III Basic Digital Filters

3.1. Non recursive digital filters

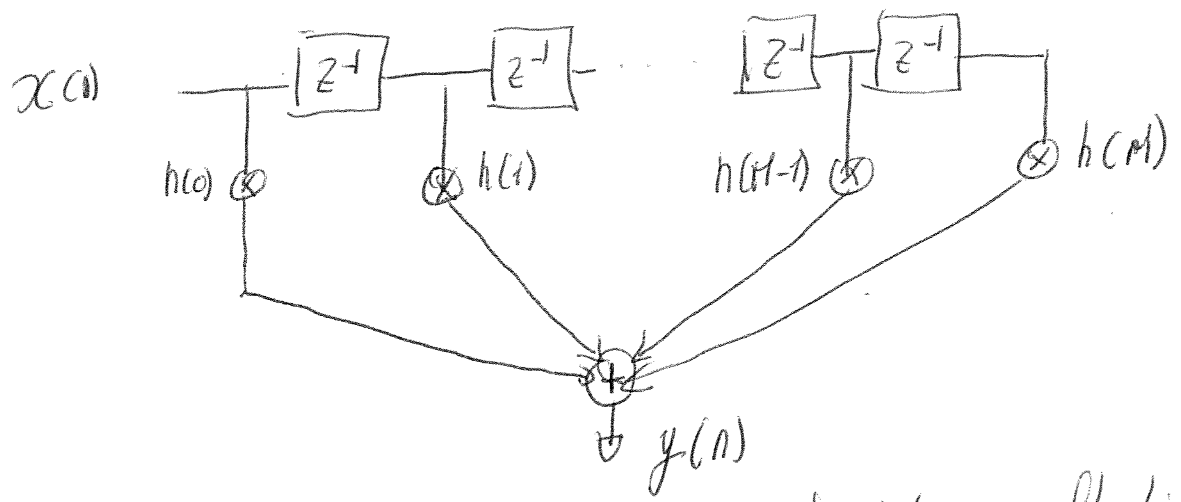
$$y(n) = \sum_{\ell=0}^{M} h(\ell)\, x(n-\ell)$$

named also FIR : Finite Impulse Response

$\Rightarrow$ the impulse response is $h(\ell)$, which is finite in time

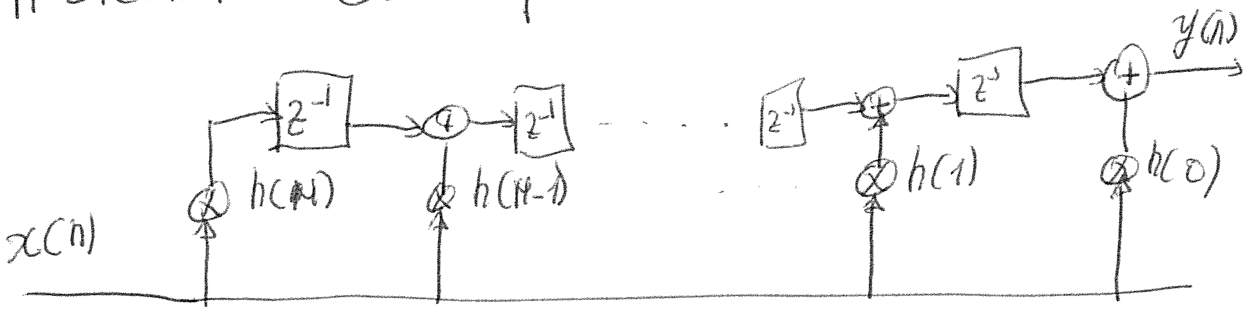
$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \sum_{\ell=0}^{M} h(\ell) z^{-\ell}$$

