Tic-Tac-Toe on a Finite Plane

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Everyone knows how to play tic-tac-toe. On an $n \times n$ board, if a player places n of her marks either horizontally, vertically, or diagonally before her opponent can do the same, then she wins the game. What if we keep the rules of the game the same but increase the number of ways to win? For simplicity, any configuration of n marks that produces a win, regardless of whether or not it appears straight, will be called a winning line. For example, we will add the four winning lines shown in Figure 1 when playing on the 3×3 board.

Figure 1: New winning lines for 3×3 tic-tac-toe

This brings the total number of winning lines on this board to twelve. Why did we decide to add these particular lines? If you know the rudimentaries of finite geometry, you can see that the winning lines are prescribed by the geometry of a finite affine plane. Otherwise, for now you should just notice that every new line contains exactly one mark in each row and each column. You should also notice that these new lines make it more difficult to identify a win here than in the standard game. As you will see, the reason for this complexity is that lines in an affine plane need not appear straight. With this new twist, the game that grew tiresome for us as children is transformed into an interesting, geometrically motivated game.

The geometric intuition required to understand finite planes often proves elusive as our Euclidean-trained minds have preconceived notions of lines and points. The new version of tic-tac-toe helps to develop this intuition. Moreover, this game relates geometric concepts to game-theoretic concepts as the natural question of winning strategies arises. Since more winning lines mean more possible ways to win, one might think that it would be easier to force a win in this new game. Not only is the answer to this question nonintuitive, but the difficulty encountered in providing an answer for the 4×4 board is surprising.

First, we review Latin squares and affine planes, as well as the relationship between them, in order to find the new winning lines. Once you can identify the winning lines, you are ready to play tic-tac-toe on the affine plane. Since projective planes are a natural extension of affine planes, you will also learn to play tic-tac-toe on these planes. You may recall that in the 3×3 version of tic-tac-toe we played as children, one quickly learns that there is no advantage to being the first player since the game between two skilled players always ends in a draw. While this is the case on many finite planes, we will show that there *are* planes where the first player holds the advantage. In the event that you are the *second* player on a plane where a forced draw is possible, we provide a computational method that guarantees a draw. We will also show simple configurations of points that produce a draw with very few points.

Squares and planes

Consider the 36 officer problem: There are 36 officers, each with one of six rank designations and one of six regiment designations. Can the 36 officers be arranged into six rows and six columns so that each rank and regiment is represented in each row and each column? Leonhard Euler showed this could be done for 9 and 25 officers (try it!), but conjectured correctly that it could not be done for 36 officers. In an attempt to solve this problem he introduced Latin squares [11]. A Latin square of order n is an $n \times n$ matrix with entries from $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, where each number occurs exactly once in each row and each column. Examples of Latin squares of orders 2, 3, and 4 are given in Figure 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

FIGURE 2: Latin squares of orders 2, 3, and 4

Since it is natural to explain the game of tic-tac-toe on a finite plane by the connection between planes and these squares, we begin with an explanation of Latin squares, affine planes, and the relationship between them. The material presented in this section can be found in any text on affine and projective planes [3, 18]. Readers familiar with these concepts may wish to proceed to the next section.

Latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ are orthogonal if and only if $C = [c_{ij}]$, whose entries are the ordered pairs $c_{ij} = (a_{ij}, b_{ij})$, contains all n^2 possible ordered pairs of $\mathbb{Z}_n \times \mathbb{Z}_n$. A collection of Latin squares is mutually orthogonal (MOLS) if and only if each pair is orthogonal. (The Maple command *MOLS(p,m,n)* produces n MOLS of order p^m when p is prime and $n < p^m$.) In the examples above, the two Latin squares of order 3 are orthogonal, and the three of order 4 are MOLS. Euler's 36 officer problem asks if it is possible to find a pair of orthogonal Latin squares of order 6, one representing the ranks of the 36 officers and the other representing the regiments. As illustrated by the first Latin square of order 3 in FIGURE 2, you can easily produce one Latin square of order 6 by continually shifting the elements of your first row to the right by one position and wrapping the leftover elements to the beginning. The proof of the 36 officer problem shows that you cannot produce a second Latin square orthogonal to the first. (Try it!) Exhaustive solutions [20] to this problem, as well as more sophisticated ones [9, 19], can be found in the literature. (Laywine and Mullen [15] offer many interesting questions concerning Latin squares.)

