{stab}(y_0)\diagdown TG$ with $M\times\mathfrak{g}$ and TM by setting $(\operatorname{stab}(y_0)X,u)=(h(X,y_0),u)$ and $\operatorname{stab}(y_0)(Xu)=T_Xh_{y_0}(Xu)$. The projected control system on M can be written as $\tilde f\colon M\times\mathfrak{g}\to TM$, $\tilde f(y,u)=T_Xh_{y_0}(Xu)$ for $h(X,y)=y_0$. Note that this shows that (4) is well-defined and smooth. The operations above can be illustrated by the following commutative diagram:

$$\begin{array}{ccccc} G\times\mathfrak{g} & \longrightarrow & \operatorname{stab}(y_0)\diagdown G\times\mathfrak{g} & = & M\times\mathfrak{g} \\ {\scriptstyle Xu}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle T_Xh_{y_0}(Xu))} \\ TG & \longrightarrow & \operatorname{stab}(y_0)\diagdown TG & = & TM \end{array}$$

Since the output map h is equivariant under the action of $\operatorname{stab}(y_0)$ it induces a map $\tilde h$ between the orbit spaces $\operatorname{stab}(y_0)\diagdown G$ and $M\diagup\operatorname{stab}(y_0)$. As $\operatorname{stab}(y_0)$ acts only trivially on M, we have that $M\diagup\operatorname{stab}(y_0)=M$. The quotient $\operatorname{stab}(y_0)\diagdown G$ can be identified with M via $\operatorname{stab}(y_0)X=h_{y_0}(X)$. The induced map $\tilde h$ has to satisfy the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{h_{y_0}} & M \\ {\scriptstyle h_{y_0}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ M & \xrightarrow{\tilde h} & M \end{array}$$

where id_M denotes the identity. Thus $\tilde h=\mathrm{id}_M$. In summary we have the following commutative diagram

$$\begin{array}{ccccccc} G\times\mathfrak{g} & \xrightarrow{Xu} & TG & \xrightarrow{\pi_G} & G & \xrightarrow{h_{y_0}} & M \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_M} \\ M\times\mathfrak{g} & \xrightarrow{T_Xh_{y_0}(Xu)} & TM & \xrightarrow{\pi_M} & M & \xrightarrow{\mathrm{id}_M} & M \end{array}$$

where $\pi_G\colon TG\to G$, $\pi_M\colon TM\to M$ are the canonical projections of the tangent bundle. It follows from the uniqueness of solutions for admissible inputs that the system (1) and (2) and the system (4) with full state output have the same input-output behaviour. □

Theorem 5 provides a significant insight into the observability of the full system. In particular, since the system can be reduced to a fully-state observable system on the observation space, the observed dynamics cannot provide any additional information on the system state.

Recall that two states $X, Y \in G$ are called indistinguishable, if for any admissible input u the solutions $X(t)$, $Y(t)$ of (1) and (2) produce the same output $h(X(t), y_0) = h(Y(t), y_0)$ [8, 32].

Corollary 6 *Consider the system* (1) *with complementary observations* (2)*. Two states $X, Y \in G$ are undistinguishable if and only if XY^{-1} is contained in the stabiliser subgroup* $\mathrm{stab}(y_0)$ *of y_0. In particular, a state $X \in G$ is undistinguishable from the identity in G if and only if $X \in \mathrm{stab}(y_0)$.*

Proof: Since the projected system on M is fully observable, two states in $X, Y \in G$ are indistinguishable if and only if they are mapped onto the same $y \in M$. Two states $X, Y \in G$ are mapped onto the same $y \in M$ if and only if $XY^{-1} \in \mathrm{stab}(y_0)$. □

Remark 7 *One can view Theorem 5 as a special case of Sussmann's result [32] of the existence of a minimal realisation. Sussmann's proof is based on factoring the state space by the equivalence relation of indistinguishable states. In our case, two states X_1, X_2 are indistinguishable if and only if $X_1 X_2^{-1} \in \mathrm{stab}(y_0)$. Hence, factoring by this equivalence relation is equivalent to factoring by* $\mathrm{stab}(y_0)$*. This is another way to see that projecting the system on G onto M yields a minimal realisation of* (1) *and* (2)*.*

4 Observers for projected systems on homogeneous spaces

In this section we consider the structure of nonlinear observers for the projected system on the homogeneous output space. We show that the concepts of synchrony and canonical errors, and a decomposition of observers into internal model and innovation terms discussed in earlier work by the authors [17, 18] can be extended to the case of complementary observations. We provide a full characterisation of the invariance properties of the innovation term that guarantees invariance of the error dynamics, an extension of recent work by Bonnabel *et al.* [6].

Consider systems of the form

$$\dot{y} = T_X h_{y_0}(Xu) \text{ with } X \in h_{y_0}^{-1}(y) = \{X \in G \mid h_{y_0}(X) = y\} \tag{5}$$

on the homogeneous space M. We call such systems **projected systems** since they arise from a projection of the system $\dot{X} = Xu$ on G as discussed in Theorem 5. A projected system, however, is defined on the homogeneous space M and can be independently analysed. A simple example is instructive.

Example 8 *Consider Example 1 of attitude estimation on $SO(3)$ for the case of a complementary observation on the homogeneous space S^2. The measured output is modelled by $y = R^\top y_0$ for a known 'reference' $y_0 \in S^2$. Consider an arbitrary $R_y \in SO(3)$ such that $R_y^\top y_0 = y$ and note that $R_y^\top y_0 = y = R^\top y_0$. For any such R_y the projected system dynamics are given by*

$$\dot{y} = T_{R_y} h_{y_0}(R_y \Omega_\times) = -\Omega_\times R_y^\top y_0 = -\Omega_\times y,$$

the kinematics of an inertial direction on S^2 expressed with respect to the body-fixed-frame. The projected system can be studied independently as a kinematic system $\dot{y} = -\Omega_\times y$ on S^2 without reference to the Lie-group kinematics $\dot{R} = R\Omega_\times$ [25, 26].

The first concept we consider for observer structure for projected systems is that of synchrony of a plant and an observer. We use the notation $e\colon M \times M \to N$, to denote a smooth **error function** function to a smooth manifold N. An error function is a generalisation of the observer error $(\hat{x} - x)$ that is used for systems on $\mathbb{R}^n$. For the moment we make no further assumptions on the error function.

Definition 9 *Let M be a smooth manifold, B a vector bundle over M, $f_x\colon B \to TM$, $f_{\hat{x}}\colon B \to TM$ bundle maps and*

$$\begin{aligned} \dot{x} &= f_x(u) \\ \dot{\hat{x}} &= f_{\hat{x}}(u) \end{aligned} \tag{6}$$

a pair of general systems on M. We call the pair of systems (6) **e-synchronous** *with respect to an error function $e\colon M \times M \to N$ if for all admissible $u\colon \mathbb{R} \to \mathbb{R}^n$, all initial values x_0, $\hat{x}_0 \in M$, and all $t \in \mathbb{R}^+$*

$$\frac{d}{dt} e(\hat{x}(t; \hat{x}_0, u), x(t; x_0, u)) = 0.$$

Theorem 10 *Consider a pair of projected systems on M,*

$$\begin{aligned} \dot{x} &= T_X h_{y_0}(Xu), \quad X \in h_{y_0}^{-1}(x) \\ \dot{\hat{x}} &= T_{\hat{X}} h_{y_0}\left(\hat{X}u\right), \quad \hat{X} \in h_{y_0}^{-1}(\hat{x}). \end{aligned} \tag{7}$$

If there exists a smooth error function $e\colon M \times M \to N$ such that the systems are e-synchronous, then e has the form

$$e(\hat{x}, x) = g(h_{y_0}(\hat{X}X^{-1})) \text{ with } h_{y_0}(\hat{X}) = \hat{x}, h_{y_0}(X) = x$$

and $g\colon M \to N$ a smooth function.