3.1.a Direct form

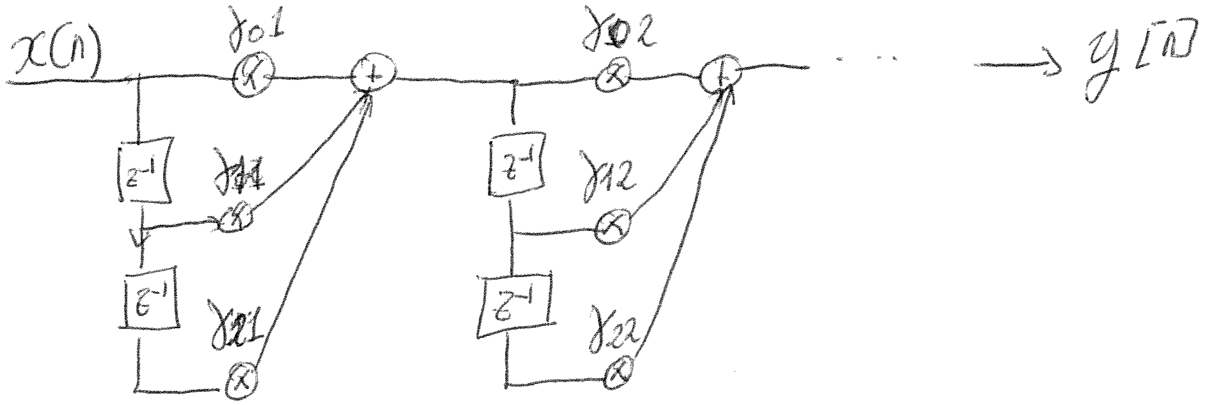


Canonic : minimum number of delays, multipliers and adders

3.1.b Alternative direct form



3.1.c Cascade Form : cascade of second order FIRs



$$H(z) = \prod_{k=1}^{N} \left(\gamma_{0k} + \gamma_{1k} z^{-1} + \gamma_{2k} z^{-2} \right)$$

3.2 Linear-phase filters

Why? → constant group delay!

$$\Rightarrow H(e^{j\omega}) = \underbrace{B(\omega)}_{\text{Real}} \, e^{-j(\omega\tau + \phi)}$$

$$\Rightarrow h(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} e^{j\phi} \int_{-\pi}^{+\pi} B(\omega) \, e^{j\omega(n-\tau)} \, d\omega$$

Let $\tau = \frac{k}{2}$ $k \in \mathbb{Z}$

$$\Rightarrow h^*(2\tau - n) = \left[\frac{e^{j\phi}}{2\pi} \int_{-\pi}^{+\pi} \underset{\text{Real}}{B(\omega)}\, e^{j\omega(2\tau - n - \tau)} d\omega\right]^*$$

$$= \frac{e^{-j\phi}}{2\pi} \int_{-\pi}^{\pi} B(\omega)\, e^{-j\omega(\tau - n)} d\omega$$

$$= h(n)$$

$\Rightarrow$ if $h(n)$ FIR $\Rightarrow$

$$\boxed{h(n) = e^{2j\phi}\, h^*(M-n)}$$

$\Rightarrow$ if $h(n)$ Real

$\Rightarrow e^{2j\phi}$ Real

$\Rightarrow \phi = \frac{k\pi}{2}$ $k \in \mathbb{Z}$ $(k = 0, 1, 2, 3)$

$$\boxed{h(n) = (-1)^k\, h(M-n)} \quad k \in \mathbb{Z}$$

$$H(e^{j\omega}) = B(\omega)\, e^{-j\left(\frac{\omega M}{2} - \frac{k\pi}{2}\right)}$$

in Practice $\rightarrow$ $k = 0, 1$

$(k = 2 \equiv k = 0$ and $B(\omega) = -B(\omega)$
$k = 3 \equiv k = 1$ and $B(\omega) = -B(\omega))$