The Euclidean plane is an example of an affine plane, and the axioms of affine planes are merely a subset of those from Euclidean geometry. Specifically, an affine plane is a nonempty set of points, P, and a nonempty collection of subsets of P (called lines), L, which satisfy the following three axioms: (1) through any two distinct points there exists a unique line;

(2) if p is a point, ℓ is a line, and p is not on line ℓ, then there exists a unique line, m, that passes through p and is parallel to ℓ, that is, $p \in m$ and $\ell \cap m = \emptyset$; (3) there are at least two points on each line, and there are at least two lines. When p is a point on line ℓ, we say that p is incident with ℓ. The Cartesian plane, with points and lines defined as usual, is the example we typically envision when reading this definition. It is an example of an infinite affine plane.

Finite affine planes are those with a finite set of points. There is no finite affine plane where P contains exactly one, two or three points. (Why not? What axiom(s) of affine planes does this violate?) The smallest finite affine plane can be given as $P = \{p, q, r, s\}$ and $L = \{\{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}\}$, which is represented by both of the graphs in FIGURE 3. Notice that an intersection of line segments does not necessarily indicate the existence of a point in P.

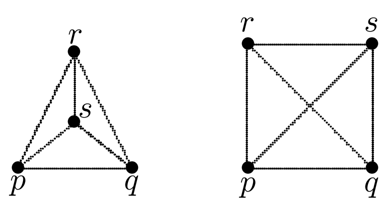

FIGURE 3: Two graphical representations of the affine plane of order 2

Using the given axioms, we invite the reader to reproduce the following elementary results: On a finite affine plane, each line must contain the same number of points and each point is incident with the same number of lines. The number of points on each line is called the order of the plane. This is why the diagrams in FIGURE 3 are described as the affine plane of order 2. In general, an affine plane of order n has n points on every line, and each point is incident with $n+1$ lines. For any such plane, $|P| = n^2$ and $|L| = n^2 + n$. Two lines are parallel if and only if they have no common points, and parallelism is an equivalence relation on the set of lines. A parallel class consists of a line and all the lines parallel to it. An affine plane of order n has $n + 1$ parallel classes, each containing n lines. As another example, FIGURE 4 shows the affine plane of order 3, where $P = \{a, b, c, d, e, f, g, h, i\}$ and

$L = \{\{a,b,c\}, \{d,e,f\}, \{g,h,i\}, \{a,d,g\}, \{b,e,h\}, \{c,f,i\}, \{a,e,i\}, \{c,e,g\}, \{a,h,f\},$ $\{g,b,f\}, \{i,b,d\}, \{c,h,d\}\}$. You can see that each line has three points, each point is incident with four lines, $|P| = 9$, and $|L| = 12$. The lines $\{c,e,g\}$, $\{a,h,f\}$ and $\{i,b,d\}$ are parallel and, therefore, form one of the four parallel classes.

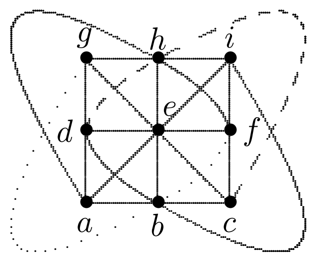

FIGURE 4: Affine plane of order 3

While we have seen affine planes of orders 2 and 3, there is no such plane of every order. In fact, determining which orders exist is exceptionally difficult, and remains a largely open problem. It *is* well known that there are affine planes of order p^k where p is prime and $k \in \mathbb{Z}^+$. This tells us, for example, that there are affine planes of orders 2, 3, 4, 5, 7, 8, and 9. How about 6 and 10? We can answer one of these using the following connection between affine planes and Latin squares: Bose [7] showed that an affine plane of order n exists if and only if there exist $n-1$ MOLS of order n. Using this result, we see that there can be no affine plane of order 6 since the solution to the 36 officer problem shows that there is no *pair* of orthogonal Latin squares of order 6. The proof of the nonexistence of the plane of order 10 is much more difficult, requiring a great deal of mathematics and an enormous computation to finish the proof. (Lam [14] gives an historical account.) It is not known whether an affine plane of order 12 exists. In fact, it is unknown whether there are any affine planes that do not have prime-power order. Some of these orders, however, are known not to exist (see Bruck-Ryser Theorem [3]).