Proof: We first show that e has to be invariant under the action of G on $M \times M$, i.e. $e(h(S, \hat{x}), h(S, x)) = e(\hat{x}, x)$ for all $x, \hat{x} \in M$, $S \in G$. Let $S \in G$ and $x_0, \hat{x}_0 \in M$. We choose a smooth curve $T\colon [0,1] \to G$, $T(0) = e$, $T(1) = S$ and set $u(t) = T(t)^{-1}\dot{T}(t)$. Note that for any $x \in M$, $t \in [0,1]$, $R \in G$ with $h(R, y_0) = x$ we have

$$\begin{aligned} \frac{d}{dt} h(T(t), x) &= \frac{d}{dt} h(T(t), h(R, y_0)) \\ &= \frac{d}{dt} h_{y_0}(RT(t)) \\ &= T_{RT(t)} h_{y_0}(R\dot{T}(t)) \\ &= T_{RT(t)} h_{y_0}(RT(t)u(t)). \end{aligned}$$

Hence $h(T(t), x)$ is a solution of $\dot{x} = T_X h_{y_0}(Xu), X \in h_{y_0}^{-1}(x)$ with initial value x and $u(t) = T(t)^{-1}\dot{T}(t)$. Thus $(h(T(t), x_0), h(T(t), \hat{x}_0))$ is the solution of (7) with inital value $(x_0, \hat{x}_0)$ and $u(t)$ as above. Since the systems are e-synchronous, e is constant on solutions of (7) and $e(x_0, \hat{x}_0) = e(h(T(1), x_0), h(T(1), \hat{x}_0))$. Thus for all $S \in G$

$$e = e \circ h_S$$

and e is invariant under the action of G. For $x, \hat{x} \in M$ let now $X, \hat{X} \in G$ such that $h_{y_0}(X) = x, h_{y_0}(\hat{X}) = \hat{x}$. Then

$$e(\hat{x}, x) = e(h(\hat{X}, y_0), h(X, y_0)) = e(h(X^{-1}, h(\hat{X}, y_0)), y_0) = e(h_{y_0}(\hat{X}X^{-1}), y_0).$$

Thus e has indeed the form as claimed in the theorem with a smooth $g\colon M \to N$, $g(z) = e(z, y_0)$. □

Theorem 18 justifies the consideration of a **canonical error function**

$$e(\hat{x}, x) = h(\hat{X}X^{-1}, y_0) \text{ with } h_{y_0}(\hat{X}) = \hat{x}, h_{y_0}(X) = x. \tag{8}$$

The canonical error function is well-defined, smooth and non-degenerate, in the sense that the differential of e with respect to either the first or second variables is full rank. Note that "no error" corresponds to $e(\hat{x}, x) = y_0$.

Remark 11 *The canonical error is the projection of the right-invariant error considered in the Lie group case [18] onto the homogeneous space M. The right-invariant error was associated with synchrony of left-invariant systems on the Lie group.*

Definition 12 *Two systems on a homogenous space are* **synchronous***, if they are e-synchronous with respect to the canonical error* (8)*.*

Theorem 13 *Consider the projected system* (5) *on a homogeneous space M, and a general second system on M of the form*

$$\dot{\hat{y}} = f_{\hat{y}}(\hat{y}, u, t).$$

Then the pair of systems are synchronous if and only if

$$f_{\hat{y}}(\hat{y}, u, t) = T_{\hat{X}} h_{y_0}(\hat{X}u),$$

for all $\hat{X} \in G$ such that $h(\hat{X}, y_0) = \hat{y}$.

Proof: Consider firstly a pair of systems

$$\dot{y} = T_X h_{y_0}(Xu) \tag{9a}$$
$$\dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u). \tag{9b}$$

Consider the systems on G

$$\dot{X} = Xu, \qquad h(X(0), y_0) = y(0) \tag{10a}$$
$$\dot{\hat{X}} = \hat{X}u, \qquad h(\hat{X}(0), y_0) = \hat{y}(0). \tag{10b}$$

It is straightforward to see that the solutions of (10) project to the solutions of (9). From the Lie group case [18], it is known that the systems

(10) are synchronous with respect to the right invariant error $E_r = \hat{X}X^{-1}$. Hence their projected solutions (9) must be synchronous with respect to the projection of E_r, the canonical error on M. It follows that the plants are synchronous.

On the other hand, consider a pair of *synchronous* systems

$$\begin{aligned}\dot{y} &= T_X h_{y_0}(Xu)\\ \dot{\hat{y}} &= f_{\hat{y}}(\hat{y}, u, t).\end{aligned}$$

Consider the system (10a) along with

$$\dot{\hat{X}} = (f_{\hat{y}}(\hat{y}, u, t))^H ,$$

where v^H denotes[1] the horizontal lift with respect to an arbitrary, smooth horizontal distribution to the fibres of M in G. For the right invariant error on G, one has

$$\begin{aligned}\frac{d}{dt}\hat{X}X^{-1} &= T_{\hat{X}}R_{X^{-1}}(f_{\hat{y}}(\hat{y}, u, t))^H - T_{X^{-1}}L_{\hat{X}}T_eL_{X^{-1}}T_XR_{X^{-1}}Xu\\ &= T_{\hat{X}}R_{X^{-1}}(f_{\hat{y}}(\hat{y}, u, t))^H - T_{\hat{X}}R_{X^{-1}}T_eL_{\hat{X}}u.\end{aligned}$$

Since these systems project down onto the original systems on M, the equivariance of h_{y_0} yields that

$$\begin{aligned}\frac{d}{dt}h_{y_0}(\hat{X}X^{-1}) &= T_{\hat{X}X^{-1}}h_{y_0}T_{\hat{X}}R_{X^{-1}}(f_{\hat{y}}(\hat{y}, u, t))^H - T_{\hat{X}X^{-1}}h_{y_0}T_{\hat{X}}R_{X^{-1}}\hat{X}u\\ &= T_{\hat{y}}h_{X^{-1}}T_{\hat{X}}h_{y_0}(f_{\hat{y}}(\hat{y}, u, t))^H - T_{\hat{y}}h_{X^{-1}}T_{\hat{X}}h_{y_0}\hat{X}u\\ &= T_{\hat{y}}h_{X^{-1}}f_{\hat{y}}(\hat{y}, u, t) - T_{\hat{y}}h_{X^{-1}}T_{\hat{X}}h_{y_0}\hat{X}u.\end{aligned}$$

Since this result holds for all initial values $y(0) \in M$, the synchrony condition of the systems yield

$$f_{\hat{y}}(\hat{y}, u, t) = T_{\hat{X}}h_{y_0}(\hat{X}u).$$

□

From the definition of synchrony it is possible to develop and decomposition of an observer into a synchronous internal model and an innovation or nonlinear output injection term.

