Decidable Properties of 2D Cellular Automata

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1D Cellular Automata

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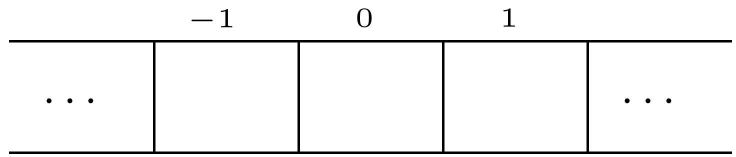
- infinite cells over a regular lattice $\mathcal{L}$

lattice $\mathcal{L} = \mathbb{Z}$



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$\cdots$				$\cdots$

- each cell assumes a state from an alphabet A

configuration $c \in A^{\mathbb{Z}}$

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$\cdots$	c_{-1}	c_0	c_1	$\cdots$

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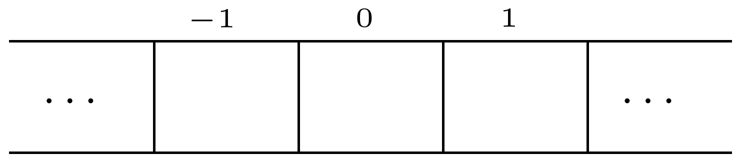

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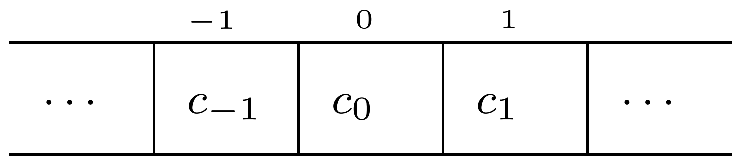
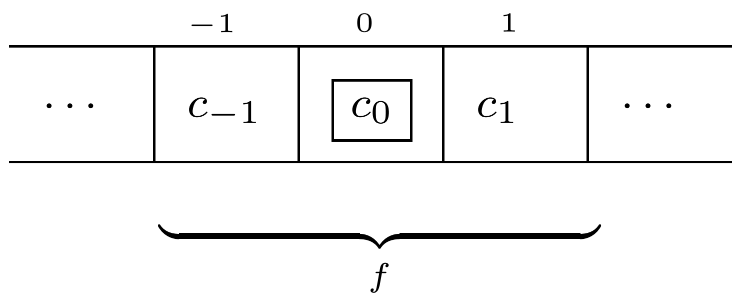


- a r-radius rule $f : A^{2r+1} \longrightarrow A$ updates the state by looking at a neighborhood

update of a cell



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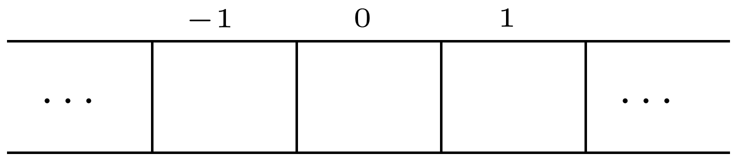

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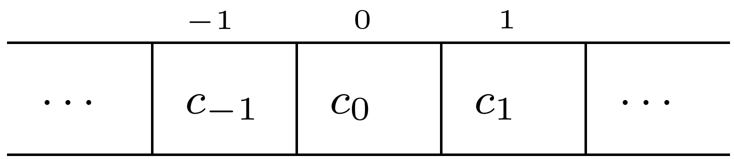

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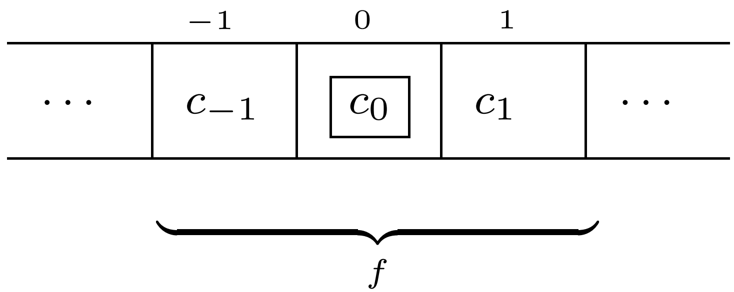

- Global rule $F : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$: local, uniform, and synchronous update

$$\forall c \in A^{\mathbb{Z}}, \quad \forall i \in \mathbb{Z}, \quad F(c)_i = f(c_{i-r}, \ldots, c_{i+r})$$

1D CA as Discrete Dynamical Systems

- The configuration set is usually equipped with the metric d

$$\forall c, c' \in A^{\mathbb{Z}},\ d(c,c') = 2^{-n},\ \text{where } n = \min\left\{i \geq 0\ :\ c_i \neq c'_i \text{ or } c_{-i} \neq c'_{-i}\right\}\ .$$

- the global map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is continuous *w.r.t.* d

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So the pair $(A^{\mathbb{Z}}, F)$ is a Discrete Dynamical System.

If A is finite, the configuration set $A^{\mathbb{Z}}$ is compact.

Compactness is essential to prove the most important results on the basic and dynamical properties of CA.

Some Basic and Dynamical Properties

- **Surjectivity:**
  F is a surjective map.
- **Openness:**
  for any open set $O \subseteq A^{\mathbb{Z}}$, $F(O)$ is open.

  For 1D CA with A finite, openness is equivalent to the following condition:
  all configurations have the same number of pre-images.
- **mixing:**
  for any non empty open $O, O' \subseteq A^{\mathbb{Z}}$ there exists $n \in \mathbb{N}$ such that for all $t \geq n$
  $F^t(O) \cap O' \neq \emptyset$.
- **Denseness of Periodic Orbits (DPO):**
  the set of all periodic points of F is dense in $A^{\mathbb{Z}}$.