This connection between affine planes of order n and the $n-1$ MOLS of order n can be used to find the lines of the plane quite easily. After arranging the n^2 points of a finite affine plane in an $n \times n$ grid, we will first identify its $n+1$ parallel classes, which in turn reveals all of the lines. The n horizontal lines form one parallel class, and the n vertical lines form another. Each of the remaining $n-1$ parallel classes corresponds to one of the $n-1$ MOLS as follows: the ith line in any parallel class is formed by the positions of symbol i in the corresponding Latin square. (Here, $i = 0, 1, \ldots, n-1$.) For example, using FIGURE 4 and the two orthogonal 3×3 Latin squares in FIGURE 2, we see that the four parallel classes for the affine plane of order 3 are (i) the horizontal lines $\{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}\}$ (ii) the vertical lines $\{\{a, d, g\}, \{b, e, h\}, \{c, f, i\}\}$ (iii) the lines indicated by the first Latin square $\{\{c, e, g\}$, $\{a, h, f\}$, $\{i, b, d\}\}$, and (iv) the lines indicated by the second Latin square $\{\{g, b, f\}, \{c, h, d\}, \{a, e, i\}\}$. At this point you might notice that the four lines that do not appear straight correspond precisely to the winning lines we added to the 3×3 tic-tac-toe board, as shown in FIGURE 1.

There is one other type of plane on which we will play tic-tac-toe, namely finite projective planes. A projective plane is easily constructed from an affine plane of order n by adding $n+1$ points (the points at infinity) and one line (the line at infinity). The points are added in this way: each point at infinity must be incident with the n lines of a unique parallel class. (Now you see that $n+1$ points must be added since there are $n+1$ parallel classes on the affine plane of order n.) The line at infinity, ℓ_∞, simply consists of the $n+1$ points at infinity. For example, the projective plane of order 2 can be constructed from the affine plane of order 2 given in FIGURE 3 by adding $\ell_\infty = \{a, b, c\}$, as shown in both of the graphs in FIGURE 5. Here we see point a is added to the parallel lines $\{r, p\}$ and $\{s, q\}$, b is added to the parallel lines $\{r, s\}$ and $\{p, q\}$, and c is added to the parallel lines $\{q, r\}$ and $\{p, s\}$.

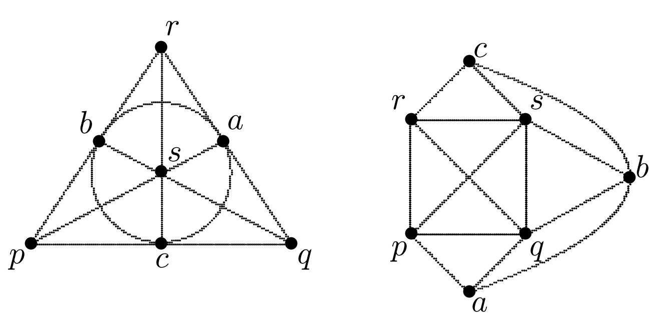

FIGURE 5: Two graphical representations of the projective plane of order 2

Of course, we could have discussed projective planes before affine planes. By definition, a projective plane is a nonempty set of points, P, and a nonempty set of lines, L, which satisfy the following three axioms: (1) any two distinct lines meet in a unique point; (2) any two distinct points have a unique line through them; (3) there are at least three points on each line, and there are at least two lines. We invite the reader to reproduce the following elementary results: On a finite projective plane, each line must contain the same number of points. In particular, a projective plane of order n has $n + 1$ points on each line. For any such plane, $|P| = |L| = n^2 + n + 1$. For example, in FIGURE 5 we can identify the seven lines of the plane of order 2 as $\{r, p, a\}$, $\{s, q, a\}$, $\{r, s, b\}$, $\{p, q, b\}$, $\{q, r, c\}$, $\{p, s, c\}$ and $\{a, b, c\}$. Since the number of points is not a perfect square, you should notice that we will not be playing tic-tac-toe on an $n \times n$ grid for these planes. Lastly, just as we were able to construct a projective plane from an affine plane, we can do the opposite. Starting with a projective plane of order n, removing any line and all of the points with which it is incident forms an affine plane of order n.