[1] A construction of a suitable horizontal distribution is given in the proof of Theorem 16.

Definition 14 *Consider a pair of systems* $f_y\colon B\times\mathbb{R}\to TM$*,* $f_{\hat{y}}\colon B\times M\times\mathbb{R}\to TM$*,*

$$\begin{aligned}\dot{y} &= f_y(u,t)\\ \dot{\hat{y}} &= f_{\hat{y}}(w,y,t)\end{aligned}$$

on a homogeneous space.

1. *We say that* $\hat{y}$ *has an internal model of* y *if for all admissible inputs, all* $y_0\in M$ *and all* $t\in\mathbb{R}^+$

$$\hat{y}(t,y_0,y(t,y_0,u),u)=y(t,y_0,u)$$

2. *We define an innovation term to be a map* $\alpha\colon M\times TM\times M\to TM$ *such that*

   (a) $\alpha(\hat{y},w,y,t)\in T_{\hat{y}}M$

   (b) $\alpha(\hat{y}(t,y_0,y(t,y_0,u),u),f_y(u,t),y(t,y_0,u),t)=0$ *for all* $y_0\in M$*,* $t\in\mathbb{R}$ *and all admissible inputs* u*.*

Proposition 15 *Consider the projected system* (5) *on* M*. Then any observer with an internal model of this system has the form*

$$\dot{\hat{y}}=T_{\hat{X}}h_{y_0}(\hat{X}u)+\alpha(\hat{y},u,y,t) \tag{11}$$

where α *is an innovation term.*

Proof: Consider an observer

$$\dot{\hat{y}}=f(\hat{y},u,y,t)$$

with an internal model of the observed system. Then we can define

$$g(\hat{y},u,y,t)=f(\hat{y},u,y,t)-T_{\hat{X}}h_{y_0}(\hat{X}u).$$

A calculation analogous to the proof of Theorem 13 shows that

$$\frac{d}{dt}h_{y_0}(\hat{X}X^{-1})=T_{\hat{X}X^{-1}}h_{y_0}T_{\hat{X}}R_{X^{-1}}g(\hat{y},u,y,t).$$

Since $\hat{y}$ has an internal model of y, this implies that

$$g(y,u,y,t)=0$$

for all $y \in M$, $t \in \mathbb{R}$ and admissible $u \in \mathfrak{g}$, i.e. g is an innovation. $\square$

We call an innovation term $\alpha\colon M \times \mathfrak{g} \times M \times \mathbb{R} \to TM$ **equivariant** if for all $S \in G$, $y, \hat{y} \in M$, $u \in \mathfrak{g}$, $t \in \mathbb{R}$

$$T_{\hat{y}} h_S \alpha(\hat{y}, u, y, t) = \alpha(h_S(\hat{y}), u, h_S(y), t).$$

Note that in this definition we consider only the specific action $(S, \hat{y}, u) \mapsto (h_S(\hat{y}), u)$ on the control bundle $B = M \times \mathfrak{g}$.

Theorem 16 *The dynamics of the canonical error of an observer with an internal model,* (11)*, is autonomous if and only if the innovation term α is equivariant and does not depend on u or t. The autonomous error dynamics has the form*

$$\frac{d}{dt} e = \alpha(e, y_0).$$

Proof: Choose a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g} = \ker T_e h_{y_0} + \mathfrak{h}$ is the direct sum of $\mathfrak{h}$ and the Lie-algebra of the stabiliser subgroup. Define a horizontal distribution H on G by $H(X) := T_e R_X \mathfrak{h}$. Let $u\colon \mathbb{R} \to \mathfrak{g}$ be a fixed, admissible input and $y(t)$, $\hat{y}(t)$ two solutions of (5) and (11) corresponding to this input. We choose $X_0, \hat{X}_0 \in G$ such that $h_{y_0}(X_0) = y(0)$ and $h_{y_0}(\hat{X}_0) = \hat{y}(0)$. Let $X(t)$ and $\hat{X}(t)$ be curves with initial values X_0, $\hat{X}_0$, which satisfy the differential equations

$$\begin{aligned} \dot{X} &= Xu \\ \dot{\hat{X}} &= \hat{X}u - (\alpha(\hat{y}, u, y, t))^H, \end{aligned}$$

where $(v)^H$ denotes the horizontal lift of a vector $v \in T_{h_{y_0}(X)} M$ to $T_X G$ via $H(X)$. The curves project to $y(t)$ and $\hat{y}(t)$, i.e. $h_{y_0}(X(t)) = y(t)$ and $h_{y_0}(\hat{X}(t)) = \hat{y}(t)$ hold. Let us first consider the right-invariant error $\hat{X}X^{-1}$ on the Lie group. Then we get that

$$\begin{aligned} &\frac{d}{dt} \hat{X} X^{-1} \\ &= T_{\hat{X}} R_{X^{-1}} \hat{X} u + T_{\hat{X}} R_{X^{-1}} (\alpha(\hat{y}, u, y, t))^H - T_{X^{-1}} L_{\hat{X}} T_e L_{X^{-1}} T_X R_{X^{-1}} X u \\ &= T_{\hat{X}} R_{X^{-1}} (\alpha(\hat{y}, u, y, t))^H \end{aligned}$$

Using the equivariance of the canonical projection we get for the canonical error on the group that

$$\begin{aligned}\frac{d}{dt}e &= T_{\hat{X}X^{-1}}h_{y_0}\left(T_{\hat{X}}R_{X^{-1}}\left(\alpha(\hat{y},u,y,t)\right)^H\right)\\ &= T_{\hat{y}}h_{X^{-1}}T_{\hat{X}}h_{y_0}\left(\left(\alpha f(\hat{y},u,y,t)\right)^H\right)\\ &= T_{\hat{y}}h_{X^{-1}}\alpha(\hat{y},u,y,t).\end{aligned}$$

If α is equivariant and does not depend on u or t, then

$$\frac{d}{dt}e = \alpha(h(X^{-1},\hat{y}),y_0) = \alpha(e,y_0),$$

i.e. the dynamics of e is autonomous. On the other hand, if the dynamics of e is autonomous, then there is a function $F\colon M \to TM$ with for all $e, y, \hat{y} \in M$ $X, \hat{X} \in G$ with $h_{y_0}(X) = y$, $h_{y_0}(\hat{X}) = \hat{y}$, $h_{y_0}(\hat{X}X^{-1}) = e$, that

$$F(e) = T_{\hat{y}}h_{X^{-1}}\alpha(\hat{y},u,y,t).$$

It follows immediately that α does not depend on u or t, i.e. $\alpha(\hat{y},u,y,t) = \alpha(\hat{y},y)$. Since the canonical error is invariant under the natural action of G on $M \times M$, one has for all $S \in G$ that

$$T_{h_S(\hat{y})}h_{(SX)^{-1}}\alpha(h_S(\hat{y}),h_S(y)) = F(e) = T_{\hat{y}}h_{X^{-1}}\alpha(\hat{y},y).$$

In particular, this holds for $y = y_0$. Thus, for all $S \in G$, $\hat{y} \in M$

$$T_{\hat{y}}h_S\alpha(\hat{y},y_0) = \alpha(h_S(\hat{y}),h_S(y_0)).$$

For $x, y \in M$, $R \in G$ with $y = h(Y,y_0)$, we can set $\hat{y} = h(Y^{-1},x)$ and $S = RY$, and obtain

$$T_y h_R\alpha(x,y) = \alpha(h_R(x),h_R(y)).$$

Hence α is equivariant. □

5 Observer synthesis

In this section we consider the question of observer synthesis. The approach taken is to build observers from a synchronous term along with an gradient-like innovation derived from a cost function on the homogeneous space. Thus, we begin by considering gradient-like observers for projected systems and then look at lifting these observers up to the Lie group.

