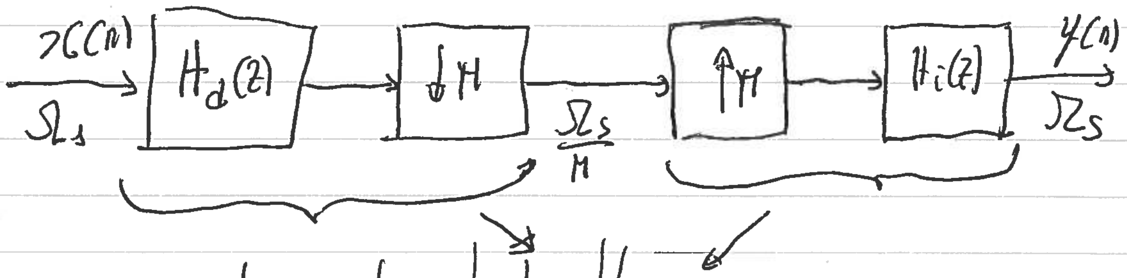
4.8 Efficient filter implementation

4.8.1 Narrow band filters

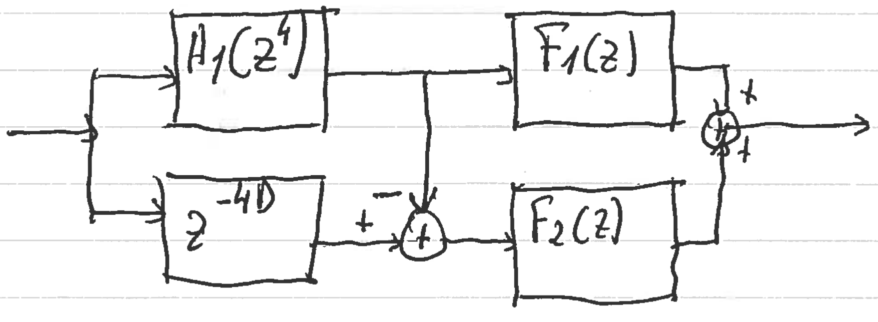


can be implemented with
+ Low complexity (see TD 5 for example)
+ Less stringent characteristics of the filter $H_{d\,polyphase}(z)$

Roughly, using one over M samples reduces complexity by $\frac{M}{2}$.

4.8.2. Wideband FIR filters

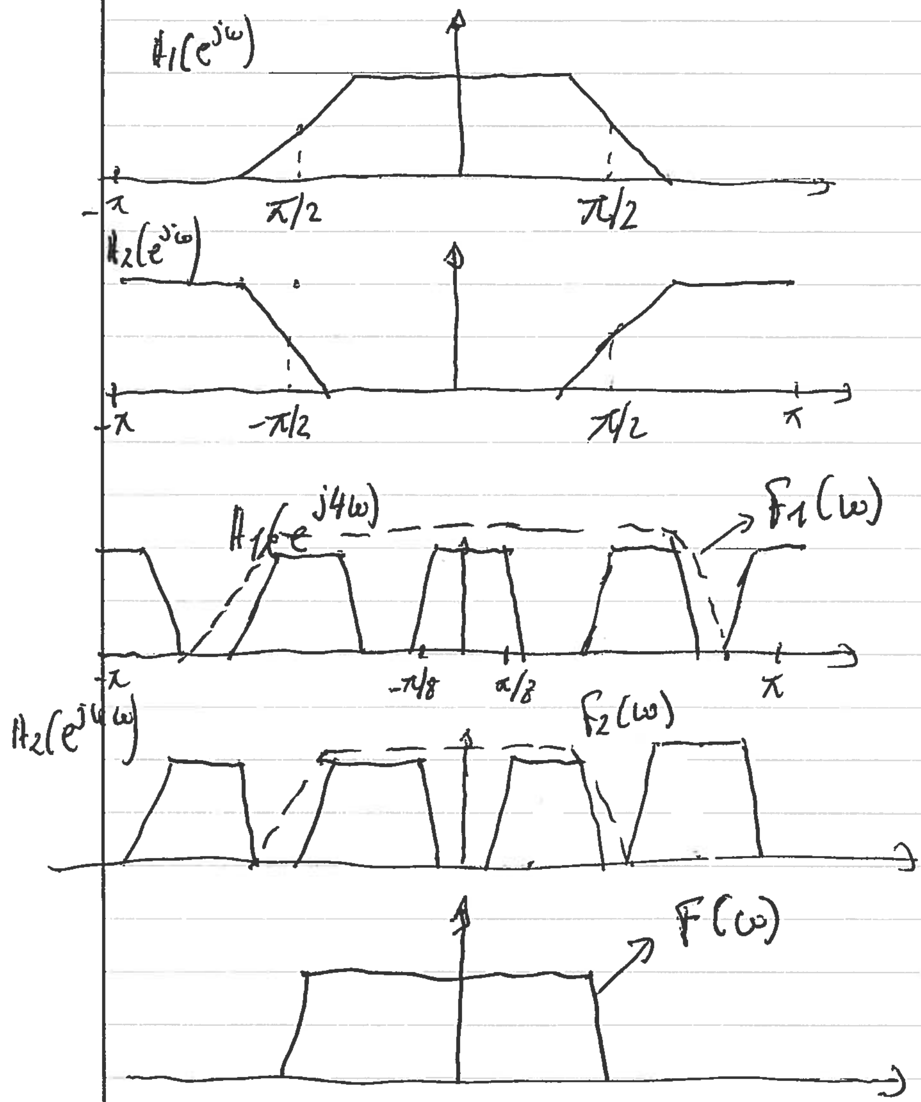
+ uses oversampling in the design of sharp cutoff filters
+ Example with L=4