Throughout this work the word plane will refer to either an affine or projective plane. Affine planes of order n will be represented as π_n, and projective planes as Π_n. All statements about uniqueness are always understood to mean up to isomorphism.

The game

A zero-sum game is a game where one player's loss is a gain for the other player(s). The standard game of tic-tac-toe is a two-player, zero-sum game on an 3×3 board where players alternately mark one open cell with an X or an O. For simplicity, we will refer to player X as Xeno, player O as Ophelia, and assume that Xeno always makes the first move. A player wins by being the first to place three matching marks on a line. If a game is complete and no player has won, the game is a draw. Tic-tac-toe is an example of a game of perfect information since each choice made by each player is known by the other player. Poker is not such a game since players do not reveal their cards.

A *strategy* is an algorithm that directs the next move for a player based on the current state of the board. A winning strategy for Xeno, for example, is a strategy that is guaranteed to produce a win for him. Although there is a best way to play standard tic-tac-toe, there is no winning strategy since each player can guarantee that the other cannot win. In this case, we say that both players have a drawing strategy, that is, an algorithm that leads to a draw. The assumption that both players are knowledgeable and play correctly is a standard game-theoretic assumption called the principle of rationality, that is, at each move, each player will make a choice leading to a state with the greatest utility for that player.

We give the following definition for tic-tac-toe on a plane of order n. Xeno and Ophelia alternately place their marks on any point of the plane that has not already been labelled. The first player to claim all of the points on a line wins the game. The game is a draw if all points are claimed and neither player has completed a line. In order to show the order of play, we will denote Xeno's first move as X_1, his second move as X_2, and so on. Ophelia's moves are likewise designated. We shall refer to a game as an ordered pair $[(X_1, \ldots, X_s), (O_1, \ldots, O_r)]$ where $r = s - 1$ or $r = s$. A complete game is one that has resulted in a win or a draw.

As suggested in the previous section, when playing on π_n we will arrange the n^2 points in an $n \times n$ grid. In this way, the cells of the standard $n \times n$ tic-tac-toe grid have become the points of π_n. When Xeno marks an open cell in the grid with his X, he is essentially claiming a point on the affine plane. The $n^2 + n$ lines of π_n are found in this way: n lines are horizontal, n lines are vertical, and the remaining $n^2 - n$ lines are identified by consulting the $n-1$ MOLS of order n. Remember, each of the MOLS of order n defines n lines (displayed as identical symbols). Since there are $n-1$ MOLS, each defining n lines, we have our remaining $n(n-1)$ lines. For example, the game shown on the left in FIGURE 6 is a win for Xeno on the affine plane of order 3 since $\{X_2, X_3, X_4\}$ forms a line, as can be verified by viewing FIGURE 4 or consulting the second Latin square of order 3 given in FIGURE 2. The game on the right is a win for Ophelia on the affine plane of order 4 since $\{O_1, O_3, O_6, O_7\}$ forms a line, as can be verified by consulting the second Latin square of order 4 in FIGURE 2.

X_1	X_4	
X_3	O_2	O_1
O_3		X_2

X_1	X_2	O_3	X_3
O_1	X_6	X_4	
X_7	O_6	O_2	O_5
O_4	X_5		O_7

FIGURE 6: Win for Xeno on π_3 and win for Ophelia on π_4

Of the many interesting graph-theoretic, game-theoretic, and combinatorial questions this game generates, we will first consider two fundamental questions.

Question 1: For which planes are there winning strategies?

Question 2: For which planes can play end in a draw?