3.2.2 Four type of linear phase F.I.Rs

Type I $\quad k=0 \qquad M$ even

$$\Rightarrow h(n) = h(M-n)$$

$$\Rightarrow H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} + h\left(\frac{M}{2}\right) z^{-\frac{M}{2}} + \sum_{n=\frac{M}{2}+1}^{M} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} + z^{-(M-n)}\right] + h\left(\frac{M}{2}\right) z^{-\frac{M}{2}}$$

In Frequency $\rightarrow z = e^{j\omega}$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega M + j\omega n}\right] + h\left(\frac{M}{2}\right) e^{-j\omega\frac{M}{2}}$$

$$= e^{-j\omega\frac{M}{2}}\left(\sum_{n=0}^{\frac{M}{2}-1} 2h(n)\cos\left[\omega\left(n-\frac{M}{2}\right)\right] + h\left(\frac{M}{2}\right)\right)$$

$n \leftarrow \frac{M}{2} - n$

$$= e^{-j\omega\frac{M}{2}}\left(\sum_{m=1}^{M/2} 2h\left(\frac{M}{2}-m\right)\cos\omega m + h\left(\frac{M}{2}\right)\right)$$

$$= e^{-j\omega\frac{M}{2}} \sum_{m=0}^{M/2} a(m)\cos(\omega m)$$

$$\begin{cases} a(0) = h\left(\frac{M}{2}\right) \\ a(m) = 2h\left(\frac{M}{2}-m\right) & m = 1, \dots, \frac{M}{2} \end{cases}$$

Type 2 $\quad k=0 \quad M$ odd

$$h(n) = h(M-n)$$

$$H(z) = \sum_{n=0}^{\frac{M-1}{2}} h(n) z^{-n} + \sum_{n=\frac{M+1}{2}}^{M} h(n) z^{-n}$$

$$= \sum_{n=0}^{(M-1)/2} h(n)\left[z^{-n} + z^{-(M-n)}\right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} 2h(n) \cos\left[\omega\left(n-\frac{M}{2}\right)\right]$$

$$\left(n \to \left(\frac{M+1}{2} - m\right)\right) \quad = e^{-j\omega\frac{M}{2}} \sum_{m=1}^{\frac{M+1}{2}} b(m) \cos\left(\omega\left(m-\frac{1}{2}\right)\right)$$

with $b(m) = 2\, h\left(\frac{M+1}{2} - m\right)$

note: $H(e^{j\pi}) = 0 \quad (\Sigma \text{ cosines at } \pm\frac{\pi}{2})$

Type 3 $\quad k=1 \quad M$ even

$$h(n) = -h(M-n)$$

$$H(z) = \sum_{n=0}^{M/2-1} h(n)\left[z^{-n} - z^{-(M-n)}\right]$$

$$H(e^{j\omega}) = e^{-j\left(\omega\frac{M}{2}-\frac{\pi}{2}\right)} \sum_{n=0}^{M/2-1} -2h(n) \sin\left[\omega\left(n-\frac{M}{2}\right)\right]$$

$$= e^{-j\left(\omega\frac{M}{2}-\frac{\pi}{2}\right)} \sum_{m=1}^{M/2} c(n)\, \underline{\sin \omega m}$$

$$\Downarrow$$

$$H(e^{j0}) = H(e^{j\pi}) = 0$$

Type 4

$k = 1$ M odd

$$h(n) = -h(M-n)$$

$$\Rightarrow H(z) = \sum_{n=0}^{\frac{M-1}{2}} h(n)\left[z^{-n} - z^{-(M-n)}\right]$$

$$H(e^{j\omega}) = e^{-j(\omega\frac{M}{2} - \frac{\pi}{2})} \sum_{m=1}^{\frac{M+1}{2}} d(m) \sin(\omega(m-\tfrac{1}{2}))$$