objective → Lowpass with $\omega_p = 5\pi/8$
$\omega_s = 11\pi/16$

$\rightarrow \omega_s - \omega_p = \frac{\pi}{16}$

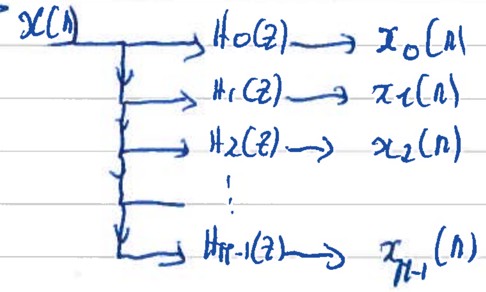
Filt 43


$H_1(e^{j\omega})$
-π
-π/2
π/2
$H_2(e^{j\omega})$
-π
-π/2
π/2
π
$H_1(e^{j4\omega})$
$F_1(\omega)$
-π
-π/8
π/8
π
$H_2(e^{j4\omega})$
$F_2(\omega)$
$F(\omega)$

5. Filter Banks

+ Split signal in several frequency bands
+ Usually → More samples than the original signal
+ → decompose into critically decimated freq. band
  → Recover with minimum or no error

Example

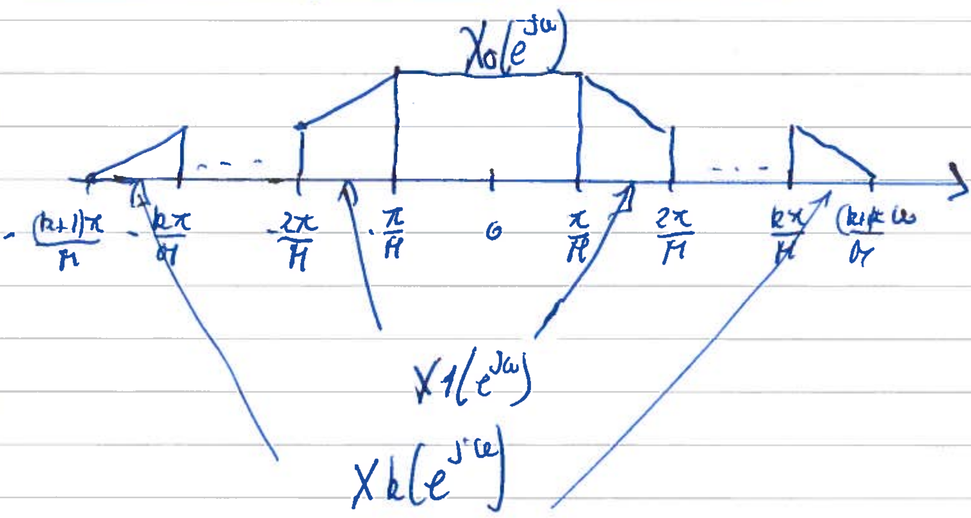


→ M x more samples

→ $H_i(z)$ → possibly narrow band (Multiplex)

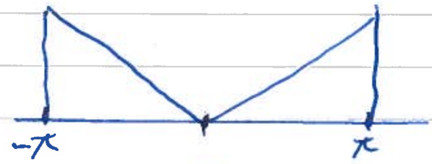
→ $X_i(z)$ → narrow band → can be decimated by M

