Filtrage Numérique

Syllabus

1. Background Review: Frequency response, phase, group delay, minimum phase and all pass systems.
2. Sampling / Sampling rate conversion
3. Quantization and Oversampled Noise Shaping
4. IIR / FIR filter structures
5. Filter design: FIR filters
6. Raised Cosine filters - Matched filters
7. Multirate Systems and Polyphase Structu[res]
8. Modulated Filter Banks

Evaluation

3 Quizzes → each 10%
1 Homework 40%
1 Final test 30%

Reference Book

Discrete-Time Signal Processing, A. Oppenheim, R.W. Schaffer
J.R. Buck - Prentice Hall, 1999

I. Background Review

I.1 Frequency Response of Linear Time Invariant (L.T.I) systems

L.T.I. : $y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$

z-transform : $Y(z) = H(z) \cdot X(z)$

Fourier transform : $Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$

$$\Rightarrow \begin{cases} |Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})| \\ \measuredangle Y(e^{j\omega}) = \measuredangle H(e^{j\omega}) + \measuredangle X(e^{j\omega}) \end{cases}$$

$|H(e^{j\omega})|$: magnitude of the response (= gain of the sy.

$\measuredangle H(e^{j\omega})$: phase response (= phase shift)

a) Ideal Frequency-Selective Filters

Low-Pass $\quad H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < \infty$$

High-Pass $\quad H_{hp}(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c < |\omega| \leq \pi \end{cases}$

$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega}) \Rightarrow h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}$

Non computationally realizable !

b) Phase Distortion and Delay

ideal delay $\quad h_{id}[n] = \delta[n - n_d]$

$$\Rightarrow H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$\Leftrightarrow$$

$$|H_{id}(e^{j\omega})| = 1 \quad , \quad \measuredangle H_{id}(e^{j\omega}) = -\omega n_d \quad |\omega| < \pi,$$

linear phase distortion accepted

Use for ideal filter

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d} & , \ |\omega| < \omega_c \\ 0 & , \ \omega_c < |\omega| < \pi \end{cases}$$

$$h_{lp}[n] = \frac{\sin \omega_c(n - n_d)}{\pi(n - n_d)} \qquad -\infty < n < \infty$$

→ discuss Truncation.

Phase Wrapping

$$H(e^{j\omega}) = e^{-2j\omega}$$

$$\measuredangle H(e^{j\omega}) = -2\omega$$

→ draw on blackboard

c) Group delay

→ Effect of the phase on a narrow band system

Suppose $x[n] = s[n] \cos(\omega_0 n)$, and $X(e^{j\omega}) \neq 0$ around $\omega = \omega_0$

$\Rightarrow \measuredangle H(e^{j\omega}) \simeq -\phi_0 - \omega n_d$ (first order approximation)

$\Rightarrow x[n] \simeq |H(e^{j\omega_0})| \, s[n-n_d] \cdot \cos(\omega_0 n - \phi_0 - \omega_0 n_d)$

! if $H(e^{j\omega})$ continuous in ω

group delay

$$\tau(\omega) = \text{grd}(H(e^{j\omega})) = -\frac{d}{d\omega}\left[\arg(H(e^{j\omega}))\right]$$

cas idéal → $\tau(\omega)$ constant

if $\tau(\omega)$ non constant → NON LINEARITY of the phase

Filt. 5

In general, phase is ambiguous since $e^{j(\theta+2\pi n)} = e^{j\theta}$ for any integer n. OSB Figure 5.7 shows the continuous phase (denoted as arg) and the wrapped phase (denoted as ARG) of an LTI system.

Group Delay

Typically, it is hard to infer much from a phase plot, and a group delay plot gives more useful information.

$$\tau(\omega) = grd[H(e^{j\omega})] \equiv -\frac{d}{d\omega}[\arg H(e^{j\omega})] = -\frac{d}{d\omega}[ARG\ H(e^{j\omega})]$$

The last equality holds except at discontinuities of $[ARG\ H(e^{j\omega})]$.