5.1 Observer synthesis on the projected system

In order to compute gradient-like terms for construction of the observer it is necessary to define a Riemannian metric on the homogeneous output space. Furthermore, it is natural, and indeed necessary for the results that follow, that the metric is invariant with respect to the group action. Not all homogeneous spaces necessarily admit such a metric and it is necessary to assume this additional structure to continue with the proposed approach. That is, we assume that the homogeneous output space M is a **reductive homogeneous space**, or equivalently that it admits an invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on M [9]. One has that for all $S \in G$, $y \in M$, $u, v \in T_yM$

$$\langle u, v\rangle = \langle T_y h_S u, T_y h_S v\rangle.$$

Let $f\colon M \times M \to \mathbb{R}$ be a smooth cost function on M. That is, the diagonal $\Delta = \{(y, y) \mid y \in M\}$ is a closed minimal level set of the function f. A smooth cost function $f\colon M \times M \to \mathbb{R}$ is said to be invariant function with respect to the group action h if

$$f(h(S, \hat{y}), h(S, y)) = f(\hat{y}, y)$$

for all $y, \hat{y} \in M$, $S \in G$.

We propose the observer design

$$\dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u) - \operatorname{grad}_1 f(\hat{y}, y) \tag{12}$$

for the system (5) on M. Here, $\operatorname{grad}_1 f$ denotes the gradient of the function $\hat{y} \mapsto f(\hat{y}, y)$. Note that since f is minimal on the diagonal Δ then $\hat{y} \mapsto f(\hat{y}, y)$ has a local minima at $\hat{y} = y$ and $\operatorname{grad}_1 f(y, y) = 0$. This shows that the gradient term is an innovation according to Definition 14. The following lemma shows that the innovation $\operatorname{grad}_1 f$ is itself equivariant if the cost function and metric are invariant.

Lemma 17 *Consider a group action $h : G \times M \to M$ on a reductive homogeneous space M. Let g be an invariant metric on M and f be an invariant cost function with respect to h. Then for all $S \in G$, $y, \hat{y} \in M$*

$$\operatorname{grad}_1 f(h(S, \hat{y}), h(S, y)) = T_{\hat{y}} h_S \operatorname{grad}_1 f(\hat{y}, y).$$

That is $\operatorname{grad}_1 f$ is an equivariant vector field on M.

Proof: By the invariance of f and the Riemannian metric we have for all $v \in T_{\hat{y}}M$

$$\begin{aligned} \langle \operatorname{grad}_1 f(\hat{y}, y), v \rangle &= \mathrm{d}_1 f(\hat{y}, y)(v) \\ &= \mathrm{d}_1 f(h(S, \hat{y}), h(S, y))(T_{\hat{y}} h_S(v)) \\ &= \langle \operatorname{grad}_1 f(h(S, \hat{y}), h(S, y)), T_{\hat{y}} h_S(v) \rangle \\ &= \left\langle (T_{\hat{y}} h_S)^{-1} \operatorname{grad}_1 f(h(S, \hat{y}), h(S, y)), v \right\rangle, \end{aligned}$$

with d_1 denoting the differential with respect to the first variable. □

Since any equivariant innovation term yields autonomous error dynamics (Theorem 16), we obtain the following corollary for the proposed gradient-like observer (12).

Theorem 18 *Consider system* (5) *on a reductive homogeneous space equipped with an invariant metric. Assume that there is an invariant cost function f and consider the observer* (12). *Then the canonical error $e = h(\hat{X}X^{-1}, y_0)$ has the dynamics*

$$\dot{e} = -\operatorname{grad}_1 f(e, y_0).$$

Proof: By Lemma 17 the term $\operatorname{grad}_1 f$ is equivariant. By Theorem (16) the error is autonomous and the the form as given above. □

Since the projected system is fully observed and the observer design is based on a gradient construction, it is possible to obtain strong almost global stability results for the observer.

Corollary 19 *Consider system* (5) *on a reductive homogeneous space equipped with an invariant metric. Assume that there is an invariant cost function f and consider the observer* (12). *Assume furthermore that $\hat{y} \mapsto f(\hat{y}, y_0)$ is a Morse-Bott function with unique global minimum y_0 and no further local minima. Then the error e converges to y_0 for almost all initial conditions and arbitrary, admissible inputs u.*

It is relevant to note that although the innovation term is equivariant the observer itself is not. This is due to the fact that the synchronous term is not equivariant with respect to direct to direct application of the group action.

This situation is analogous to the case observed for Lie group systems with full measurements [18].

Since the system state of the projected system is fully observed then the choice of a gradient innovation term based on a function of the full state on M is very natural. The most significant difficulty in the proposed observer design is the requirement to find a suitable cost function f. The two requirements for a cost function candidate are the invariance property (used in the structure of the observer design) and the Morse-Bott requirement (used in the convergence proof Corollary 19). The easiest approach to generating cost functions is to start by searching for a Morse-Bott function candidate without demanding the invariance property. That is one looks to construct a function $\hat{f} : M \to \mathbb{R}$ such that $\hat{f}(y)$ is Morse-Bott and has an isolated global minima at y_0. Such a function is always locally available by considering a least squares cost with respect to local coordinates on the homogeneous output space M. In practice, local coordinates generally yield a poor choice and the best Morse-Bott function candidates are obtained by studying the global geometric structure of the homogeneous space case by case. Since such homogeneous spaces are generated as symmetry spaces of Lie groups there is often significant structure that can be exploited in the construction of global Morse-Bott candidate functions. Once such a function $\hat{f}$ is found it can be used to generate an invariant cost function by setting

$$f(y_1, y_2) = f(h(X, y_0), h(\hat{X}, y_0)) = \hat{f}(h(\hat{X}^{-1}X, y_0)).$$

for any X and $\hat{X}$ such that $y_1 = h(X, y_0)$ and $y_2 = h(\hat{X}, y_0)$. It is straightforward to verify that f is well defined and invariant. Moreover, if $\hat{f}$ has an unique global minimum at y_0 and no further local minima, the function f will satisfy the conditions of Corollary 19. The construction is analogous to the development given in [18].

5.2 Observers for the full system

Section 5.1 provided a constructive process for the design of observers for projected systems. Since the projected system has a fully observed state, the gradient based innovation is one of the most natural approaches to observer design. Since the projected system is a minimal observable realisation of the (1) and (2) on the Lie group, it is not to be expected that an observer constructed separately on the Lie group could yield more information about the state than that obtained from the projected system. Thus, the most 'natural' observer on the Lie group is the 'lift' of the observer on the projected system onto the Lie group. However, only the innovation is equivariant for the project system observer and the main system cannot be unique lifted. The proposed solution is to lift the innovation and use the natural synchronous internal model on the Lie group to define an observer.