5.1 Decimation of a bandpass signal

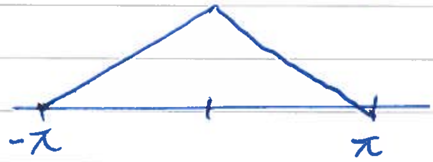


After decimation

k odd

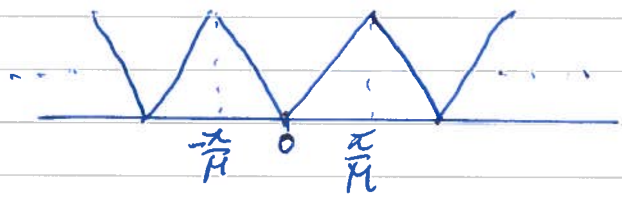


k even

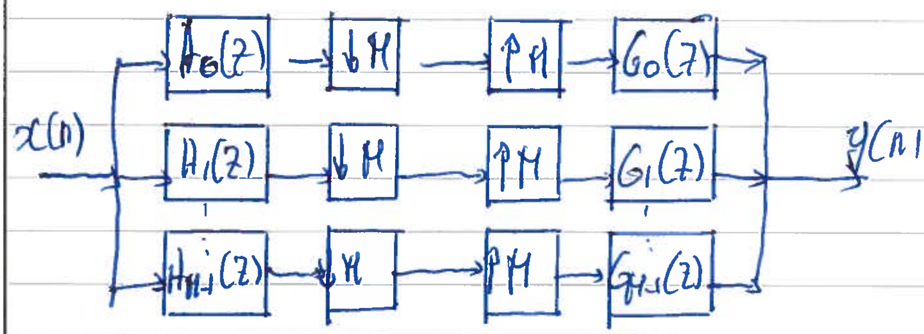


kth decimated Band upsampled

k odd



3.2 Perfect Reconstruction: Critically decimated M-band filter banks



Critically decimated M-band filter banks

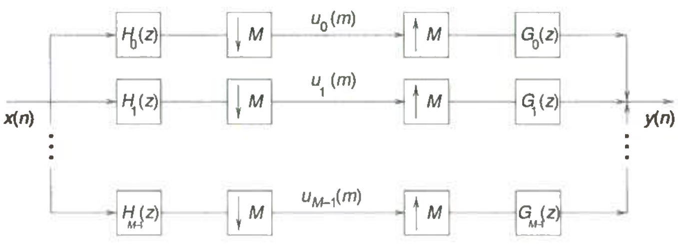


Figure 5: Block diagram of an M-band filter bank.

Critically decimated M-band filter banks

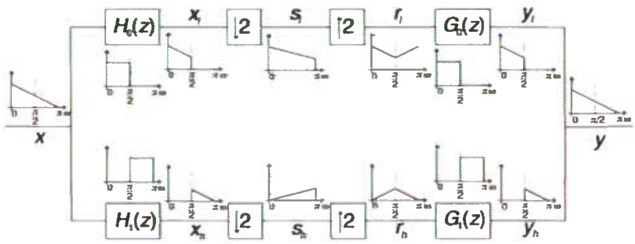


Figure 6: A 2-band perfect reconstruction filter bank using ideal filters.

Critically decimated M-band filter banks

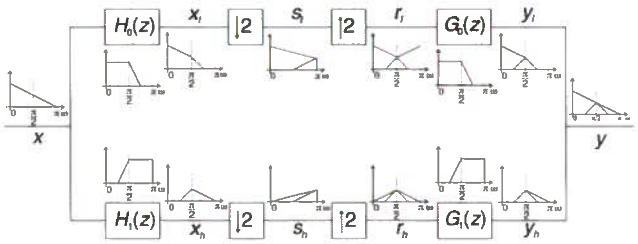


Figure 7: Two-band filter bank using realizable filters.

Critically decimated M-band filter banks

- The filters required for the M-band perfect reconstruction filter bank described above are not realizable.
- In a first analysis, the original signal would be only approximately recoverable from its decimated frequency bands.
- One can see that since the filters $H_0(z)$ and $H_1(z)$ are not ideal, the sub-bands $s_l(m)$ and $s_h(m)$ have aliasing.
- The signals $x_l(n)$ and $x_h(n)$ can not be correctly recovered from $s_l(m)$ and $s_h(m)$, respectively.

Critically decimated M-band filter banks

- Nevertheless, one can see that since $y_l(n)$ and $y_h(n)$ are added in order to obtain $y(n)$, the aliased components of $y_l(n)$ can be combined with the ones of $y_h(n)$.
- In principle, there is no reason why these aliased components could not be made to cancel each other, yielding $y(n)$ equal to $x(n)$. In such a case, the original signal could be recovered from its sub-band components.
- In an M-band filter bank as shown in Figure 5, the filters $H_k(z)$ and $G_k(z)$ are usually referred to as the analysis and synthesis filters of the filter bank.

Perfect reconstruction: M-band filter banks in terms of polyphase components

- Representing $H_k(z)$ and $G_k(z)$ by their polyphase components,

$$H_k(z) = \sum_{j=0}^{M-1} z^{-j} E_{kj}(z^M) \tag{1}$$

$$G_k(z) = \sum_{j=0}^{M-1} z^{-(M-1-j)} R_{jk}(z^M) \tag{2}$$

where $E_{kj}(z)$ is the jth polyphase component of $H_k(z)$, and $R_{jk}(z)$ is the jth polyphase component of $G_k(z)$.

Perfect reconstruction: M-band filter banks in terms of polyphase components

- Matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ entries are $E_{ij}(z)$ and $R_{ij}(z)$, for $i, j = 0, 1, \ldots, (M-1)$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \mathbf{E}(z^M) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{M-1}(z) \end{bmatrix} = \mathbf{R}^T(z^M) \begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix} \quad (4)$$

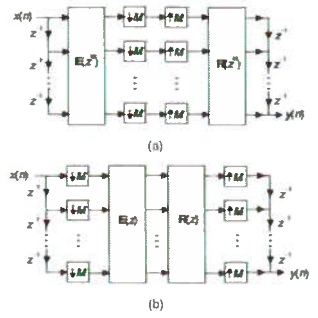


Figure 8: M-band filter bank in terms of the polyphase components: (a) before application of the noble identities; (b) after application of the noble identities.

