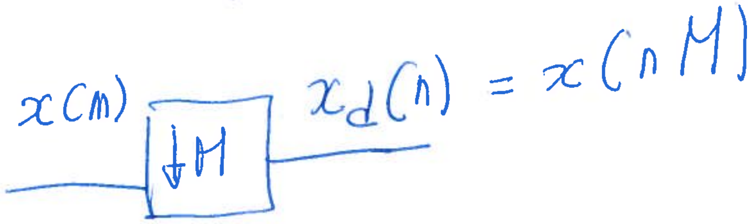
Fill 33

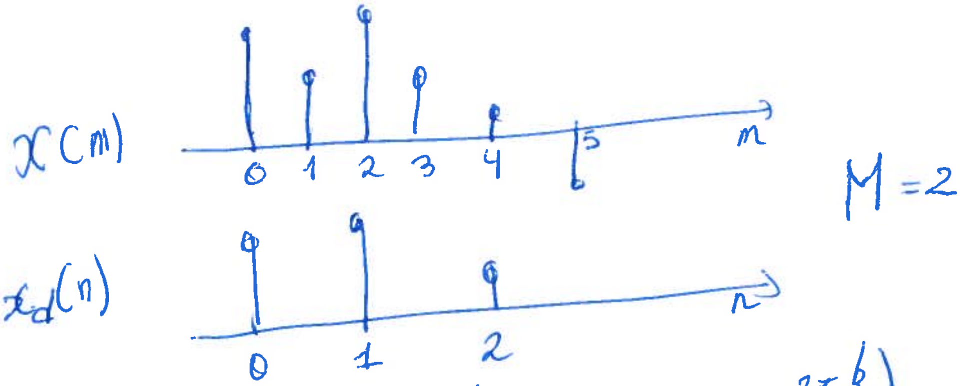
Chap. 4 Multirate Filters

4.1 Why

+ different sampling rates co-exist
+ Wideband vs. narrowband ⇒ Multiplexing
+ Multi resolution filters
+ Efficient filters

4.2 Decimation





$$\Rightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\frac{\omega-2\pi k}{M}}\right)$$

$$x'(n) = \begin{cases} x(n) & n = nM,\ n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \qquad x'(n) = x(n) \sum_{n=-\infty}^{+\infty} \delta(n-nM)$$

1 $$X_d(e^{j\omega}) = \sum_n x_d(n)\, e^{-j\omega n}$$
$$= \sum_n x(nM)\, e^{-j\omega n}$$
$$= \sum_n x'(nM)\, e^{-j\omega n}$$
$$= \sum_\ell x'(\ell)\, e^{-j\omega \frac{\ell}{M}} = X'(e^{j\omega})$$

2 $$X'(e^{j\omega}) = X(e^{j\omega}) * TF\{\sum \delta(n-nM)\}$$
$$= X(e^{j\omega}) * \frac{2\pi}{M} \sum_{k=0}^{M-1} \delta\left(\omega - \frac{2\pi k}{M}\right)$$
$$= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\omega - \frac{2\pi k}{M}\right)}\right)$$

(1) (2) $\Rightarrow$ $$X_d(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\omega - \frac{2\pi k}{M}\right)}\right)$$

⚠ Aliasing

Graph on Blackboard.

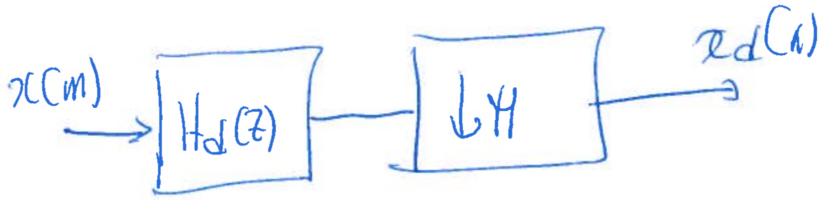
$\Rightarrow$ "Widening" of Spectrum in ω