$$d(m) = 2h\left(\frac{M+1}{2} - m\right)$$

note $H(e^{j\omega}) = 0$ for $\omega = 0$

figures $\rightarrow$ see filt_scilab_coor.sce

3.2.3 Propriétés des filtres FIR à phase linéaire

on peut écrire les filtres FIR à phase linéaire

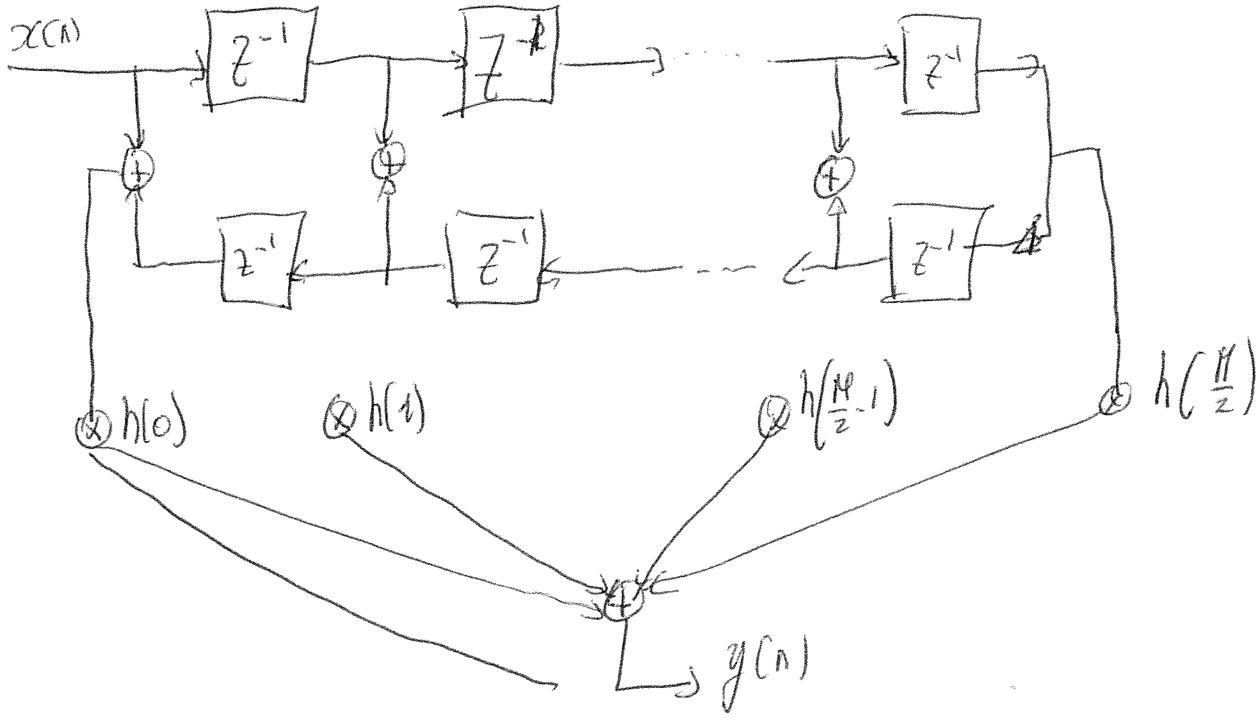
$$H(z) = z^{-\frac{M}{2}} \sum_{n=0}^{K} h(n)\left[z^{\frac{M}{2}-n} \pm z^{-(\frac{M}{2}-n)}\right]$$

$\Rightarrow$ si z_j est un zéro $\Rightarrow$ z_j^{-1} est un zéro

si $h(n)$ réel et si z_j complexe est un zéro $\Rightarrow$ z_j^* est un zéro (et donc $(z_j^*)^{-1}$)