M-band filter banks in terms of polyphase components

- In signal processing it is often advantageous to split a sequence $x(k)$ into several frequency bands prior to processing.

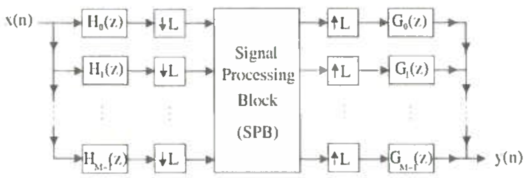


Figure 9: Signal processing in sub-bands.

M-band filter banks in terms of polyphase components

- Since the bandwidth of each analysis filter output is M times smaller than in the original signal, we can decimate each $x_i(k)$ by a factor of L smaller or equal to M and still avoid aliasing.
- For $L \leq M$, it is possible to retain all information contained in the input signal by properly designing the analysis filters in conjunction with the synthesis filters $G_i(z)$, for $i = 0, 1, \ldots, (M-1)$.
- If $L > M$ there is a loss of information due to aliasing which does not allow the recovery of the original signal.
- For $L = M$, we refer to the filter bank as maximally (or critically) decimated.
- For $L < M$, the filter bank is called oversampled (or noncritically sampled) since the set of sub-bands comprises more samples than the input signal.

Perfect reconstruction M-band filter banks

- If $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, the M-band filter bank becomes that one shown in Figure 10.

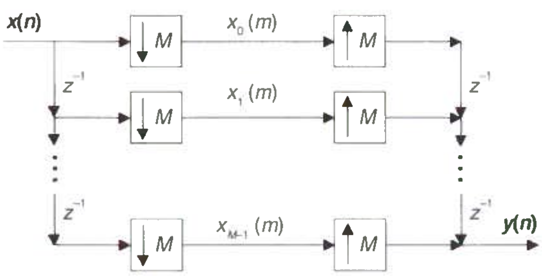


Figure 10: M-band filter bank when $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$.

Perfect reconstruction M-band filter banks

- By substituting the decimators and interpolators by the commutator models we arrive at the scheme which is clearly equivalent to a pure delay.
- Therefore, the condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ guarantees perfect reconstruction for the M-band filter bank.
- If $\mathbf{R}(z)\mathbf{E}(z)$ is equal to a pure delay one can still consider that perfect reconstruction holds.
- The weaker condition is sufficient for perfect reconstruction.

$$\mathbf{R}(z)\mathbf{E}(z) = z^{-\Delta}\mathbf{I} \quad (5)$$

- The total delay introduced by a perfect reconstruction filter bank is

$$\Delta_{\text{total}} = M\Delta + M - 1 \quad (6)$$

where $M\Delta$ is the delay originated from the polyphase matrices product and the term $(M-1)$ accounts for the delay introduced by the commutator.

Perfect reconstruction M-band filter banks

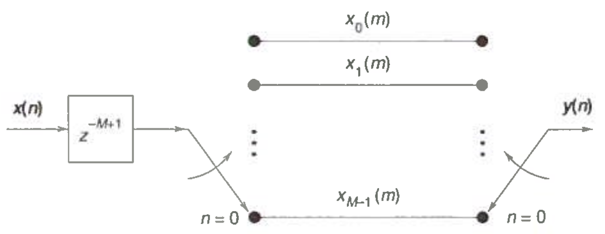


Figure 11: The commutator model of an M-band filter bank when $R(z)E(z) = I$ is equivalent to a pure delay.

Perfect reconstruction M-band filter banks

How a simple perfect reconstruction filter bank can be built?

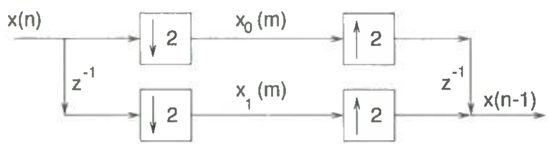


Figure 12: 2-band unit delay.

Perfect reconstruction M-band filter banks

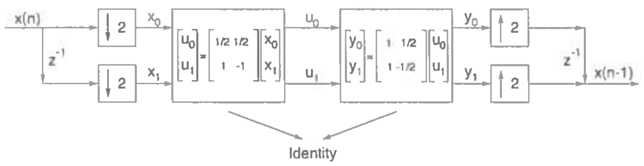


Figure 13: 2-band unit delay, including inverse matrices.