1D Closing and Permutivity

- **Left (resp., right) asymptotic configurations $c, c' \in A^{\mathbb{Z}}$:**
  for some $n \in \mathbb{Z}$, $c_{(-\infty,n]} = c'_{(-\infty,n]}$ (resp. $c_{[n,\infty)} = c'_{[n,\infty)}$)
  $(-\infty, n]$: infinite integer interval of positions $i \leq n$.
- **Right (resp. left) closing:**
  for any pair of distinct left (resp. right) asymptotic configurations $c, c' \in A^{\mathbb{Z}}$,
  $F(c) \neq F(c')$.
- **Rightmost (resp. leftmost) permutivity:**
  $\forall u \in A^{2r}$ (input block), $\forall \beta \in A$ (output symbol), $\exists \alpha \in A$ (input symbol) such that
  $f(u\alpha) = \beta$ (resp., $f(\alpha u) = \beta$).

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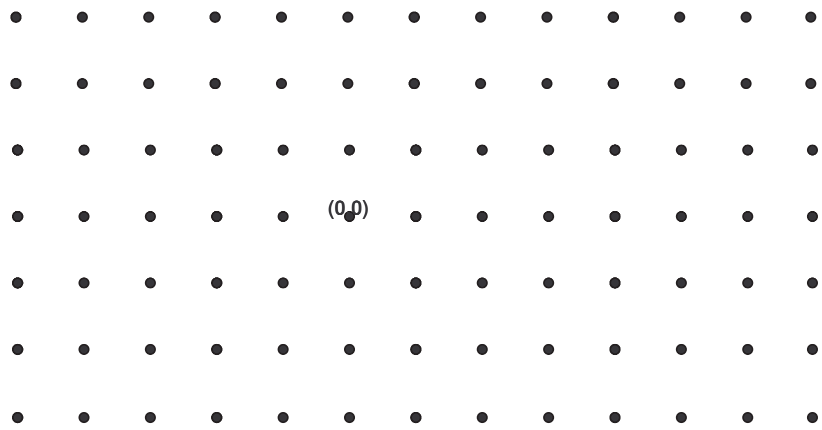
Known Results for 1D CA. If A is a finite set:

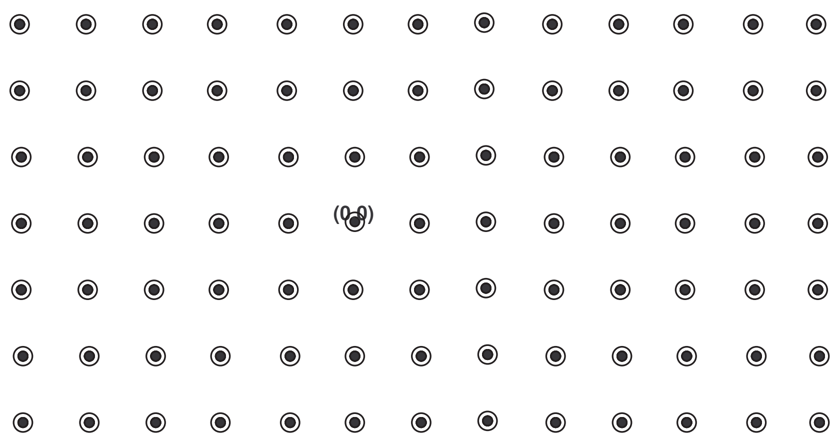
- Closingness and Permutivity are decidable;
- Closing CA have DPO and are surjective;
- A CA is open iff it is both left and right closing;
- permutive CA have DPO, and are topologically mixing and surjective.

2D Cellular Automata/1

- lattice $\mathcal{L} = \mathbb{Z}^2$

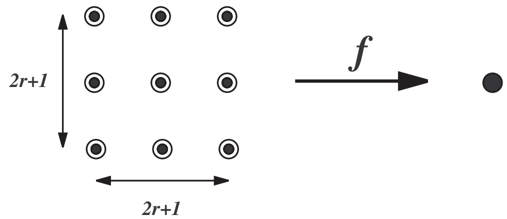


- configuration $c : \mathbb{Z}^2 \longrightarrow A$



2D Cellular Automata/2

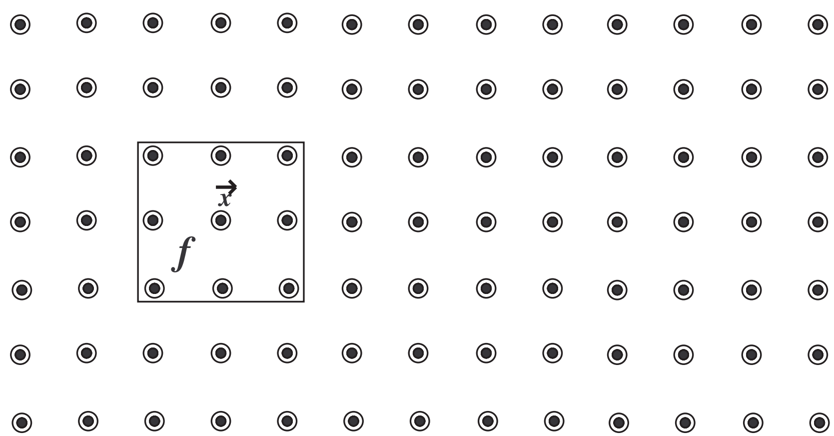
- r-radius local rule $f : \mathcal{M}_r \to A$
  $\mathcal{M}_r$: set of all the 2D $(2r+1) \times (2r+1)$ matrices with values in A



- global rule $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$: local, uniform and synchronous update

$$\forall c \in A^{\mathbb{Z}^2}, \forall \vec{x} \in \mathbb{Z}^2, \quad F(c)(\vec{x}) = f\left(M_r^{\vec{x}}(c)\right) \ ,$$

$M_r^{\vec{x}}(c) \in \mathcal{M}_r$ is the $(2r+1) \times (2r+1)$ matrix inside c centered in the position $\vec{x} \in \mathbb{Z}^2$.