The first question is essentially a game-theoretic question, whereas the second question is fundamentally a geometric question. As regards the first question, in game theory it is known that in a finite two-player game of perfect information, either one player has a winning strategy or both players can force a draw [17]. A 'strategy-stealing' argument [4, 5] proved by Hales and Jewett [12] shows that in our case it is Xeno who has a winning strategy when such a strategy exists. To show this, assume that Ophelia has the winning strategy. Let Xeno make a random first move and thereafter follow the winning strategy of Ophelia. Specifically, Xeno plays as if he *were* Ophelia by pretending that his first move has not been made. If at any stage of the game he has already made the required move, then a random move can be made. Any necessary random moves, including the first, cannot harm him since he is merely claiming another point. This leads Xeno to a win, contradicting the assumption that Ophelia has the winning strategy. (Notice that this argument does not apply to Nim, for example, since a random move may cause the first player to lose.) Hence, in tic-tac-toe either Xeno has a winning strategy or both players have drawing strategies, in which case we say Ophelia can force a draw. If no draws exist then Xeno is guaranteed to have a winning strategy. However, the existence of draws is not enough to guarantee that Ophelia can force a draw. We discuss the existence of a winning strategy for all finite planes in the two sections that follow.

Regarding the second question, a draw is possible when there exists a set T of $\lceil |P|/2 \rceil$ points such that every line in the plane has points in T and points not in T, that is, no line has its points disjoint from T nor contained in T. We will determine the planes in which play can end in a draw in the following two sections. Of course, knowing that a draw exists does not explain how Ophelia can force the draw. To this end, we give a computational method guaranteed to produce a draw in the section on weight functions, and we describe simple configurations of draws in the last section of the paper.

Planes of small order

There is a unique affine plane of order 2; in it each line has two points, as represented in FIGURE 3. Xeno has a trivial winning strategy when playing tic-tac-toe on this plane. Namely, if X_1 and O_1 are chosen arbitrarily, then X_2 produces a win for Xeno with the line containing X_1 and X_2, regardless of its placement. Hence, Xeno wins merely by being the first player, and a draw is not possible since any two points form a line.

There is a unique projective plane of order 2; as represented in FIGURE 5, each line has three points. Xeno has a winning strategy when playing on this plane as well. Namely, if X_1 and O_1 are chosen arbitrarily, then he chooses X_2 to be any point not on the line containing X_1 and O_1. Since there is a line between any two points, O_2 must be placed on the line containing X_1 and X_2 (otherwise Xeno wins on his next move). He chooses X_3 to be the point on the line containing O_1 and O_2. Then Ophelia must block either the line containing X_1 and X_3 or the line containing X_2 and X_3, (it is a simple matter to see that Ophelia does not already have these lines blocked). Xeno wins on his next move when he completes the line that O_3 did not block. Even if the principle of rationality is violated and Xeno purposely chooses a point unwisely, a draw is not possible on Π_2. Any four points on Π_2, no three of which are collinear, form an object called a hyperoval, and the complement of this hyperoval is a line. Hence, there does not exist a set $T \subset P$ with $|T| = 4$ such that T and its complement intersect each line.

There is a unique affine plane of order 3; as represented in FIGURE 4, each line has three points. When playing on this plane the winning strategy for Xeno is identical to the winning strategy on Π_2. (It is interesting to note that playing on this plane is the same as playing on a torus version of tic-tac-toe [21].) If the principle of rationality is violated then the game *could* end in a win for Ophelia, but a draw is impossible since there are no draws on π_3. To show this, assume that a draw *is* possible and let T be the set of five points that meets each line in L without containing any line completely. Let ℓ_1, ℓ_2, and ℓ_3 be the three lines of one of the parallel classes of π_3. Without loss of generality, assume T meets both ℓ_1 and ℓ_3 at two points and ℓ_2 at one point. Let T meet ℓ_1 at points x and y, and ℓ_2 at point z. The line between x and z intersects ℓ_3, say at p. The line between y and z also intersects ℓ_3, say at q. Notice that both of these lines go through z, and ℓ_3 is not parallel to either of these lines. Therefore, we have $p \neq q$ since no two lines intersect in more than one point. Since T intersects ℓ_3 in two points, if p is not in T then q must be in T. So, either line $\{x, z, p\}$ or $\{y, z, q\}$ is in T, which contradicts our assumption of the existence of a draw.