Perfect reconstruction M-band filter banks

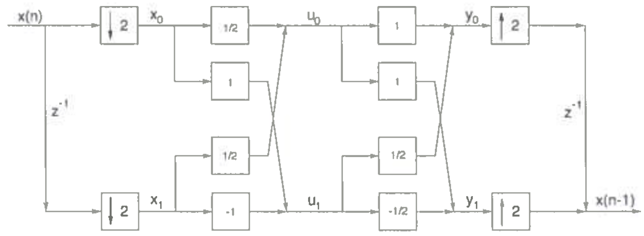


Figure 14: 2-band unit delay, with explicit realization of the matrix products.

Perfect reconstruction M-band filter banks

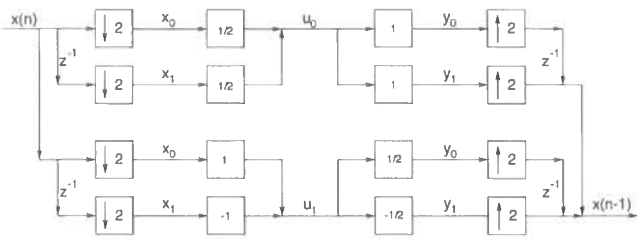


Figure 15: 2-band unit delay, splitting the decimators and interpolators.

Perfect reconstruction M-band filter banks

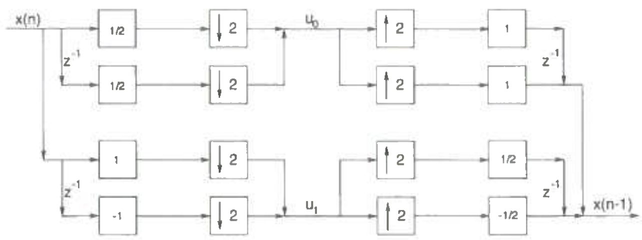


Figure 16: 2-band unit delay, moving the decimators and interpolators.

Perfect reconstruction M-band filter banks

By "merging" the decimators/interpolators we reach the realization of the unit delay as shown in Figure 17. Figure 17 is equivalent to the filter bank in Figure 18.

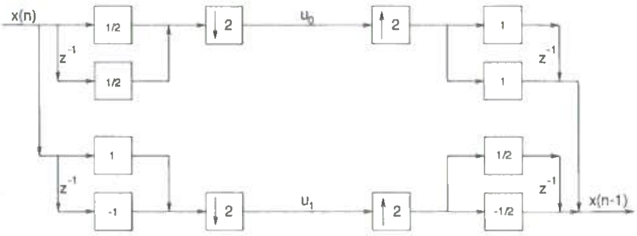


Figure 17: 2-band unit delay, merging the decimators and interpolators.

Perfect reconstruction M-band filter banks

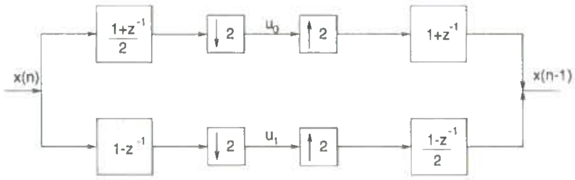


Figure 18: 2-band filter bank with perfect reconstruction.

M-band filter banks in terms of polyphase components

Example 9.1

Let $M = 2$, and

$$\mathbf{E}(z) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \tag{7}$$

$$\mathbf{R}(z) = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \tag{8}$$

Show that these matrices characterize a perfect reconstruction filter bank, and find the analysis and synthesis filters and their corresponding polyphase components.

M-band filter banks in terms of polyphase components

Solution

- Clearly $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, and the filter bank yields perfect reconstruction. The polyphase components $E_{kj}(z)$ of the analysis filters $H_k(z)$, and $R_{jk}(z)$ of the synthesis filters $G_k(z)$ are then

$$E_{00}(z) = \frac{1}{2}, \quad E_{01}(z) = \frac{1}{2}, \quad E_{10}(z) = 1, \quad E_{11}(z) = -1 \tag{9}$$

$$R_{00}(z) = 1, \quad R_{01}(z) = \frac{1}{2}, \quad R_{10}(z) = 1, \quad R_{11}(z) = -\frac{1}{2} \tag{10}$$

M-band filter banks in terms of polyphase components

- We can find $H_k(z)$, and $G_k(z)$.

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) \tag{11}$$

$$H_1(z) = 1 - z^{-1} \tag{12}$$

$$G_0(z) = 1 + z^{-1} \tag{13}$$

$$G_1(z) = -\frac{1}{2}(1 - z^{-1}) \tag{14}$$

- The magnitude response of $G_k(z)$ is equal to the one of $H_k(z)$ for $k = 0, 1$ except for a gain constant.
- Perfect reconstruction could be achieved with filters that are far from being ideal.
- Each sub-band is highly aliased, still we recover the original signal exactly. △

Perfect reconstruction M-band filter banks

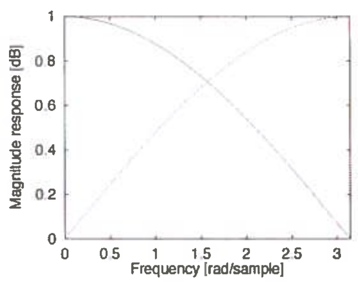


Figure 19: Magnitude responses of the filters described by equations (11) and (12): $H_0(z)$ (solid line); $H_1(z)$ (dashed line).