2D CA as Discrete Dynamical Systems

- The configuration set is usually equipped with the metric d

$$\forall c, c' \in A^{\mathbb{Z}^2}, \quad d(c,c') = 2^{-k} \quad \text{where} \quad k = \min\left\{|\vec{x}| : \vec{x} \in \mathbb{Z}^2, c(\vec{x}) \neq c'(\vec{x})\right\} \quad .$$

- the global map $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is continuous *w.r.t.* d

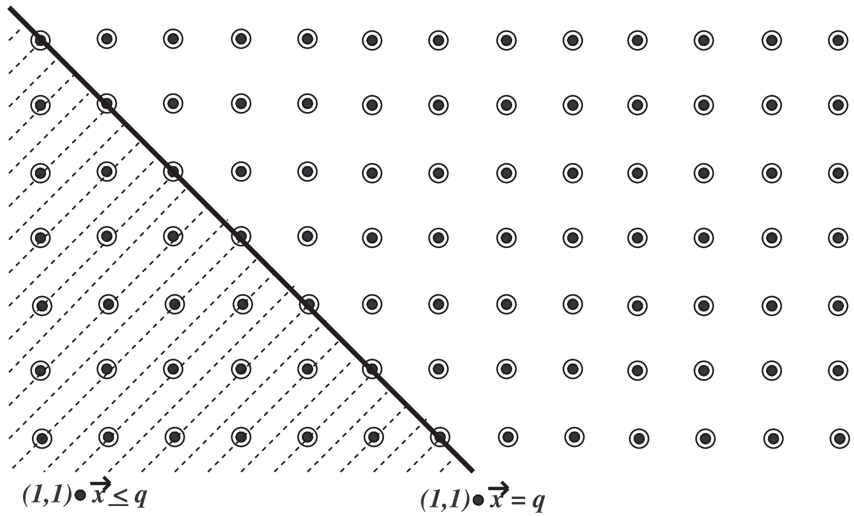
So the pair $(A^{\mathbb{Z}^2}, F)$ is a Discrete Dynamical System.

If A is finite, the configuration set $A^{\mathbb{Z}}$ is compact.

We assume a finite alphabet A when dealing with 2D CA.

2D Closing

- **South-West (SW) asymptotic configurations** $c, c' \in A^{\mathbb{Z}^2}$**:**
  there exists $q \in \mathbb{Z}$ such that $\forall \vec{x} \in \mathbb{Z}^2$ with $(1,1) \cdot \vec{x} \leq q$ it holds that $c(\vec{x}) = c'(\vec{x})$.

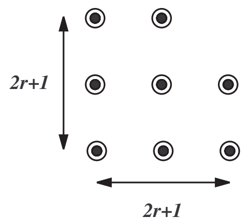


- **North-East (NE) closing:**
  for any pair of distinct SW asymptotic configurations $c, c' \in A^{\mathbb{Z}^2}$,
  $F(c) \neq F(c')$
- In a similar way, the notions of NW, SW, SE closing have been introduced.

Result: NE/NW/SE/SW closing are decidable properties.

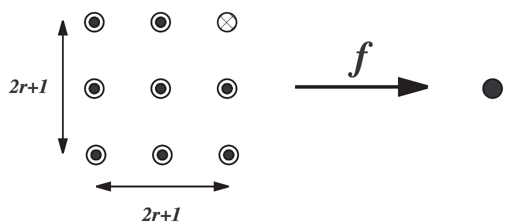
2D Permutivity

- **NE permutivity:**
  for each pair of matrices $N, N' \in \mathcal{M}_r$ with $N(\vec{x}) = N'(\vec{x})$ in all vectors $\vec{x} \neq (r, r)$, it holds that $N(r, r) \neq N'(r, r)$ implies $f(N) \neq f(N')$. Equivalently,

  for any input pattern 

 and for any output symbol 

  there exists an input symbol  such that



- In a similar way, the notions of NW, SW, SE permutivity have been introduced.

<u>**Result:**</u> NE/NW/SE/SW permutivity are decidable properties.