⚠ Complexity $\to$ Discuss complexity of

$$x(n) \to \boxed{H(z)} \to$$

vs $$x_d(n) \to \boxed{H_d(z)} \to$$

4.2.2 On Aliasing

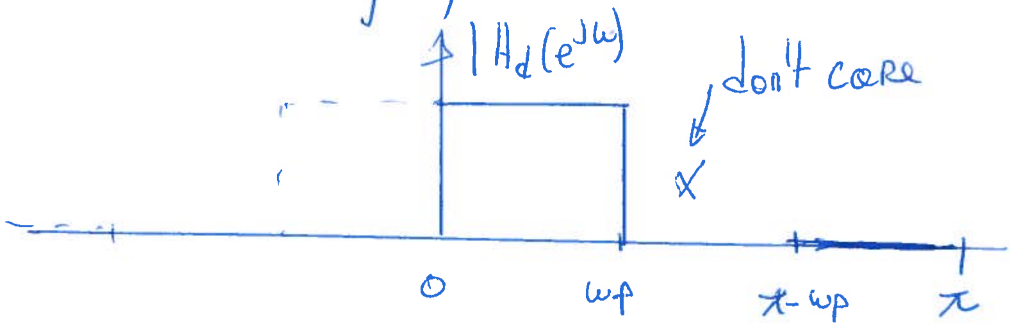


Anti aliasing Filter $H_d(z)$?

Let $\cancel{X}(z) = 0$ for $|\omega| > \omega_p$, $\omega_p < \frac{\pi}{M}$

$\rightarrow$ $H_d(e^{j\omega})$ can be relaxed to

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_p \\ 0, & |\omega| \in \left[\frac{2\pi k}{M} - \omega_p, \frac{2\pi k}{M} + \omega_p\right] \end{cases} \quad k = 1, 2, \dots M$$



$\rightarrow$ draw aliasing on black board

$\rightarrow$ show slide 21 - 23

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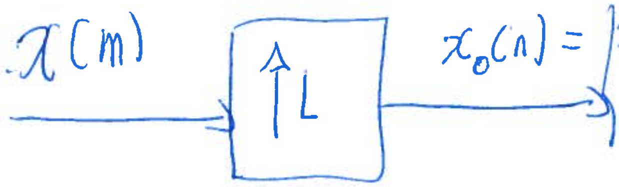
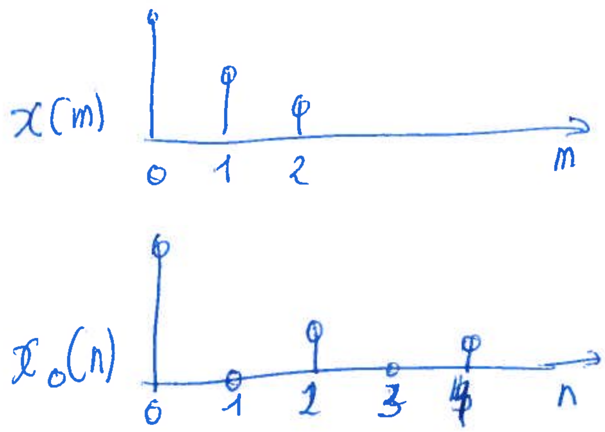
4.3 Oversampling

4.3.1 Basics



$$x_o(n) = \begin{cases} x\left(\frac{n}{L}\right) & n = kL,\ k \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$



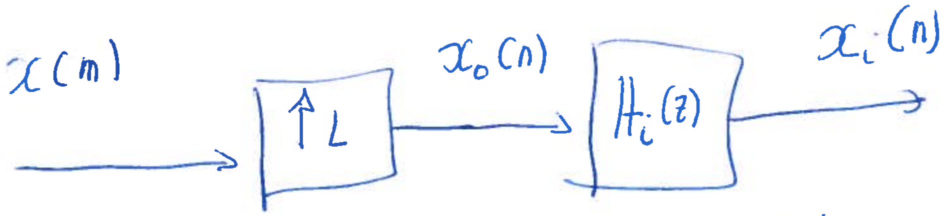
$L = 2$

$$X_o(e^{j\omega}) = X(e^{j\omega L})$$

straight forward, write $X_o(z)$ vs $X(z)$

"narrowing" of spectrum in ω

4.3.2 Interpolation

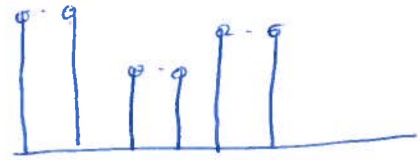


$$H_i(e^{j\omega}) = \begin{cases} L & |\omega| < \pi/L \\ 0 & \text{otherwise} \end{cases}$$

Discuss Complexity

Other interpolators

- Zero-hold



- Linear

4.4 Rational sampling-rate change

$$x(m) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow M} \longrightarrow x_c(n)$$

$H(z)$ $\begin{cases} \text{Interpolation filter} & \omega_c < \frac{\pi}{L} \\ \text{decimation filter} & \omega_c < \frac{\pi}{M} \end{cases}$