3.2.4 Structure efficace pour FIR à phase linéaire

M pair



3.3 Recursive Filters

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}}$$

In general IIR: Infinite duration Impulse Response Filter

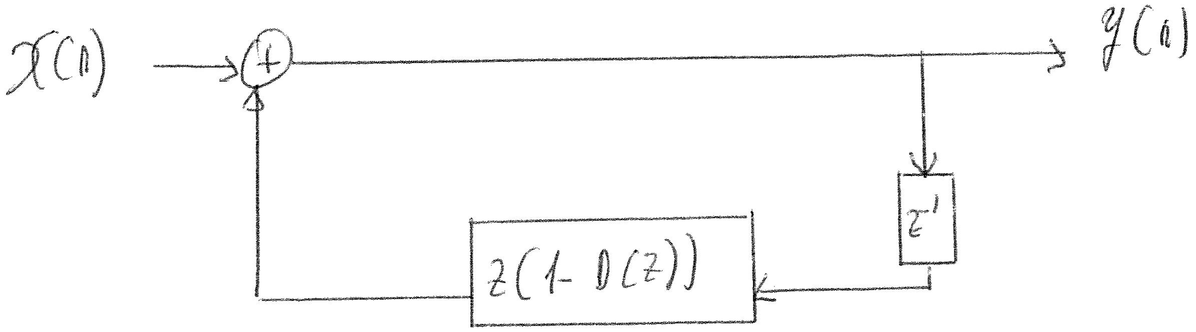
3.3.1 Basic Structure

$N(z) \to$ FIR see above

Let $D'(z) = z(1 - D(z)) = -z \sum_{i=1}^{N} a_i z^{-i}$

Structure for the realization of $\frac{1}{D(z)}$

Filt 29



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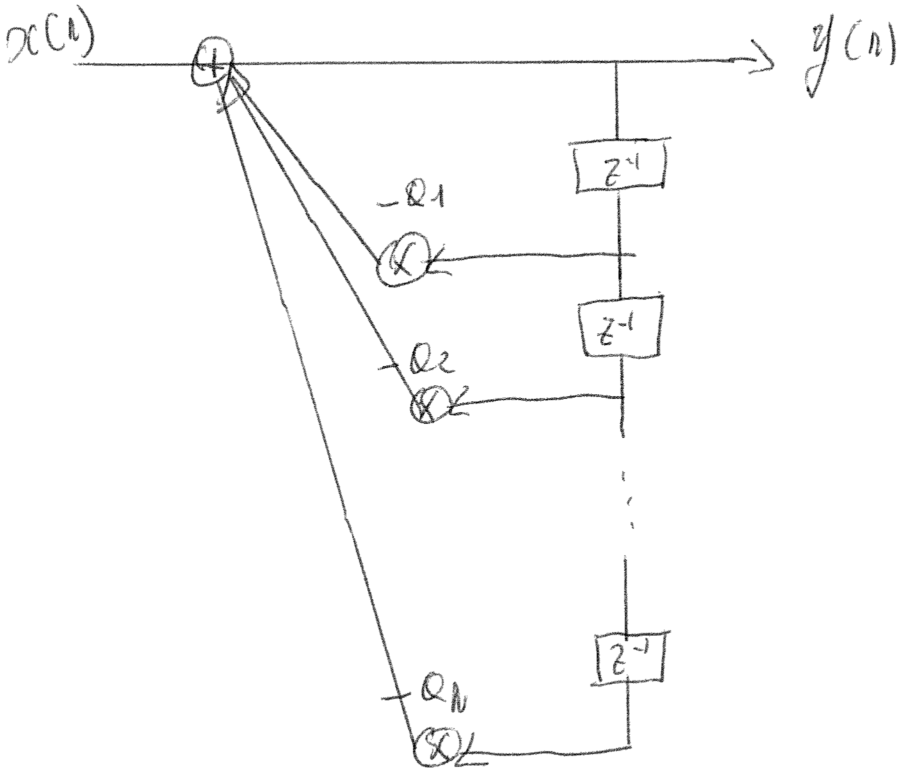
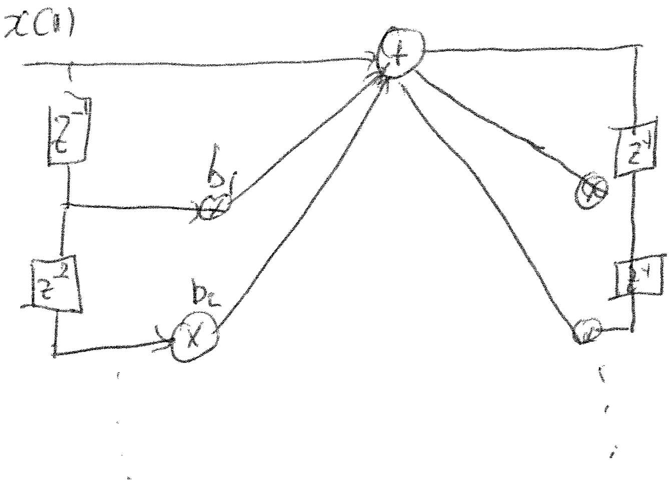