2D CA as 1D CA: slicing construction

NE slicing of the plane:

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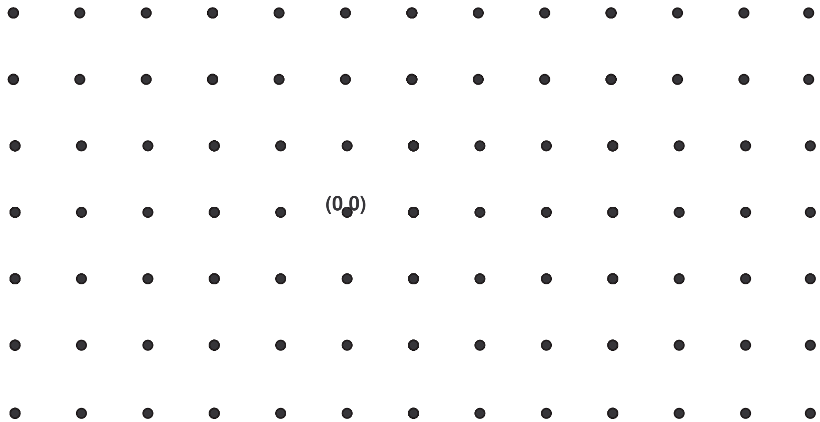

Consider the vector $(-1, 1) \in \mathbb{Z}^2$

2D CA as 1D CA: slicing construction

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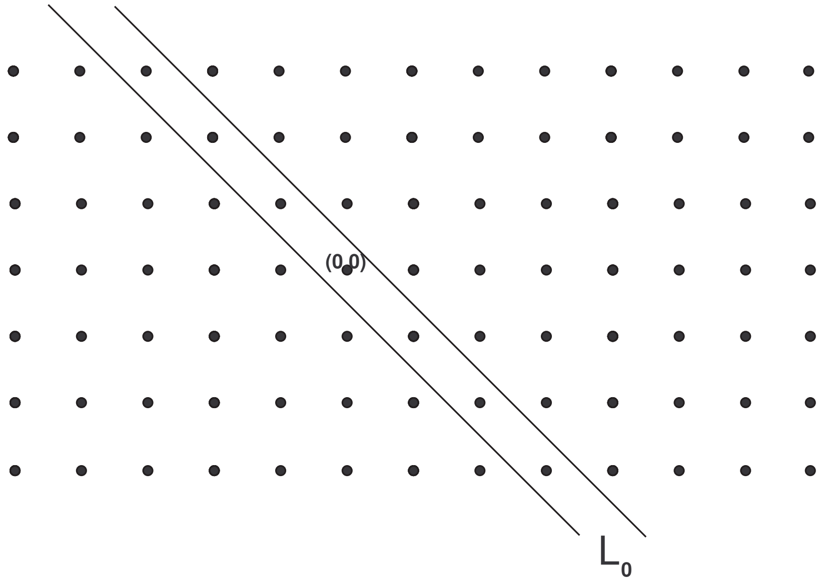

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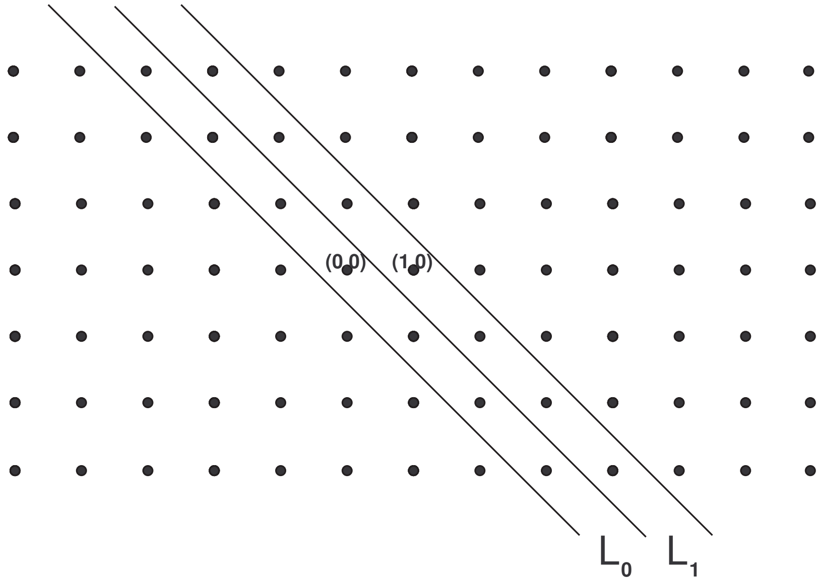

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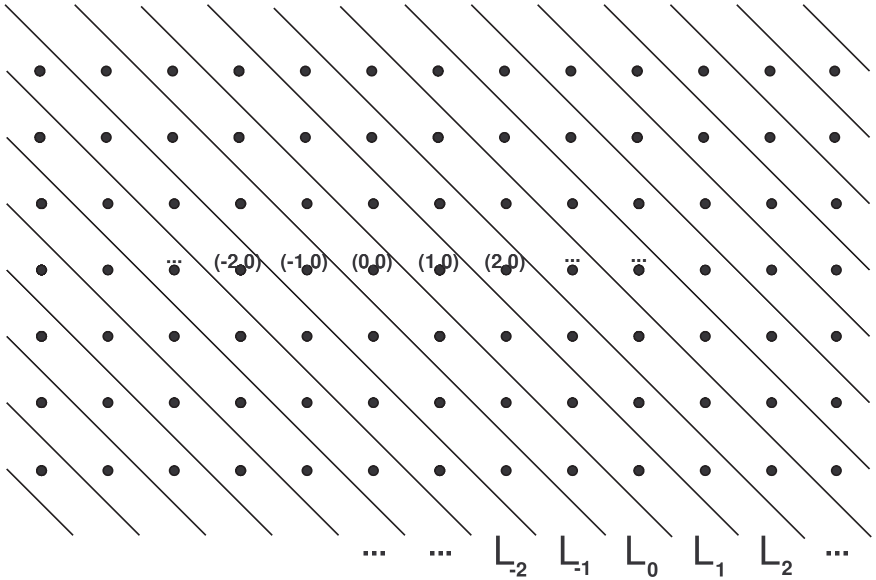