Perfect reconstruction M-band filter banks

Example 9.2

- Repeat Example 33 for the case when

$$\mathbf{E}(z) = \begin{bmatrix} \left(-\frac{1}{8}+\frac{3}{4}z^{-1}-\frac{1}{8}z^{-2}\right) & \left(\frac{1}{4}+\frac{1}{4}z^{-1}\right) \\ \left(\frac{1}{2}+\frac{1}{2}z^{-1}\right) & -1 \end{bmatrix} \quad (15)$$

$$\mathbf{R}(z) = \begin{bmatrix} 1 & \left(\frac{1}{4}+\frac{1}{4}z^{-1}\right) \\ \left(\frac{1}{2}+\frac{1}{2}z^{-1}\right) & \left(\frac{1}{8}-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right) \end{bmatrix} \quad (16)$$

M-band filter banks in terms of polyphase components

Solution Since

- $$\mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} = z^{-1}\mathbf{I} \quad (17)$$

then the filter bank has perfect reconstruction.

M-band filter banks in terms of polyphase components

- From equations (15) and (16), the polyphase components, $E_{kj}(z)$, of the analysis filters $H_k(z)$, and $R_{jk}(z)$ of the synthesis filters $G_k(z)$ are

$$\left.\begin{aligned} E_{00}(z) &= -\frac{1}{8}+\frac{3}{4}z^{-1}-\frac{1}{8}z^{-2} \\ E_{01}(z) &= \frac{1}{4}+\frac{1}{4}z^{-1} \\ E_{10}(z) &= \frac{1}{2}+\frac{1}{2}z^{-1} \\ E_{11}(z) &= -1 \end{aligned}\right\} \quad (18)$$

M-band filter banks in terms of polyphase components

-

$$\left.\begin{aligned} R_{00}(z) &= 1 \\ R_{01}(z) &= \frac{1}{4}+\frac{1}{4}z^{-1} \\ R_{10}(z) &= \frac{1}{2}+\frac{1}{2}z^{-1} \\ R_{11}(z) &= \frac{1}{8}-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2} \end{aligned}\right\} \quad (19)$$

M-band filter banks in terms of polyphase components

- From equations (18) and (1), we can find $H_k(z)$, and, from equations (19) and (2), we can find $G_k(z)$. They are

$$H_0(z) = -\frac{1}{8}+\frac{1}{4}z^{-1}+\frac{3}{4}z^{-2}+\frac{1}{4}z^{-3}-\frac{1}{8}z^{-4} \quad (20)$$

$$H_1(z) = \frac{1}{2}-z^{-1}+\frac{1}{2}z^{-2} \quad (21)$$

$$G_0(z) = \frac{1}{2}+z^{-1}+\frac{1}{2}z^{-2} \quad (22)$$

$$G_1(z) = \frac{1}{8}+\frac{1}{4}z^{-1}-\frac{3}{4}z^{-2}+\frac{1}{4}z^{-3}+\frac{1}{8}z^{-4} \quad (23)$$

The magnitude responses of the analysis filters are depicted in Figure 20.

△

Perfect reconstruction M-band filter banks

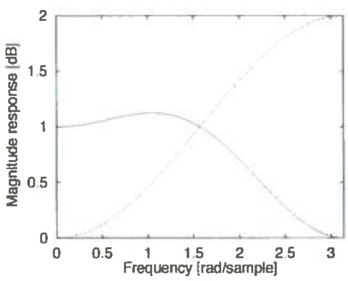


Figure 20: Magnitude responses of the filters described by equations (20) and (21): $H_0(z)$ (solid line); $H_1(z)$ (dashed line).

Perfect reconstruction M-band filter banks

Example 9.3 If the analysis filters of a perfect reconstruction filter bank are given by

$$\left.\begin{aligned} H_0(z) &= 1 + z^{-1} + \tfrac{1}{2}z^{-2} \\ H_1(z) &= 1 - z^{-1} + \tfrac{1}{2}z^{-2} \end{aligned}\right\} \quad (24)$$

determine its synthesis filters.

Solution

- From equation (24), we can write the lowpass and highpass analysis filters as

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1+\frac{1}{2}z^{-2} & 1 \\ 1+\frac{1}{2}z^{-2} & -1 \end{bmatrix}}_{E(z^2)} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \quad (25)$$

Perfect reconstruction M-band filter banks

- Then the polyphase analysis matrix is

$$E(z) = \begin{bmatrix} 1+\frac{1}{2}z^{-1} & 1 \\ 1+\frac{1}{2}z^{-1} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+\frac{1}{2}z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \quad (26)$$

Perfect reconstruction M-band filter banks

- Therefore, since the filter has perfect reconstruction, we must have that $R(z) = z^{-\Delta}E^{-1}(z)$, which gives

$$R(z) = z^{-\Delta} \begin{bmatrix} \frac{1}{1+\frac{1}{2}z^{-1}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{z^{-\Delta}}{2} \begin{bmatrix} \frac{1}{1+\frac{1}{2}z^{-1}} & \frac{1}{1+\frac{1}{2}z^{-1}} \\ 1 & -1 \end{bmatrix} \quad (27)$$