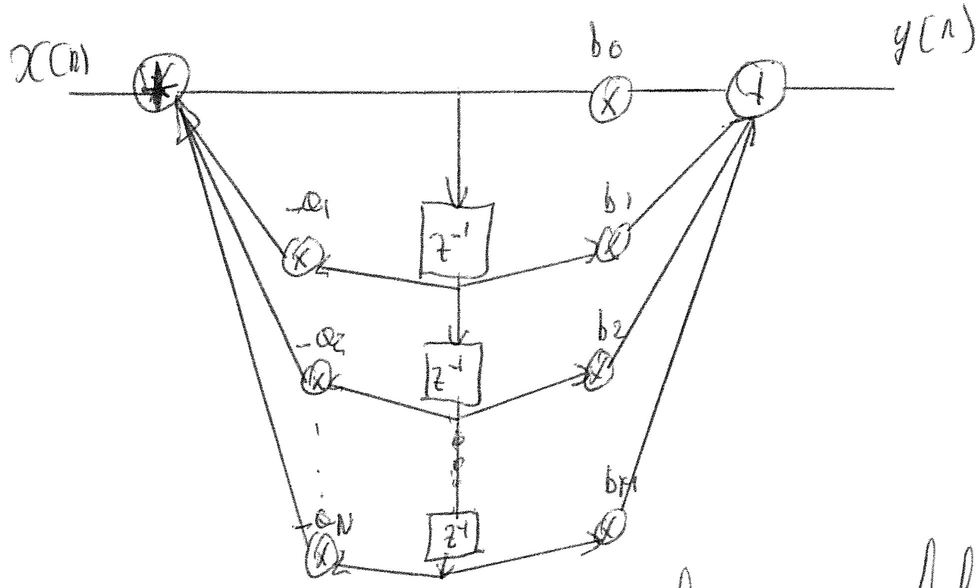
Direct form of IIR



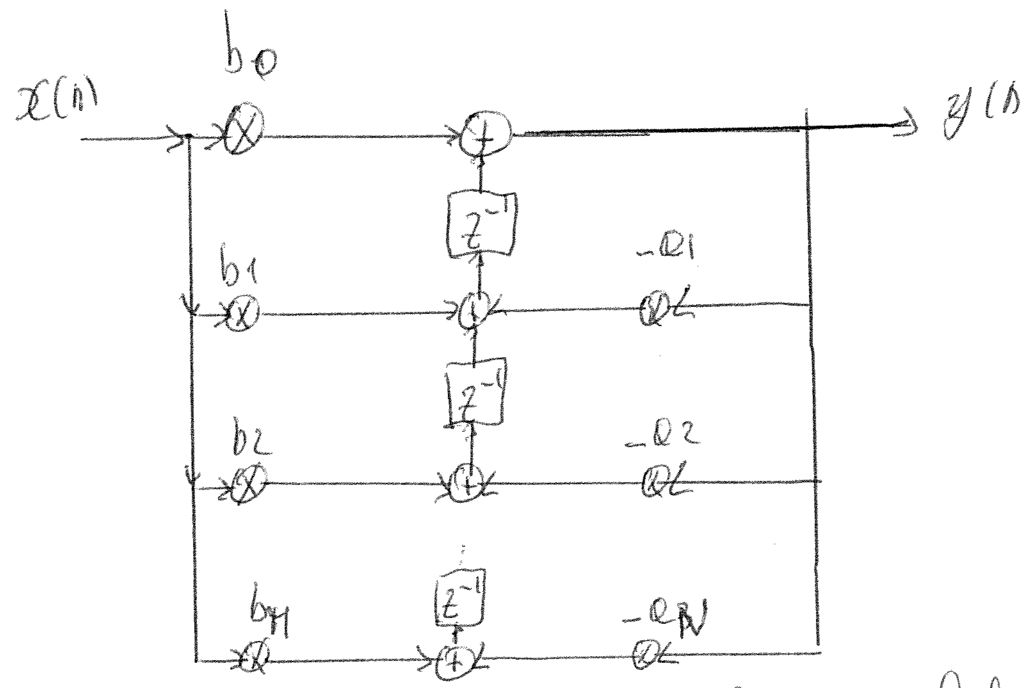
H(z) structure above need

(N + M) delays → not canonic w.r.t delays

indeed → (if N=M)



Type 1 canonic direct form for IIR filters



Type 2 canonic direct form for IIR filters

Cascade Form

$$H(z) = H_0 \prod_{k=1}^{m} \frac{z^2 + \gamma'_{1k} z + \gamma'_{2k}}{z^2 + m_{1k} z + m_{2k}}$$

Parallel Form

$$H(z) = \sum_{k=1}^{m} \frac{\gamma^p_{0k} z^2 + \gamma^p_{1k} z + \gamma^p_{2k}}{z^2 + m_{1k} z + m_{2k}}$$

$$= h_0' + \sum_{k=1}^{m} \frac{\gamma'_{0k} z^2 + \gamma'_{1k} z}{z^2 + m_{1k} z + m_{2k}}$$