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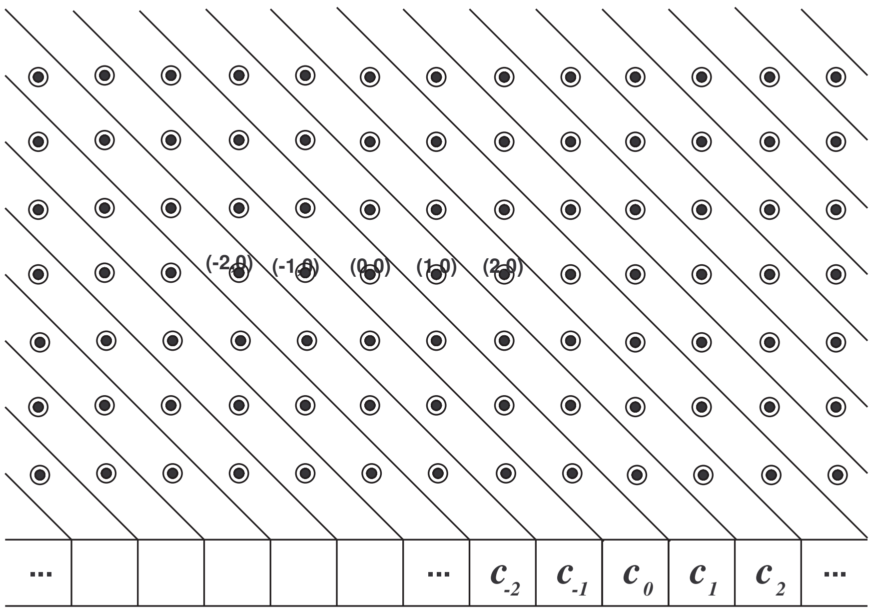
L_i is the integer line expressed in parametric form by $\vec{x} = t(1, -1) + i(1, 0)$ where $t \in \mathbb{Z}$

- $\mathcal{L} = \{L_i\}$ is in a one-to-one correspondence with $\mathbb{Z}$
- $\mathcal{L}$ induces a partition of $\mathbb{Z}^2$

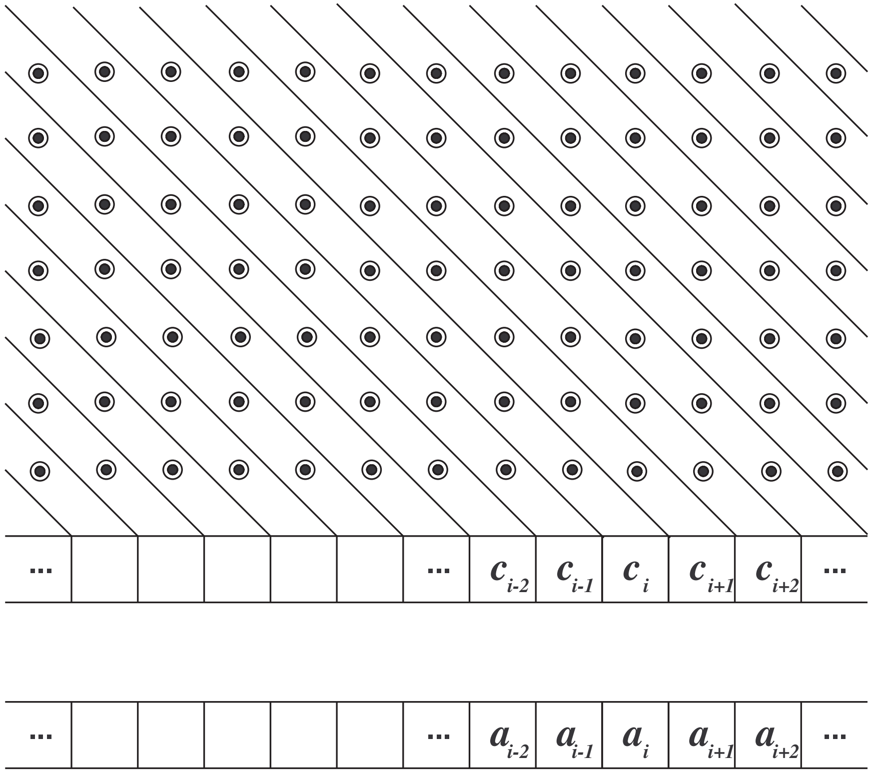
2D CA as 1D CA: slicing construction

In this way, the **NE slicing of configurations** is obtained:

- a configuration $c \in A^{\mathbb{Z}^2}$ can be view as a mapping $c : \bigcup_{i \in \mathbb{Z}} L_i \mapsto A$.
- for every $i \in \mathbb{Z}$, **the slice** c_i **over the line** L_i **of the configuration** c is the mapping $c_i : L_i \to A$, which is the restriction of c to the set $L_i \subset \mathbb{Z}^2$.
- a configuration $c \in A^{\mathbb{Z}^2}$ can be expressed as the bi-infinite one-dimensional sequence $c = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$ of its slices $c_i \in A^{L_i}$.



2D CA as 1D CA: slicing construction



Problem: for each i the slice c_i is an element of A^{L_i}. The components of c are not from the same alphabet!