The following theorem summarizes our discussion of the analysis of play on the planes of small order.

THEOREM. *Xeno has a winning strategy on π_2, π_3, and Π_2, and no draw is possible on these planes.*

Weight functions and other planes

When we venture beyond the planes of small order the complexity of the game increases dramatically. The additional points and lines generate a far greater number of possible moves for each player. This prevents an easy move-by-move analysis as we did in the previous section. This is where Erdös comes to our rescue. The two theorems that follow are special cases of a result of Erdös and Selfridge [10] that specifies conditions under which the second player can force a draw in many positional games. Our proofs are a modification of the proof of the Erdös and Selfridge theorem given by Lu [16].

To analyze the game on any plane of order n, we need a way to evaluate the state of the game at any point during play. It would be helpful to assign a number that in some way measures the utility of the state of the game for one of the players. To do this, we define functions that assign values to the state of the game when Ophelia is about to make her ith move. In order to choose the position for O_i from the unclaimed points remaining, she may first wish to consider which line has the best available point. Keep in mind that Ophelia forces a draw if she places one of her marks on every line, thereby blocking every possible winning line for Xeno. So, any line that Ophelia has already blocked can be removed from consideration. Of the unblocked lines remaining, it is most important for Ophelia to block one of the lines with the largest number of Xeno's marks. If we define the value, or weight, of an unblocked line to be 2^{-u} where u is the number of available points on that line, then the lines of greater weight are precisely the lines which have more of Xeno's marks, and are therefore urgent for Ophelia to block. As Ophelia is about to make her ith move, the weight of the game is defined as the sum of the weights of the unblocked lines. The weight of an available point is the sum of the weights of any unblocked lines incident with the point. Lastly, the weight of a pair of available points on an unblocked line is the weight of the line through these points.

Specifically, assume that the current state of play is $[(X_1, \ldots, X_i), (O_1, \ldots, O_{i-1})]$ and that L represents the set of lines. Let L_i be the collection of all lines not blocked by Ophelia at the ith move for Ophelia, with all of the points previously marked by Xeno deleted, that is, $L_i = \{\ell - \{X_1, \ldots, X_i\} \mid \ell \in L,\ \ell \cap \{O_1, \ldots, O_{i-1}\} = \emptyset\}$. So, L_i contains lines or subsets of lines that have not been blocked by Ophelia. Since the number of these collections depends upon both the order of the plane and the progress of play, we will let L_∞ denote the set when no more moves can be made, that is, the game has ended in a win or a draw. Let $P_i = P - \{X_1, \ldots, X_i, O_1, \ldots, O_{i-1}\}$, the set of points available to Ophelia at move O_i. For

$p, q \in P_i$, the weight functions are given by

$$\begin{aligned}
\text{Weight of the game} &= w(L_i) = \sum_{s \in L_i} 2^{-|s|}, \\
\text{Weight at available point } q &= w(q|L_i) = \sum_{s \in L_i, q \in s} 2^{-|s|},
\end{aligned}$$

and

$$\text{Weight at available pair } \{p, q\} = w(p, q|L_i) = 2^{-|s|} \text{ where } \{p, q\} \subseteq s \in L_i\ .$$

Let's look at some examples. Using the same game played on π_3 given in FIGURE 6, L_1 consists of eight lines of cardinality 3 and four partial lines of cardinality 2 (since X_1 has been removed), giving $w(L_1) = 4 \cdot 2^{-2} + 8 \cdot 2^{-3}$. For L_2 the state of the game is $[(X_1, X_2), (O_1)]$, and we eliminate the four lines through O_1 from consideration. Thus, L_2 consists of three lines of cardinality 3, four partial lines of cardinality 2, and one of cardinality 1, giving $w(L_2) = 2^{-1} + 4 \cdot 2^{-2} + 3 \cdot 2^{-3}$. Since there are four lines through any point on π_3, we see that $w(O_1|L_1) = w(X_2|L_1) = 2^{-2} + 3 \cdot 2^{-3}$. Also, $w(X_2, O_1|L_1) = 2^{-3}$. Continuing this example, for L_3 the state of the game is $[(X_1, X_2, X_3), (O_1, O_2)]$, and we eliminate the seven lines through O_1 or O_2 from consideration. We have $w(L_3) = 2 \cdot 2^{-1} + 3 \cdot 2^{-2}$, $w(O_2|L_2) = 2^{-1} + 2 \cdot 2^{-3}$, $w(X_3|L_2) = 2 \cdot 2^{-2} + 2^{-3}$, and $w(X_3, O_2|L_2) = 0$.