For any reductive homogeneous space there is an invariant horizontal distribution $H(X)$ and an invariant Riemannian metric on G such that the projection of the metric from $H(X)$ onto TM induces the invariant Riemannian metric on M. Details of the construction for the case of left invariant actions is provided in Cheeger *et al.* [9]. The case of right group actions and right invariant metrics is entirely analogous. The horizontal distributions arising in this construction have the form discussed in the proof of Theorem 16, but the subspace $\mathfrak{h}$ has to be chosen to satisfy some additional constraints that ensure the metric construction is well defined. Given the horizontal distribution $H(X)$ there is a unique linear map, termed the lift,

$$(\cdot)^{H(X)} : T_y M \to H(X)$$

such that $T_X h_{y_0}((v)^{H(X)}) = v$. That is $(v)^{H(X)} \in H(X)$ denotes the lift of $v \in T_y M$. Where the point $X \in G$ is clear from context, or any choice is equivalent, we will write $(v)^H = (v)^{H(X)}$.

The proposed observer for the system (1) and (2) is given by

$$\dot{\hat{X}} = \hat{X}u - \Big(\mathrm{grad}_1 f(h_{y_0}(\hat{X}), y)\Big)^H \tag{13}$$

on the group. The following proposition follows directly from its construction.

Proposition 20 *The observer* (13) *projects to the observer* (12) *on the homogeneous space.*

Only the innovation term is a horizontal lift in (13). The full system is not a horizontal lift since $\hat{X}u \notin H(\hat{X})$ need not lie in the horizontal distribution and consequently

$$\hat{X}u \neq \Big(T_{\hat{X}} h_{y_0}(\hat{X}u)\Big)^H .$$

The proposed observer, however, can be thought of as a more general lift of projected system onto G as shown in Proposition 20. The convergence properties of the observer (13) follow directly from the results on the homogeneous space.

Corollary 21 *Consider the left invariant system* (1) *with complementary outputs* (2) *on a reductive homogeneous output space. Let f be and invariant cost function such that $\hat{y} \mapsto f(\hat{y}, y_0)$ is a Morse-Bott function with unique global minimum y_0 and no further local minima. Let $E_r : G \times G \to G$, $E_r(\hat{X}, X) = \hat{X}X^{-1}$ be the right invariant error on G [18].*

Then for generic initial conditions $\hat{X}(0)$ and arbitrary, admissible inputs u the error E_r converges asymptotically to $\operatorname{stab}(y_0)$.

Proof: This result is a straightforward consequence of Corollary 19 and the fact that the canonical error on M is the projection of E_r. □

The observer (13) can also be derived directly on the Lie group. For this purpose we define the lift $\tilde{f}$ of the cost function f as the map $G \times G \to \mathbb{R}$,

$$\tilde{f}(\hat{X}, X) = f(h_{y_0}(\hat{X}), h_{y_0}(X)). \tag{14}$$

Proposition 22 *Consider a right group action on a reductive homogeneous space. Let $f : M \times M \to \mathbb{R}$ be an invariant cost function on M and let $\tilde{f}$ be given by* (14). *The gradient of the cost function f and the lifted function $\tilde{f}$ are related by*

$$\operatorname{grad}_1 \tilde{f}(\hat{X}, X) = \Big(\operatorname{grad}_1 f(h_{y_0}(\hat{X}), h_{y_0}(X))\Big)^H,$$

If f is invariant under the action of G on M, then the lifted function $\tilde{f}$ is a right invariant cost function on G.

Proof: For the first statement, note that for all $v \in T_{\hat{X}}G$

$$\mathrm{d}_1 \tilde{f}(\hat{X}, X)(v) = \mathrm{d}_1 f(\hat{X}y_0, \hat{X}y_0)(T_{\hat{X}} h_{y_0} v).$$

Hence for all $v \in T_{\hat{X}}G$

$$\langle \operatorname{grad}_1 \tilde{f}(\hat{X}, X), v\rangle = \langle \operatorname{grad}_1 f(\hat{X}y_0, \hat{X}y_0), T_{\hat{X}} h_{y_0} v\rangle.$$

In particular,

$$\langle \operatorname{grad}_1 \tilde{f}(\hat{X}, X), v\rangle = 0$$

for $v \in \ker T_{\hat{X}} h_{y_0}$. Hence, $\operatorname{grad}_1 \tilde{f}(\hat{X}, X) \in H(\hat{X})$. On the other hand we have by our conditions on the Riemannian metrics that for all $v \in H(\hat{X})$

$$\langle \operatorname{grad}_1 \tilde{f}(\hat{X}, X), v\rangle = \langle T_{\hat{X}} h_{y_0} \operatorname{grad}_1 \tilde{f}(\hat{X}, X), T_{\hat{X}} h_{y_0} v\rangle.$$

Hence $\operatorname{grad}_1 \tilde{f}$ is the horizontal lift of $\operatorname{grad}_1 f$. If f is invariant, then for all $X, \hat{X}, Z \in G$ we have

$$\begin{aligned}\tilde{f}(\hat{X}Z, XZ) = f(h_{y_0}(\hat{X}Z), h_{y_0}(XZ)) &= f(h(Z, h_{y_0}(\hat{X})), h(Z, h_{y_0}(X))) \\ &= f(h_{y_0}(\hat{X}), h_{y_0}(X)) = \tilde{f}(\hat{X}, X),\end{aligned}$$

i.e. the lift is invariant too. □

The lifted cost function is then a candidate for the observer construction for systems on a Lie group proposed in earlier work [18]. This construction yields

$$\dot{\hat{X}} = \hat{X}u - \operatorname{grad}_1 \tilde{f}(\hat{X}, X). \tag{15}$$

This construction is made based on the understanding that the full state X is measured, however, in this specific case the state X is only used in the innovation term in a manner that can be derived from the information contained in the measurement $y = h(X, y_0)$. The easiest way to see this is to note that by Proposition 22 the observer (15) is the same observer as (13). As a consequence we get the equivalence of the observer error trajectories.

Corollary 23 *Consider the left invariant system* (1) *with complementary outputs* (2) *on a reductive homogeneous output space M. Let f be and invariant cost function such that $\hat{y} \mapsto f(\hat{y}, y_0)$ is a Morse-Bott function with unique global minimum y_0 and no further local minima. Let $E_r = \hat{X}X^{-1}$ be the right invariant error on G and $e = h(E_r, y_0)$ be the induced canonical error on M. Then if $h(\hat{X}(0), y_0) = \hat{y}(0)$, the error trajectory $E_r(t)$ generated by* (13) *with initial conditions $\hat{X}(0)$ on G projects to the error trajectory $e(t)$ generated by 12 with initial conditions $\hat{y}(0)$. That is*

$$h(E_r(t), y_0) = e(t).$$

6 Example: Attitude estimation

In this section, we complete the analysis of the simple example introduced in Examples 1 and 8 to demonstrate how the developments in the paper can be applied in practice. We consider the complementary output case discussed in Example 1, that of estimating the attitude of a rigid body based on measurements of a single vectorial direction measured in the body-fixed-frame along with full measurement of angular velocity. There is an extensive literature concerning the estimation of attitude of rigid-bodies based on body-fixed frame measurements [4, 5, 6, 7, 11, 14, 17, 18, 19, 20, 22, 21, 24, 23, 25, 26, 30, 35, 34, 36, 39, 40]. We mention in particular, the recent papers [14, 20] that introduced the "explicit complementary filter" and the earlier work [26, 25] that developed observers for an inertial direction. Both these observers are obtained as specialisations of the general techniques proposed in this paper for a natural choice of cost function. We mention also the independent work [5, 6] that obtained the same observer as [20].