Solution: by the parametric form of lines L_i ($\vec{x} = t(1,-1) + i(1,0)$ where $t \in \mathbb{Z}$)

- an element of $A^{\mathbb{Z}}$ is associated with any slice c_i.
- a bi-infinite sequence $a = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in (A^{\mathbb{Z}})^{\mathbb{Z}}$ is associated with $c = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$, via an isomorphism Ψ.

2D CA as 1D CA: slicing construction

Introduction of a 1D CA over the alphabet $A^{\mathbb{Z}}$

- the global map $F^* : (A^{\mathbb{Z}})^{\mathbb{Z}} \mapsto (A^{\mathbb{Z}})^{\mathbb{Z}}$ associates any configuration $a : \mathbb{Z} \mapsto A^{\mathbb{Z}}$ with a new configuration $F^*(a) : \mathbb{Z} \to A^{\mathbb{Z}}$.
- the $2r$-radius local rule $f^* : (A^{\mathbb{Z}})^{4r+1} \to A^{\mathbb{Z}}$ takes a certain number of configurations of $A^{\mathbb{Z}}$ as input and produces a new configuration of $A^{\mathbb{Z}}$ as output.

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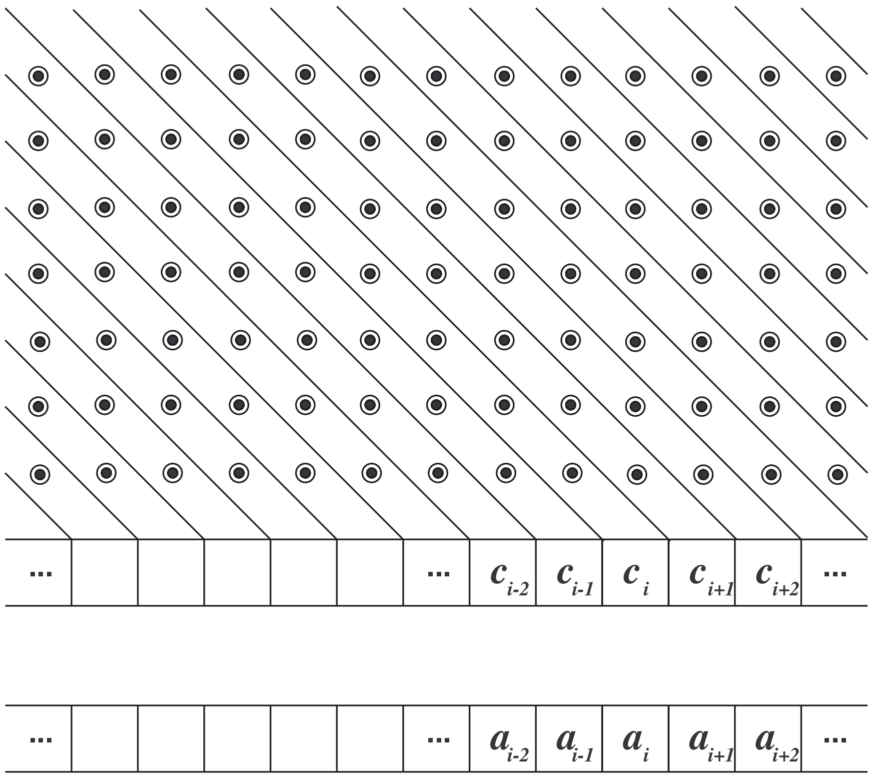

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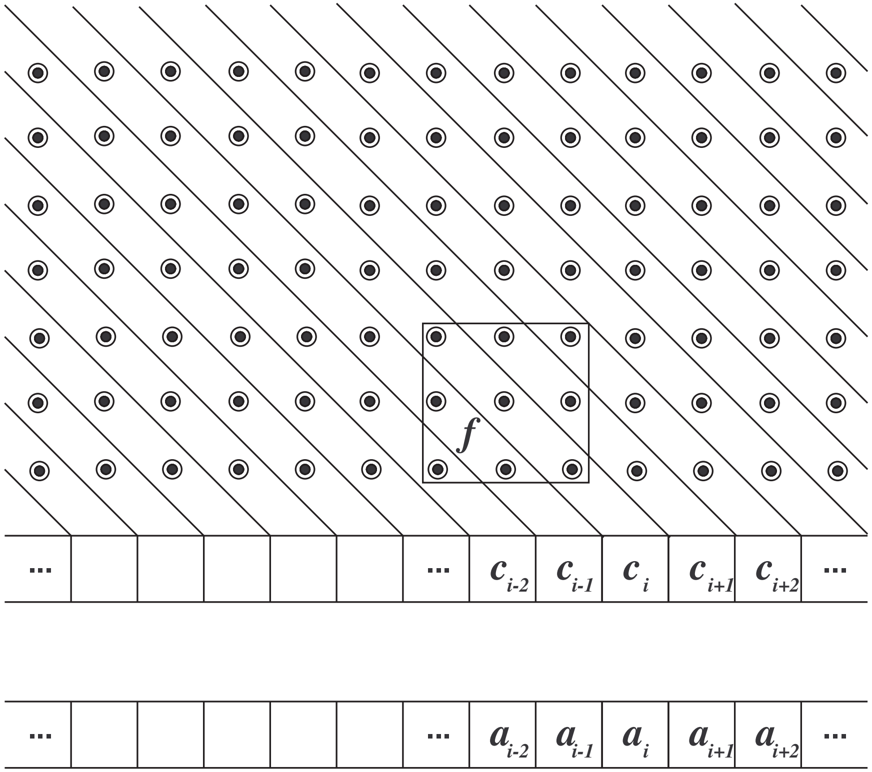