In MATLAB, group delay is calculated using the Fourier transform rather than differentiation. In the amplitude/phase representation:

$$H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)} \tag{1}$$

↑ real, can be negative!

Differentiating both sides,

$$H'(e^{j\omega}) = A'(\omega)e^{j\theta(\omega)} + A(\omega)e^{j\theta(\omega)}(j\theta'(\omega))$$

and dividing by Equation 1,

$$\frac{H'(e^{j\omega})}{H(e^{j\omega})} = \frac{A'(\omega)}{A(\omega)} + j\theta'(\omega)$$

Since both $\frac{A'(\omega)}{A(\omega)}$ and $\theta'(\omega)$ are real,

$$\tau(\omega) = -Im[\frac{H'(e^{j\omega})}{H(e^{j\omega})}].$$

If $h[n] \leftrightarrow H(e^{j\omega})$, we can apply the frequency differentiation property of the Fourier transform and get

$$-jnh[n] \leftrightarrow H'(e^{j\omega}).$$

Denoting Fourier transform of $h[n]$ as $F.T.(h[n])$,

$$\tau(\omega) = -Im[\frac{F.T.(-jnh[n])}{F.T.(h[n])}]$$

and finally,

$$\tau(\omega) = Re[\frac{F.T.(nh[n])}{F.T.(h[n])}].$$

Fil. 6

Systems with Linear Phase

Systems with constant group delay are referred to as linear phase systems. These systems are desirable when we want to minimize the distortion on the shape of a signal. The three systems given below are linear phase systems.

System 1: $h_1[n]$ is symmetric about zero.

$$h_1[n] = h_1[-n]$$

$H_1(e^{j\omega})$ is real, and phase is zero in the amplitude/phase representation. Thus, the group delay is zero, and $h_1[n]$ is linear phase.

System 2: $h_2[n]$ is symmetric about an integer n_0, i.e. $h_1[n]$ delayed by n_0.

$$h_2[n] = h_1[n - n_0]$$

$$H_2(e^{j\omega}) = e^{-j\omega n_0} H_1(e^{j\omega})$$

$H_1(e^{j\omega})$ is real, so the group delay of $H_2(e^{j\omega})$ is n_0, and $h_2[n]$ is also linear phase.

Since $h_1[n]$ is even,

$$h_2[n + n_0] = h_1[n] = h_1[-n] = h_2[-n + n_0],$$

and $h_2[n]$ satisfies the following:

$$h_2[n] = h_2[2n_0 - n]$$

System 3: $h_3[n] = h_3[2\alpha - n]$, where 2α is an integer.

$$H_3(e^{j\omega}) = H_3^*(e^{j\omega})e^{-j2\alpha\omega},$$

since $h_3[n] = h_3[-(n - 2\alpha)]$. Multiplying both sides by $e^{j\omega\alpha}$,

$$e^{j\omega\alpha}H_3(e^{j\omega}) = H_3^*(e^{j\omega})e^{-j\omega\alpha} = \overline{e^{j\omega\alpha}H_3(e^{j\omega})}.$$

This implies that both sides must be real, so $H_3(e^{j\omega})$ can be expressed as

$$H_3(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha},$$

where $A(e^{j\omega})$ is real (possibly bipolar). Thus, the phase response of $H_3(e^{j\omega})$ is linear, and the group delay is constant.

Filt 7

Since $h[n]$ is only defined at integer values, we need 2α to be an integer to represent $h[2\alpha - n]$. If α is an integer, then System 3 corresponds to System 2 and $h[n]$ is symmetric about α (see OSB Figure 5.35 (a)). If 2α is an integer but α is not an integer, the point of symmetry lies between samples (see OSB Figure 5.35 (b)). When 2α is not an integer, $h[n]$ is not symmetric, but the envelop is symmetric (see OSB Figure 5.35 (c)).