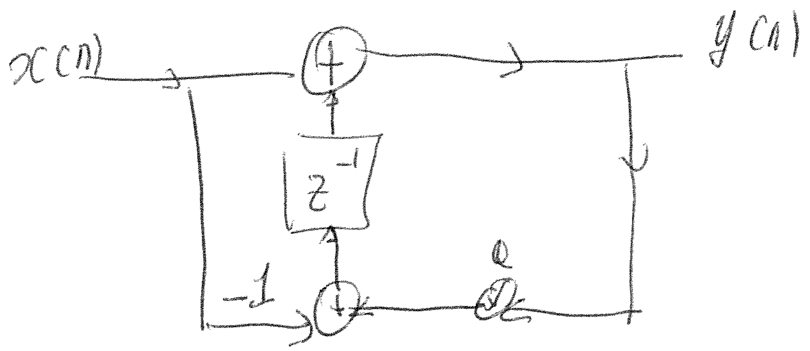
3.4 Special "Filters"

3.4.1 Digital oscillator

$$H(z) = \frac{z \sin \omega_0}{z^2 - 2 \cos \omega_0 z + 1}$$

$\Rightarrow$ poles on the unit circle

3.4.2 Comb Filter $\Rightarrow$ Multiple Passband



$\rightarrow \quad H(z) = \dfrac{1 - z^{-1}}{1 - a z^{-1}}$

$\Rightarrow$ zéro at $z = 1$

pole at $z = a$

Replace z^{-1} with z^{-L}

$\rightarrow \quad H(z) = \dfrac{1 - z^{-L}}{1 - a z^{-L}}$ $\rightarrow$ Zeros at $e^{j\frac{2\pi}{L}}$

Poles at $a^{\frac{1}{L}} e^{j\frac{2\pi}{L}}$