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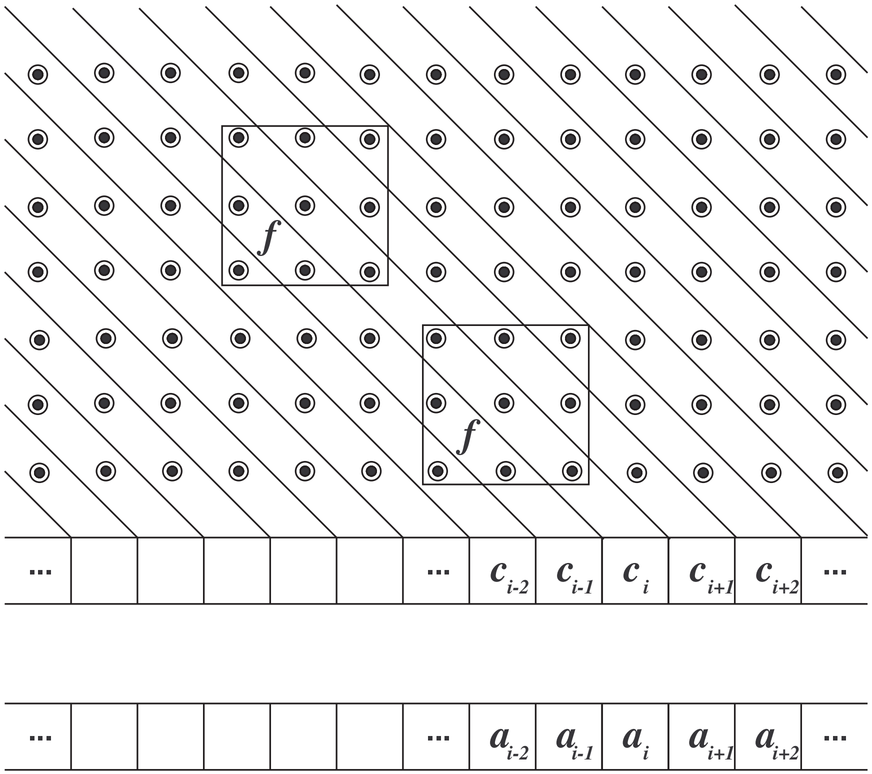

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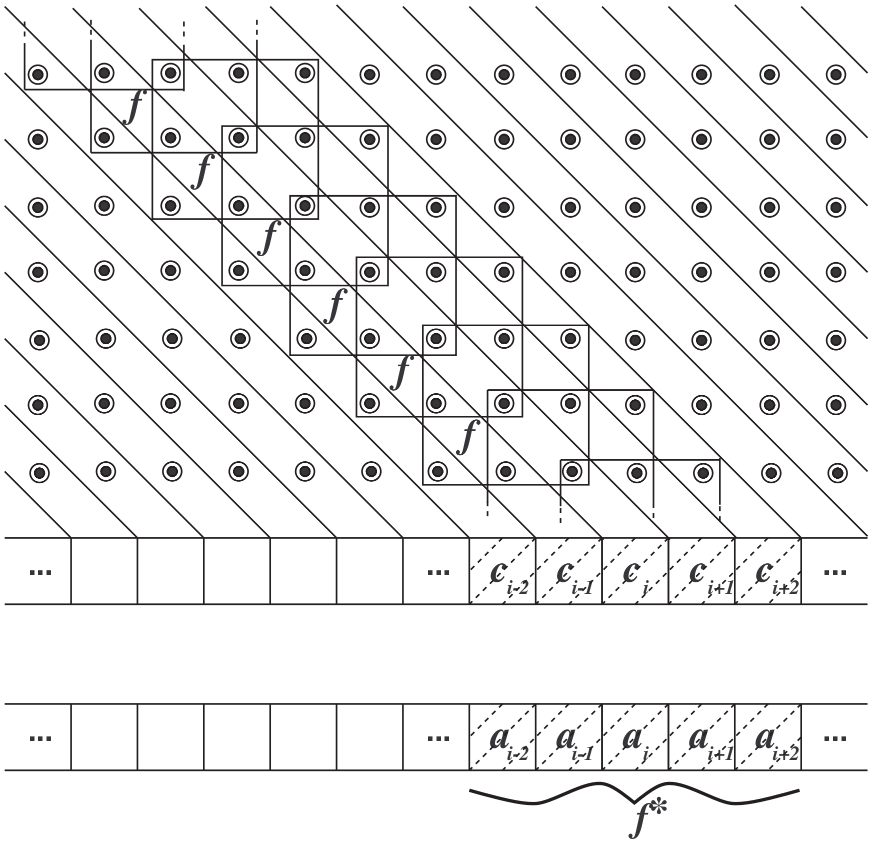