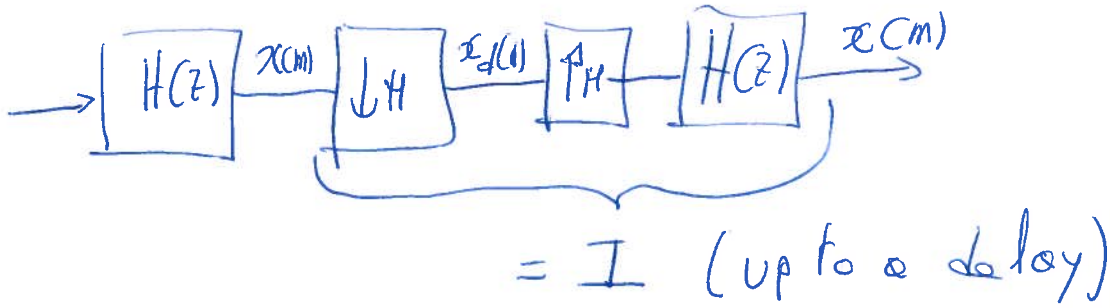
$$\Rightarrow H(z) = \begin{cases} L & |\omega| < \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) \\ 0 & \text{otherwise} \end{cases}$$

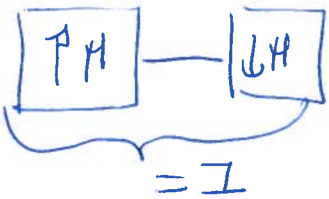
→ Relaxation Possible

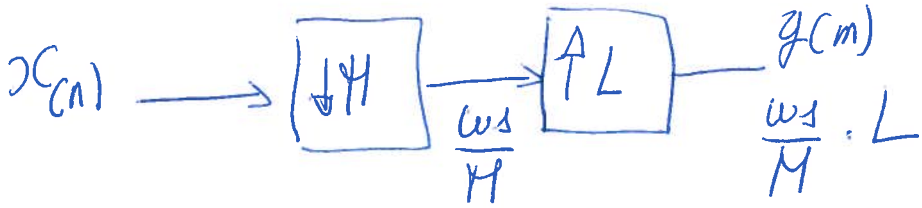
$$H(z) = L \qquad |\omega| < \min\left(\frac{\omega_p}{L}, \frac{\pi}{M}\right)$$

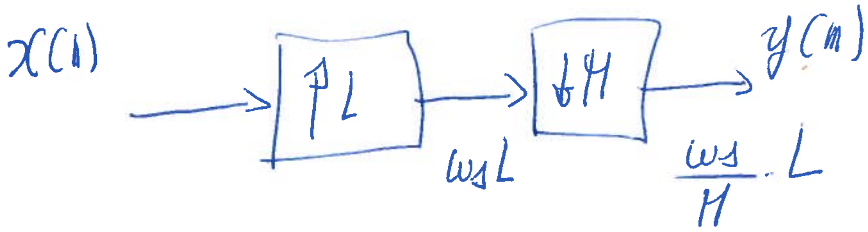
$$= 0 \qquad \min\left\{\frac{2\pi}{L} - \frac{\omega_p}{L}, \frac{2\pi}{M} - \frac{\omega_p}{L}\right\} \leq |\omega| \leq \pi$$

4.5 Inverse Operations









4.6 Noble Identities

$$x(m) \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow \;\equiv\; \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow$$

$$\rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow M} \rightarrow \;\equiv\; \boxed{\uparrow M} \rightarrow \boxed{H(z^M)} \rightarrow$$

4.7 Polyphase decomposition

We can write $H(z)$ as

$$H(z) = \sum h(k) z^{-k}$$

$$= \sum_{l} h(Ml) z^{-Ml} + \sum_{l} h(Ml+1) z^{-(Ml+1)} + \cdots + \sum_{l} h(Ml+M-1) z^{-Ml}$$

$$= \sum h(Ml) z^{-Ml} + z^{-1} \sum_{l} h(Ml+1) z^{-Ml} + \cdots + z^{-M+1} \sum_{l} h(Ml+M-1) z^{-M}$$