Perfect reconstruction M-band filter banks

- Then, from equation (4), the synthesis filters are

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = R^T(z) \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} = \frac{z^{-\Delta}}{2} \begin{bmatrix} \frac{1}{1+\frac{1}{2}z^{-2}} & 1 \\ \frac{1}{1+\frac{1}{2}z^{-2}} & -1 \end{bmatrix} \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} \quad (28)$$

that is,

$$\left.\begin{aligned} G_0(z) &= z^{-\Delta}\frac{1+z^{-1}+\frac{1}{2}z^{-2}}{2+z^{-2}} \\ G_1(z) &= z^{-\Delta}\frac{-1+z^{-1}-\frac{1}{2}z^{-2}}{2+z^{-2}} \end{aligned}\right\} \quad (29)$$

Note that the FIR solution is not possible. △

Perfect reconstruction M-band filter banks

Example 9.3 Assume the analysis filters of a 3-band perfect reconstruction filter bank are given by

$$\left.\begin{aligned} H_0(z) &= z^{-2} + 6z^{-1} + 4 \\ H_1(z) &= z^{-1} + 2 \\ H_2(z) &= 1 \end{aligned}\right\} \quad (30)$$

Determine its synthesis filters.

Perfect reconstruction M-band filter banks

Solution

- From equations (1) and (3), the polyphase description of the given analysis filters is

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E(z^3)} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

- Then

$$E(z^3) = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (31)$$

Perfect reconstruction M-band filter banks

- We can have perfect reconstruction if $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$. Thus,

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -6 & 8 \end{bmatrix} \quad (32)$$

- From equation (4), it follows that

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \mathbf{R}^T(z^3) \begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -2 & 8 \end{bmatrix} \begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix} \quad (33)$$

Perfect reconstruction M-band filter banks

- Therefore, the transfer functions of the synthesis sub filters are given by

$$\left.\begin{aligned} G_0(z) &= 1 \\ G_1(z) &= z^{-1} - 6 \\ G_2(z) &= z^{-2} - 2z^{-1} + 8 \end{aligned}\right\} \quad (34)$$

We note that the synthesis filters are all FIR. This is only possible because the determinant of the polyphase matrix of the analysis filter is proportional to a pure delay (equal to 1 in this case).

△

Analysis of M-band filter banks

- The "signal processing block" could consist, for example, of quantization, filtering, or other types of signal transformations.

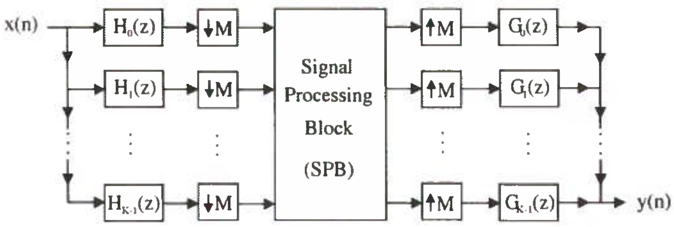


Figure 21: M-band filter bank.

Modulation matrix representation

Using this concept, the analysis of the M-band filter bank can be performed in three different, but equivalent ways:

- Using the polyphase decomposition: When the polyphase decomposition is used in the analysis and synthesis banks, the resulting polyphase matrices are very useful to establish design conditions for perfect reconstruction filter banks.
- Using the modulation matrix representation: By representing the sub-band signals in the frequency domain, it is possible to describe the input-output relation of a filter bank. This representation leads to the so called modulation matrix representation. It is particularly effective in exposing the aliasing effects generated by the decimators. Although this formulation is useful to design alias-free filter banks, it is not the easiest formulation for design purposes.

Modulation matrix representation

- Using time-domain analysis: It represents the input-output relation of the filter banks in the time domain in terms of the impulse responses of the analysis and synthesis sub-filters.
  - It is very effective in exposing the properties of perfect reconstruction filter banks as defining bases of vector spaces.
  - The analysis operations are seen as signal projections onto bases, while the synthesis operations are seen as signal expansions using bases of the vector space.
  - Properties such as orthogonality and biorthogonality come into play and provide useful insights on filter bank analysis and design.

Modulation matrix representation

- Noting that, the decimated signal $X_d(z)$ is the sum of $X(z^{\frac{1}{M}})$ and its $(M-1)$ aliased components, $X(z^{\frac{1}{M}} e^{-j\frac{2\pi}{M}k})$, for $k = 1, 2, \ldots, (M-1)$, that is

$$X_d(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{\frac{1}{M}} e^{-j\frac{2\pi}{M}k}) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{\frac{1}{M}} W_M^k) \quad (35)$$

where $W_M = e^{-j\frac{2\pi}{M}}$.