The system is posed on $SO(3)$ (see Example 1)

$$\dot{R} = R\Omega_\times \tag{16a}$$

$$y = R^\top y_0 \tag{16b}$$

with $y_0 \in \{A\}$ the 'reference' direction in the inertial frame. The complementary output $y \in \{B\}$ is the observed orientation of the vectorial direction measured in the body-fixed-frame. The measurement equation (16b) models the coordinate transformation induced by the relative orientation of $\{B\}$ with respect to $\{A\}$.

The homogeneous output space is the unit sphere S^2. By Theorem 5, the system (16) projects to a system on S^2. Recalling the derivation in Example 8 the projected system can be written

$$\dot{y} = -\Omega_\times y, \quad y(0) = h(X(0), y_0) \tag{17}$$

on S^2. We continue by designing an observer for system (17) as discussed in Section 5.

Consider the cost function $f(\hat{y}, y) = (k/2)\|\hat{y} - y\|^2$ for $k > 0$ a positive constant. It is straightforward to verify that f satisfies the conditions of Corollary 19. That is, $f\colon S^2 \times S^2 \to \mathbb{R}$ is invariant under the right action of $SO(3)$ on $S^2 \times S^2$, and f is Morse-Bott in the first argument with a unique global minima at $\hat{y} = y$. The constant k is introduced to provide a tunable gain in the observer design: adjusting k scales the eigenvalues of the Hessian of f and tunes the exponential rate of the convergence of the error dynamics. Note that

$$f(\hat{y}, y) = \frac{k}{2}\|\hat{y} - y\|^2 = k(1 - \langle \hat{y}, y\rangle).$$

The term $(1 - \langle \hat{y}, y\rangle) = 1 - \cos(\angle(\hat{y}, y))$ is locally quadratic in the angle $\angle(\hat{y}, y))$. The references [26, 25, 14, 20] used $(1 - \langle \hat{y}, y\rangle)$ directly as a Lyapunov function. This cost is also closely related to the quaternion cost used in earlier works [40, 36] as discussed in the appendix of [20].

The Riemannian metric on S^2 induced by the direct embedding of S^2 in $\mathbb{R}^3$ is invariant under the right action $(R, x) \mapsto R^\top x$ of $SO(3)$ on $\mathbb{R}^3$. In particular, $(R^\top v)^\top R^\top w = v^\top R R^\top w = v^\top w$ for $v, w \in T_y S^2 \subset \mathbb{R}^3$. The gradient of f is computed from the relationships $\langle \operatorname{grad}_1 f(\hat{y}, y), w \rangle = \mathrm{d}_1 f(\hat{y}, y)[w]$ and $\operatorname{grad}_1 f(\hat{y}, y) \in T_{\hat{y}} S^2$. Here $\mathrm{d}_1 f(\hat{y}, y)$ denotes the differential of f with respect to the first variable, i.e. $\hat{y}$. One has

$$\mathrm{d}_1 f(\hat{y}, y)[w] = w^\top (\hat{y} - y) = k w^\top \underbrace{(I - \hat{y}\hat{y}^\top)(\hat{y} - y)}_{\in T_{\hat{y}} S^2}$$

since $w^\top = w^\top (I - \hat{y}\hat{y}^\top)$ as $w \in T_{\hat{y}} S^2$. It follows that

$$\operatorname{grad}_1 f(\hat{y}, y) = -k \left(I - \hat{y}\hat{y}^\top \right) y$$

since $\hat{y}$ is in the kernel of the projection $(I - \hat{y}\hat{y}^\top)$.

The proposed observer for the projected output system is (12)

$$\dot{\hat{y}} = -\Omega_\times \hat{y} + k(I - \hat{y}\hat{y}^\top) y. \tag{18}$$

Note that this observer equation is fully posed as a system on S^2 and does not depend on the underlying system on $SO(3)$. Since $\hat{y} \mapsto f(\hat{y}, y)$ is a Morse function with only one global minimum and no further local minima, Corollary 19 applies and the canonical error on S^2 for this observer converges for generic initial conditions to y_0.

It is of interest to provide an equivalent form for Eq. 18. Observe that

$$k(y\hat{y}^\top - \hat{y}y^T)\hat{y} = k(y - \hat{y}y^T\hat{y}) = k \left(I - \hat{y}\hat{y}^\top \right) y = -\operatorname{grad}_1 f(\hat{y}, y) \tag{19}$$

Moreover, it is easily verified that

$$(\hat{y}y^\top - y\hat{y}^\top) = (\hat{y} \times y)_\times \tag{20}$$

Thus, (18) can be written

$$\dot{\hat{y}} = -\Omega_\times \hat{y} + k(\hat{y} \times y) \times \hat{y}. \tag{21}$$

This observer was originally studied by Metni *et al.* [26, 25]. The derivation given in earlier work is based on a Lyapunov analysis and includes an integral term for compensation of gyro bias that does not fit the analysis framework presented in this paper, however, the main proportional terms in [26, 25] are exactly (21).

To lift (18) to $SO(3)$ it is necessary to construct a right invariant horizontal distribution for which the right invariant metric on $SO(3)$ projects to the metric $\langle u, v\rangle$ on S^2 via the differential of the group action. Define

$$\mathfrak{h} = \{\omega_\times \in \mathfrak{g} \,\big|\, \mathrm{tr}(\omega_\times^\top (y_0)_\times) = 0\} \equiv \{\omega_\times \,\big|\, \omega \in \mathbb{R}^3, \omega^\top y_0 = 0\}$$

The subspace $\mathfrak{h}$ is chosen as the orthogonal complement of the subalgebra of the stabiliser of h under the inner product $\langle \Omega_\times, \Psi_\times\rangle = \mathrm{tr}(\Omega_\times^\top \Psi_\times)$ defined on $\mathfrak{g}$. Physically, this set corresponds to angular velocities expressed in the inertial frame $\{A\}$ that are orthogonal to the reference direction y_0. The horizontal distribution $H(\hat{R})$ is defined in the standard manner[2]

$$H(\hat{R}) = \{\omega_\times \hat{R} \,\big|\, \omega_\times \in \mathfrak{h}\}.$$

For any $\omega_\times \in \mathfrak{h}$ there is an element $\Omega_\times = \mathrm{Ad}_{\hat{R}^\top} \omega_\times$, where $\mathrm{Ad}_{\hat{R}^\top} \omega_\times = \hat{R}^\top \omega_\times \hat{R}$ is the adjoint operator, associated with the angular velocity ω expressed in the body-fixed-frame. Thus, one can also write

$$H(\hat{R}) = \{\hat{R}\Omega_\times \,\big|\, \Omega_\times = \mathrm{Ad}_{\hat{R}^\top} \omega_\times, \omega_\times \in \mathfrak{h}\}.$$

Next we consider how to compute the lift $(v)^H$ of a tangent vector in $T_{\hat{y}} S^2$ into $H(\hat{R})$. Let $v \in T_{\hat{y}} S^2$ and $\hat{R} \in SO(3)$ such that $h(\hat{R}, y_0) = \hat{y}$. Define $\bar{\Omega}(v)$ by the solution to

$$\bar{\Omega}(v) \times \hat{y} = v, \quad \bar{\Omega}(v)^\top \hat{y} = 0.$$

Since $v \in T_{\hat{y}} S^2$ is orthogonal to $\hat{y}$ then it is clear that this vector exists and is unique. Physically, $\bar{\Omega}(v)$ is the instantaneous body-fixed-frame velocity that generates the observed velocity v for the projected dynamics of $\hat{y}$, and has zero yaw component around $\hat{y}$. The horizontal lift $(v)^H$ is defined by

$$(v)^H := \left(\mathrm{Ad}_{\hat{R}} \bar{\Omega}(v)_\times\right) \hat{R} = \hat{R}\bar{\Omega}(v)_\times.$$

The element $\omega(v)_\times = \mathrm{Ad}_{\hat{R}} \bar{\Omega}(v)_\times \in \mathfrak{h}$ corresponds to the element in the subspace $\mathfrak{h}$ representing the lifted angular velocity expressed in the inertial frame.