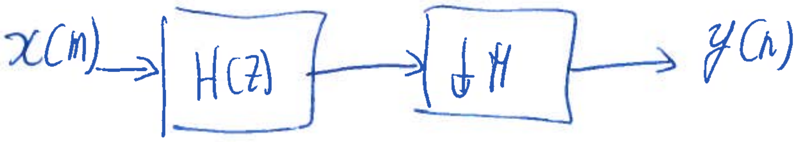
$$= \sum_{j=0}^{M-1} z^{-j} E_j(z^M)$$

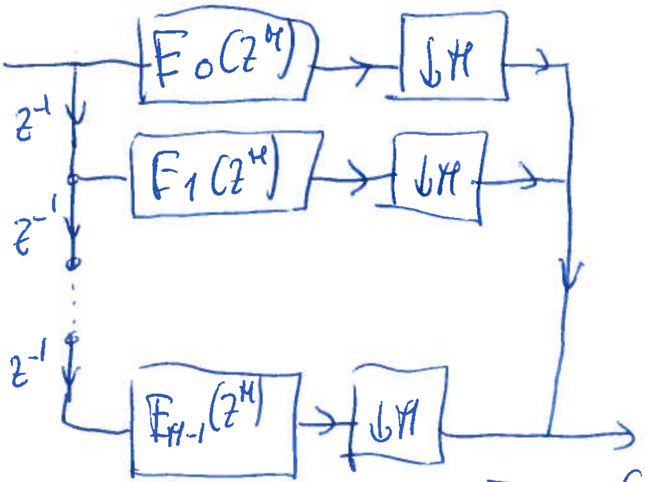
where $E_j(z^M) = \sum_{l=-\infty}^{+\infty} h(Ml+j) z^{-lM}$

= type I polyphase filter

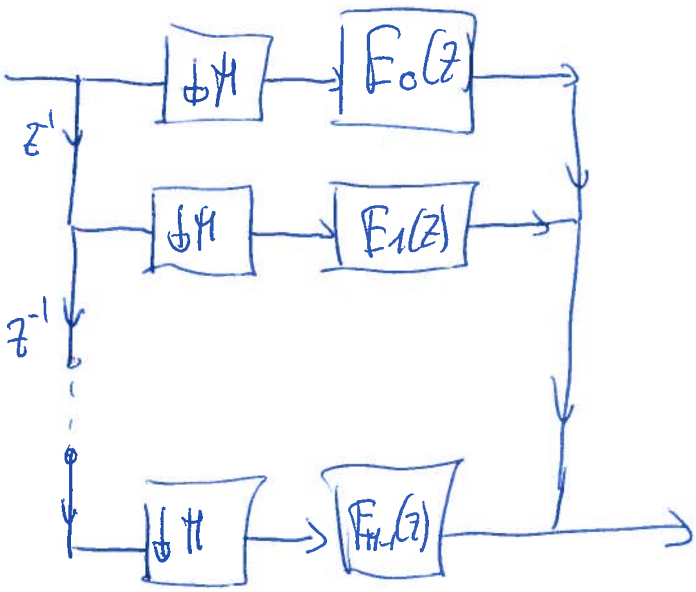
Filt 40

Decimation by Polyphase dec.





≡ (Noble)



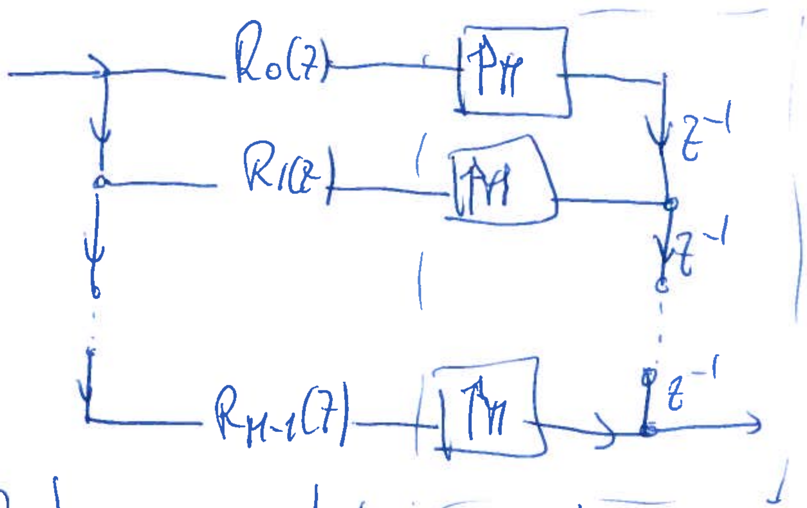
Polyphase Component of type II

$$R_j(z) = E_{M-1-j}(z)$$

$$\Rightarrow H(z) = \sum_{i=0}^{M-1} z^{-(M-1-j)} R_j(z^M)$$

Interpolation by Polyphase doc.

$$x(n) \rightarrow \boxed{\uparrow M} \rightarrow \boxed{G(z)} \rightarrow y(n)$$



Rotary switches