Modulation matrix representation

- Using the above equation we can express the decimated output of the analysis filters in Figure 5, $U_k(z)$, for $k = 0, 1, \ldots, (M-1)$, as

$$\begin{aligned} U_k(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}}W_M^k)H_k(z^{\frac{1}{M}}W_M^k) \\ &= \frac{1}{M}\mathbf{x}_m^T(z^{\frac{1}{M}})\begin{bmatrix} H_k(z^{\frac{1}{M}}) \\ H_k(z^{\frac{1}{M}}W_M) \\ \vdots \\ H_k(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix} \end{aligned} \quad (36)$$

where

$$\mathbf{x}_m(z^{\frac{1}{M}}) = \begin{bmatrix} X(z^{\frac{1}{M}}) & X(z^{\frac{1}{M}}W_M) & \ldots & X(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix}^T \quad (37)$$

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Modulation matrix representation

- Therefore, we can define the auxiliary vector

$$\mathbf{U}^T(z) = \begin{bmatrix} U_0(z) & U_1(z) & \ldots & U_{M-1}(z) \end{bmatrix} = \frac{1}{M}\mathbf{x}_m^T(z^{\frac{1}{M}})\mathbf{H}_m(z^{\frac{1}{M}}) \quad (38)$$

- With

$$\mathbf{H}_m(z^{\frac{1}{M}}) = \begin{bmatrix} H_0(z^{\frac{1}{M}}) & H_1(z^{\frac{1}{M}}) & \ldots & H_{M-1}(z^{\frac{1}{M}}) \\ H_0(z^{\frac{1}{M}}W_M) & H_1(z^{\frac{1}{M}}W_M) & \ldots & H_{M-1}(z^{\frac{1}{M}}W_M) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(z^{\frac{1}{M}}W_M^{M-1}) & H_1(z^{\frac{1}{M}}W_M^{M-1}) & \ldots & H_{M-1}(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix} \quad (39)$$

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Modulation matrix representation

- Applying the noble identities, we see that the filter bank output as a function of the sub-bands $U_k(z)$ is given by

$$Y(z) = \sum_{k=0}^{M-1} U_k(z^M)G_k(z) = \mathbf{U}^T(z^M)\mathbf{g}(z) \quad (40)$$

- Where

$$\mathbf{g}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \ldots & G_{M-1}(z) \end{bmatrix}^T \quad (41)$$

Then, from equations (38) and (40), we can express the input-output relation of an M-band filter bank as

$$Y(z) = \frac{1}{M}\mathbf{x}_m^T(z)\mathbf{H}_m(z)\mathbf{g}(z) \quad (42)$$

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Modulation matrix representation

- Since $Y(z)$ above is a scalar, we have that $\mathbf{x}_m^T(z)\mathbf{H}_m(z)\mathbf{g}(z) = \mathbf{g}^T(z)\mathbf{H}_m^T(z)\mathbf{x}_m(z)$, and then

$$\begin{aligned} Y(z) &= \frac{1}{M}\mathbf{g}^T(z)\mathbf{H}_m^T(z)\mathbf{x}_m(z) \\ &= \frac{1}{M}\begin{bmatrix} G_0(z) & G_1(z) & \ldots & G_{M-1}(z) \end{bmatrix} \\ &\times \begin{bmatrix} H_0(z) & H_0(zW_M) & \ldots & H_0(zW_M^{M-1}) \\ H_1(z) & H_1(zW_M) & \ldots & H_1(zW_M^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1}(z) & H_{M-1}(zW_M) & \ldots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}\begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix} \end{aligned} \quad (4$$

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Modulation matrix representation

- In equation (43), often referred to as the modulation matrix representation of the filter bank, if

$$\mathbf{g}^T(z)\mathbf{H}_m^T(z) = \begin{bmatrix} B(z) & 0 & \ldots & 0 \end{bmatrix} \quad (44)$$

- Then aliasing is canceled, since

$$Y(z) = \frac{1}{M}\begin{bmatrix} B(z) & 0 & \ldots & 0 \end{bmatrix}\begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix} = \frac{1}{M}B(z)X(z) \quad (45)$$

- It can also be inferred that if $B(z) = Mcz^{-\Delta}$, then the output of the filter bank is just a delayed version of the input scaled by a constant c, that is, the filter bank has perfect reconstruction.

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Modulation matrix representation

Example 9.5 Find the perfect reconstruction conditions for all 2-band filter banks using the modulation matrix approach.

Solution

- For the 2-band case, perfect reconstruction requires, since $W_2 = -1$, that

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix}\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \begin{bmatrix} 2cz^{-\Delta} & 0 \end{bmatrix} \quad (46)$$

which implies

$$\left.\begin{aligned} H_0(z)G_0(z) + H_1(z)G_1(z) &= 2cz^{-\Delta} \\ H_0(-z)G_0(z) + H_1(-z)G_1(z) &= 0 \end{aligned}\right\} \quad (47)$$

Then the output of the filter bank to be equal to the input delayed by Δ and scaled by a constant c. $\triangle$