Generalized Linear Phase

Consider a system which is antisymmetric about zero:

$$h[n] = -h[n]$$

This implies that $H(e^{j\omega})$ is purely imaginary, therefore in the amplitude/phase representation,

$$H(e^{j\omega}) = jI(\omega) = I(\omega)e^{j\pi/2},$$

where $I(\omega)$ is real. Thus, we see that the phase of $H(e^{j\omega})$ is $\pi/2$, and the group delay is zero.

If the system has an odd symmetric impulse response

$$h[n] = -h[2\alpha - n]$$

where 2α is an integer, then it follows that

$$H(e^{j\omega}) = -H^*(e^{j\omega})e^{-j2\alpha\omega}.$$

Multiplying both sides by $e^{j\omega\alpha}$,

$$e^{j\omega\alpha}H(e^{j\omega}) = -H^*(e^{j\omega})e^{-j\omega\alpha} = -\overline{e^{j\omega\alpha}H(e^{j\omega})},$$

which implies that both sides must be purely imaginary, so

$$e^{j\omega\alpha}H(e^{j\omega}) = jI(\omega)$$

where $I(\omega)$ is real. Thus, $H(e^{j\omega})$ has the form

$$H(e^{j\omega}) = I(\omega)e^{j\pi/2}e^{-j\omega\alpha}.$$

Therefore, we see that the phase of $H(e^{j\omega})$ consists of a constant term added to a linear function, and the group delay of the system is α, a constant.

Systems with linear phase in this sense are called generalized linear phase.

I.2 All Pass Systems

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$

$$H_{ap}(e^{j\omega}) = e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - a e^{-j\omega}}$$

$$\left(|e^{-j\omega}| = 1 \quad 1 - a^* e^{j\omega} = \operatorname{conj}(1 - a e^{-j\omega}) \right.$$
$$\Rightarrow \left| \frac{1 - a^* e^{j\omega}}{1 - a e^{-j\omega}} \right| = 1$$

general real-valued all pass: $H_{ap}(z) = A \cdot \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$

d_k: Real poles

e_k: complex poles

→ sketch a zero-pole graph.

I.3 Minimum-Phase Systems

$H(z)$ stable and causal → poles in unit circle

$\frac{1}{H(z)}$ stable and causal → zeros of $H(z)$ in unit circle.

Ex: design of filter with

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) H^*(e^{j\omega})$$
$$= H(z) H^*\left(\frac{1}{z^*}\right)\Big|_{z = e^{j\omega}}$$

if $H(z)$ rational

$$\rightarrow H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})}$$

$$\rightarrow C(z) = H(z)\, H^*\!\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N} (1-d_k z^{-1})(1-d_k^* (z))}$$

if $H(z)$ minimum phase

$\rightarrow$ consists of all poles and zeros of $C(z)$ inside the unit circle.

Min-Phase / All-Pass decomposition

$$\forall\, H(z) \quad : \quad H(z) = H_{min}(z)\, H_{ap}(z)$$

(demonstration OSB p. 280)

Example $\quad H_1(z) = \dfrac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$ $\qquad$ zero : $z=-3$; pole : $z=-\frac{1}{2}$

$$H_{ap}(z) = \frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}$$

$$H_{min}(z) = 3\, \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

Properties of Min. Phase Systems

a] Minimum Phase-Lag Property

$$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \underbrace{\arg[H_{ap}(e^{j\omega})]}_{<0 \;\; \forall\, 0 \leq \omega \leq \pi}$$

$\Rightarrow$ Reflection of zeros of H_{min} in unit circle to outside of unit circle decreases the phase

increases $\underbrace{\text{the negative of the phase}}_{= \text{phase-lag}}$

b] Minimum Group-delay Property

$$grd[H(e^{j\omega})] = grd[H_{min}(e^{j\omega})] + \underbrace{grd[H_{ap}(e^{j\omega})]}_{>0}$$

c] Minimum Energy-delay Property

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{min}[m]|^2$$

$\rightarrow$ Energy concentrated around $n=0$

(note $\sum_{m=0}^{\infty} |h[m]|^2 = \sum_{m=0}^{\infty} |h_{min}[m]|^2$!)