It remains to show that the metric $\langle u, v\rangle = u^\top v$ on S^2 is induced by a right invariant metric on $H(\hat{R})$. Observe that for $v, w \in T_{\hat{y}} S^2$ then for any $\hat{R}$ such

[2] See also the proof of Theorem 16.

that $h(\hat{R}, y_0) = \hat{y}$, one has

$$\begin{aligned}
v^\top w =& \hat{y}^\top \bar{\Omega}(v)_\times^\top \bar{\Omega}(w)_\times \hat{y} = \bar{\Omega}(v)^\top \hat{y}_\times^\top \hat{y}_\times \bar{\Omega}(w) \\
=& \bar{\Omega}(v)^\top (I - \hat{y}\hat{y}^\top) \bar{\Omega}(w) = \bar{\Omega}(v)^\top \bar{\Omega}(w) \\
=& \frac{1}{2} \operatorname{tr}\left(\bar{\Omega}(v)_\times^\top \bar{\Omega}(w)_\times \right) = \frac{1}{2} \langle \bar{\Omega}(v)_\times \hat{R}, \bar{\Omega}(w)_\times \hat{R} \rangle
\end{aligned} \tag{22}$$

where the final line is the natural right invariant metric on the Lie group $SO(3)$ scaled by $1/2$. The scaling factor is associated to the structure of the matrix representation used. The above argument verifies that the projection $T_{\hat{R}} h_{y_0}$ projects the right invariant metric (22) restricted to $H(\hat{R})$ to the embedded Euclidean metric on $T_{\hat{y}} S^2$.

From Equation 13 the lift of the observer (15) is

$$\dot{\hat{R}} = \hat{R}\Omega_\times + k \left((I - \hat{y}\hat{y}^\top) y \right)^H \tag{23}$$

From Proposition 20 it follows that this observer projects to the observer on S^2, Eq. 18. Corollary 21 states that $\hat{R}$ converges to the set $\{\hat{R} \in SO(3) \mid h(\hat{R}, y_0) = y\}$, that is, $\hat{R}$ is identified up to the unobservable rotation around the measured direction.

It is of interest to provide an explicit form for Eq. 23. Recalling (19) and (20) one has

$$\operatorname{grad}_1 f(\hat{y}, y) = k(\hat{y} \times y)_\times \hat{y}$$

It is easily verified that $\bar{\Omega}(\operatorname{grad}_1 f)_\times = k(\hat{y} \times y)_\times$. Thus, the lifted observer can be written

$$\begin{aligned}
\dot{\hat{R}} =& \hat{R}u - k \left(\operatorname{Ad}_{\hat{R}} (\hat{y} \times y)_\times \right) \hat{R} \\
=& \hat{R} \left(u - k(\hat{y} \times y)_\times \right) = \hat{R} \left(u + k(y \times \hat{y})_\times \right)
\end{aligned} \tag{24}$$

Equation 24 is the explicit complementary filter proposed in [20, Eq. (32)] excluding the integral introduced in that paper to compensate gyro bias.

The observer (23) can also be obtained as a direct gradient of a lifted cost function. The lifted cost function $\tilde{f}$ of f is given by

$$\tilde{f}(\hat{R}, R) = \frac{k}{2} \|\hat{R}^\top y_0 - R^\top y_0\|^2 = \frac{k}{2} \operatorname{tr}\left((\hat{R} - R)(\hat{R} - R)^\top y_0 y_0^\top \right).$$

The differential of the lifted cost function with respect to the first coordinate is

$$\mathrm{d}_1 \tilde{f}(\hat{R}, R)[\hat{R}\Omega_\times] = k \operatorname{tr}\left(\hat{R}\Omega_\times (\hat{R} - R)^\top y_0 y_0^\top \right) - k \operatorname{tr}\left((\hat{R} - R)\Omega_\times \hat{R}^\top y_0 y_0^\top \right)$$

Rearranging terms and using the notation $\mathbb{P}_{\mathfrak{so}(3)}(A) = (A - A^\top)/2$ for matrix projection onto the subspace of anti-symmetric matrices, the differential can be rewritten

$$\begin{aligned} \mathrm{d}_1\, \tilde{f}(\hat{R}, R)(\hat{R}\Omega_\times) =& k \operatorname{tr}\left(\Omega_\times \mathbb{P}_{\mathfrak{so}(3)}\left((\hat{R} - R)^\top y_0 y_0^\top \hat{R}\right)\right) \\ =& 2k \operatorname{tr}\left(\Omega_\times \mathbb{P}_{\mathfrak{so}(3)}\left((\hat{y} - y)\hat{y}^\top\right)\right) \\ =& 2k \operatorname{tr}\left(\Omega_\times (\hat{y}y^\top - y\hat{y}^\top)\right). \end{aligned}$$

Using the metric (22) one obtains the gradient

$$\operatorname{grad}_1 \tilde{f}(\hat{R}, R) = -k\hat{R}(\hat{y}y^\top - y\hat{y}^\top) = -k\hat{R}(\hat{y} \times y).$$

Note that factor of 1/2 in the metric cancels the factor of 2 in the differential. The gradient-like observer on the Lie-group is

$$\hat{R} = \hat{R}u - k\hat{R}(\hat{y} \times y) = \hat{R}u - (\operatorname{grad}_1 f(\hat{R}, R))^H$$

(recalling (21)) and one obtains the observer (24) as expected.

7 Conclusions

This paper provides a comprehensive analysis of the design of observers for invariant systems posed on finite-dimensional connected Lie groups with measurements generated by a complementary group action on an associated homogeneous space. The observer synthesis problem can be tackled for the projected system on the homogeneous output space based on a canonical decomposition of the observer dynamics into a synchronous internal model and an equivariant innovation. A gradient-like construction for the innovation term was proposed that leads to strong stability properties of the observer, both for the projected system on the homogeneous output space, and for the lifted system on the Lie-group (stable to the unobservable subgroup).

References

[1] B. Anderson and J. Moore. *Optimal Filtering.* Dover Publications, Inc., 2005.

[2] N. Apostolou and D. Kazakos. On local observability for invariant systems on Lie groups and coset spaces. *Proc. R. Soc. Lond.*, 452:2801–2805, 1996.

[3] G. Baldwin, R. Mahony, J. Trumpf, T. Hamel, and T. Cheviron. Complementary filter design on the Special Euclidean group SE(3). In *European Control Conference (ECC 2007)*, pages 3763–3770, Kos, Greece, 2007.