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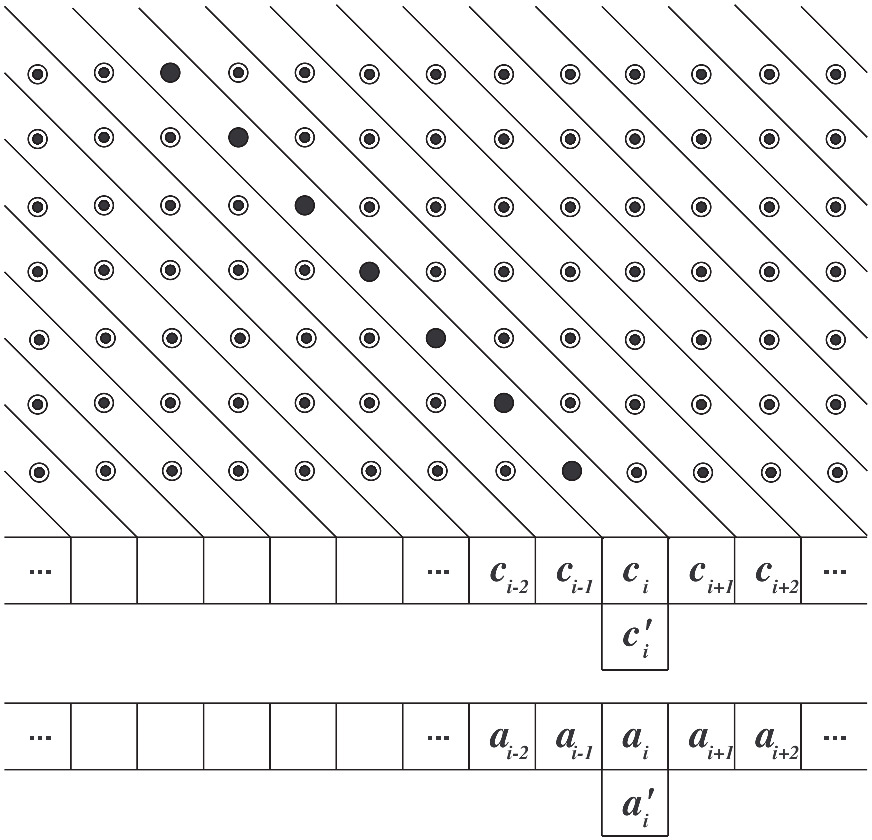


2D CA as 1D CA: slicing construction

The following diagram commutes:

$$\begin{array}{ccc} (A^{\mathbb{Z}})^{\mathbb{Z}} & \xrightarrow{F^*} & (A^{\mathbb{Z}})^{\mathbb{Z}} \\ \Psi^{-1}\downarrow & & \downarrow\Psi^{-1} \\ A^{\mathbb{Z}^2} & \xrightarrow[F]{} & A^{\mathbb{Z}^2} \end{array}$$

- the map Ψ^{-1} is continuous
- the 2D CA F is a factor of the 1D CA F^* on the infinite alphabet $A^{\mathbb{Z}}$

Result: If a 2D CA F is NE permutive then the 1D CA F^* is right permutive.

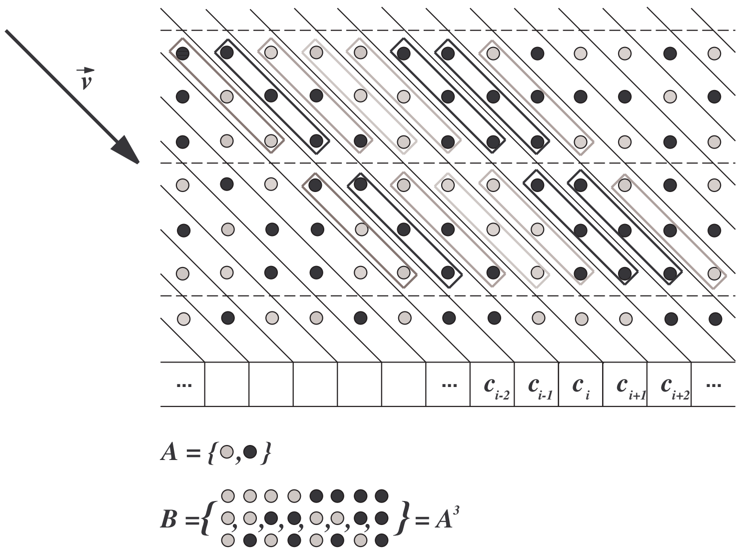
Result: If a 2D CA F is NE permutive then it is topologically mixing.
proof (idea):
- if F is NE permutive then F^* is right permutive.
- right permutivity implies mixing also for CA with infinite alphabet, so F^* is mixing.
- since F is a factor of F^*, then F is mixing too.

Slicing plus finite alphabet

For any 2D CA F we can build an associated 1D sliced version F^* with finite alphabet.

- fix a vector $\vec{v} \perp (1,1)$
- let $\sigma^{\vec{v}} : A^{\mathbb{Z}^2} \longrightarrow A^{\mathbb{Z}^2}$ the shift map of vector $\vec{v}$
- consider the set $S_{\vec{v}}$ of configurations c such that $\sigma^{\vec{v}}(c) = c$ (translation invariant)
- $F(S_{\vec{v}}) \subseteq S_{\vec{v}}$ and so $(S_{\vec{v}}, F)$ is a dynamical system

Slicing construction on $S_{\vec{v}}$:



Result: $(S_{\vec{v}}, F)$ is top. conjugated to the 1D CA $(B^{\mathbb{Z}}, F^*)$ on the finite alphabet $B = A^{|\vec{v}|}$.
proof (idea):the slices of configurations in $S_{\vec{v}}$ are in one-to-one correspondence with symbols of the alphabet B.

Consequences

Results:

- NE (or NW, or SE, or SW) closing 2D CA have DPO and are surjective
- 2D CA which are closing *w.r.t.* all the 4 directions are open
- NE (resp., NW, resp., SE, resp., SW) permutivity implies NE (resp., NW, resp., SE, resp., SW) closing and then DPO.

Future Works

Continue the study of 2D CA by means of the slicing constructions!