[4] S. Bonnabel, P. Martin, and P. Rouchon. A non-linear symmetry-preserving observer for velocity-aided inertial navigation. In *Proceedings of the American Control Conference*, pages 2910–2914, June 2006.

[5] S. Bonnabel, P. Martin, and P. Rouchon. Invariant asymptotic observers. *IEEE Transactions on Automatic Control*, 2008. Accepted for publication.

[6] S. Bonnabel, P. Martin, and P. Rouchon. Non-linear observer on Lie groups for left-invariant dynamics with right-left equivariant output. In *Proceedings of the 17th World Congress The International Federation of Automatic Control*, Seoul, Korea, July 6-11 2008.

[7] S. Bonnabel and P. Rouchon. *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, volume 322 of *Lecture Notes in Control and Information Sciences*, chapter On Invariant Observers, pages 53–67. Springer-Verlag, 2005.

[8] R. Brockett. System theory on group manifolds and coset spaces. *SIAM J. Control*, 10(2):265–284, 1972.

[9] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry.* North-Holland, Amsterdam, 1975.

[10] D. Cheng, W. P. Dayawansa, and C. F. Martin. Observability of systems on Lie groups and coset spaces. *SIAM J. Contr. Optim.*, 28:570–581, 1990.

[11] J. L. Crassidis, F. L. Markley, and Y. Cheng. Nonlinear attitude filtering methods. *Journal of Guidance, Control,and Dynamics*, 30(1):12–28, January 2007.

[12] A. Doucet, N. de Freitas, and N. Gordon, editors. *Sequential Monte Carlo methods in practice.* Statistics for Engineering and Information Science. Springer, 2001.

[13] J. W. Grizzle and S. I. Marcus. The structure of nonlinear control systems possessing symmetries. *IEEE Trans. Auto. Control*, 30(3):248–258, 1985.

[14] T. Hamel and R. Mahony. Attitude estimation on $SO(3)$ based on direct inertial measurements. In *Proceedings of the 2006 IEEE International Conference on Robotics and Automation*, pages 2170–2175, Orlando Fl., USA, April 2006. Institue of Electrical and Electronic Engineers.

[15] V. Jurdjevic and I. Kupka. Control systems on semi-simple Lie groups and their homogeneous spaces. *Annales de l'institut Fourier*, 31(4):151–179, 1981.

[16] V. Jurdjevic and H. J. Sussmann. Control systems on Lie groups. *Journal of Differential Equations*, 12:313–329, 1972.

[17] C. Lageman, R. Mahony, and J. Trumpf. State observers for invariant dynamics on a Lie group. In *Proceedings of the international conference on the Mathematical Theory of Networks and Systems*, 2008.

[18] C. Lageman, J. Trumpf, and R. Mahony. Gradient-like observers for invariant dynamics on a Lie group. *IEEE Transactions on Automatic Control*, 2008. conditionally accepted for publication August 2008.

[19] R. Mahony, T. Hamel, and J.-M. Pflimlin. Complementary filter design on the special orthogonal group SO(3). In *Proceedings of the 44th Conference on Decision and Control*, pages 1477–1484, Seville, Spain, 2005.

[20] R. Mahony, T. Hamel, and J.-M. Pflimlin. Non-linear complementary filters on the special orthogonal group. *IEEE Transactions on Automatic Control*, 53(5):1203–1218, 2008.

[21] D. Maithripala, J. Berg, and W. Dayawansa. An intrinsic observer for a class of simple mechanical systems on a Lie group. In *American Control Conference*, volume 2, pages 1546–1551, 2004.

[22] D. Maithripala, J. Berg, and W. Dayawansa. Almost-global tracking of simple mechanical systems on a general class of Lie groups. *IEEE Transactions on Automatic Control*, 51(2):216 – 225, 2006.

[23] P. Martin and E. Salaün. Invariant observers for attitude and heading estimation from low-cost inertial and magnetic sensors. In *Proceedings*

of the 46th IEEE Conference on Decision and Control, pages 1039–1045, 2007.

[24] P. Martin and E. Salaün. An invariant observer for earth-velocity-aided attitude heading reference systems. In *Proceedings of the 17th World Congress The International Federation of Automatic Control*, Seoul, Korea, July 6-11 2008.

[25] N. Metni, J.-M. Pflimlin, T. Hamel, and P. Soueeres. Attitude and gyro bias estimation for a VTOL UAV. *Control Engineering Practice*, 14(12):1511–1520, December 2006.

[26] N. Metni, J.-M. Pflimlin, T. Hamel, and P. Soueres. Attitude and gyro bias estimation for a flying UAV. In *Intelligent Robots and Systems, 2005. (IROS 2005). 2005 IEEE/RSJ International Conference on*, pages 1114–1120, 2-6 Aug. 2005.

[27] H. Nijmeijer and T. I. Fossen, editors. *New Directions in Nonlinear Observer Design*. Lecture Notes in Control and Information Sciences. Springer, 1999.

[28] H. Nijmeijer and A. J. van der Schaft. Partial symmetries for nonlinear systems. *Mathematical Systems Theory*, 18:79–86, 1985.

[29] H. Rehbinder and B. Ghosh. Pose estimation using line-based dynamic vision and inertial sensors. *IEEE Transactions on Automatic Control*, 48(2):186–199, 2003.

[30] S. Salcudean. A globally convergent angular velocity observer for rigid body motion. *IEEE Transactions on Automatic Control*, 46(12):1493–1497, 1991.

[31] H. J. Sussmann. On quotients of manifolds: A generalization of the closed subgroup theorem. *Bull. AMS*, 80(3):573–575, 1974.

[32] H. J. Sussmann. Existence and uniqueness of minimal realizations of nonlinear systems. *Mathematical Systems Theory*, 10:263–284, 1977.

[33] P. Tabuda and G. Pappas. Quotients of fully nonlinear control systems. *SIAM J. Contr. Optim.*, 43(6):1844–1866, 2005.

[34] A. Tayebi and S. McGilvray. Attitude stabilization of a VTOL quadrotor aircraft. *IEEE Transactions on Control Systems Technology*, 14(3):562–571, May 2006.

[35] A. Tayebi, S. McGilvray, A. Roberts, and M. Moallem. Attitude estimation and stabilization of a rigid body using low-cost sensors. In *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 6424–6429, 2007.

[36] J. Thienel and R. M. Sanner. A coupled nonlinear spacecraft attitude controller and observer with an unknow constant gyro bias and gyro noise. *IEEE Transactions on Automatic Control*, 48(11):2011–2015, 2003.

[37] A. J. van der Schaft. Symmetries and conservation laws for hamiltonian systems with inputs and outputs: A generalization of Noether's theorem. *Sys. Contr. Lett.*, 1:108–115, 1981.

[38] J. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira. Landmark based nonlinear observer for rigid body attitude and position estimation. In *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 1033–1038, 2007.

[39] J. Vasconcelos, C. Silvestre, and P. Oliveira. A nonlinear observer for rigid body attitude estimation using vector observations. In *Proceedings of the 17th World Congress The International Federation of Automatic Control*, pages 8599–8604, Seoul, Korea, 2008.

[40] B. Vik and T. Fossen. A nonlinear observer for GPS and INS integration. In *Proceedings of the 40th IEEE Conference on Decision and Control*, pages 2956–2961, 2001.