Consider the difference in weights between two successive states of the game, $w(L_i) - w(L_{i+1})$. The only change between L_i and L_{i+1} is that Ophelia's ith move and Xeno's $(i+1)$st move have been made. So, the weights of any lines that do not contain O_i and X_{i+1} do not change and will therefore cancel each other out. With only the lines through these two points remaining, the weights of the lines through X_{i+1} must be subtracted from the weights of the lines through O_i in order to find $w(L_i) - w(L_{i+1})$. Since this eliminates the weight of the line that passes through both points, the weight of this line must be added back. Thus, it can be seen that

$$w(L_i) - w(L_{i+1}) = w(O_i|L_i) - w(X_{i+1}|L_i) + w(X_{i+1}, O_i|L_i). \tag{1}$$

The examples given above can be used to demonstrate (1) when $i = 1$ and $i = 2$.

These weight functions enable us to check if we have a draw at any stage of play. First notice that if $\emptyset \in L_i$ then $w(L_i) \geq 2^{-0} = 1$, Xeno has completed a line and thus, has won. On the other hand, if $w(L_i) < 1$ then $\emptyset \notin L_i$, and Xeno has not completed a line. Also, notice that if $w(L_\infty) < 1$ then $\emptyset \notin L_\infty$ and there is a draw. Moreover, these weight functions provide strategies for Xeno and Ophelia that will help us determine the outcome of play on all planes of higher order. Namely, Xeno should minimize $w(L_i) - w(L_{i+1})$ in an attempt to keep the weight of L_j, at any stage j of the game, above 1, whereas Ophelia should maximize this difference in order to drag the overall weight below 1. Hence, by equation (1), Ophelia chooses O_i by maximizing $w(O_i|L_i)$, and Xeno chooses X_{i+1} by maximizing $w(X_{i+1}|L_i) - w(X_{i+1}, O_i|L_i)$. The power and utility of these weight functions is demonstrated in the proof of the following theorem, where the drawing strategy for Ophelia is specified for infinitely many projective planes.

Draw Theorem for Π_n. *Ophelia can force a draw on every projective plane of order n with $n \geq 3$.*

Proof. To prove that Ophelia can force a draw, we must produce an algorithm that prescribes Ophelia's move at any point in the game, and then show that this strategy leads to a draw. As noted above, if $w(L_\infty) < 1$ then Ophelia has forced a draw. This is equivalent to showing

$$\text{there exists } N, \text{ where } \ 1 \cdot \ \ N < \infty, \text{ such that } w(L_N) < 1$$

and

$$w(L_{i+1}) \cdot \ \ w(L_i) \ \text{ for all } \ i \geq N.$$

Suppose that the current state of play is $[(X_1, \ldots, X_i), (O_1, \ldots, O_{i-1})]$, and Ophelia must make her ith move. Since the weight functions assign more weight to lines on which Xeno is closer to winning, Ophelia should choose a point of maximal weight. So, choose $O_i \in P_i$ such that $w(O_i|L_i) = max\{\ w(q|L_i) \ : \ q \in P_i\}$. By the choice of O_i and (1), we see that the second condition is always satisfied since $w(O_i|L_i) \geq w(X_{i+1}|L_i)$.

For a projective plane of order n, L_1 consists of $n+1$ partial lines of cardinality n (once

X_1 is removed) and $(n^2+n+1)-(n+1)$ lines of cardinality $n+1$. So, we have

$$w(L_1) = \sum_{i=1}^{n+1} 2^{-n} + \sum_{i=1}^{n^2} 2^{-(n+1)} \;=\; \frac{n^2+2n+2}{2^{n+1}}.$$

We see that $w(L_1) < 1$ when $n \geq 4$. Thus, Ophelia forces a draw on the projective planes of order $n \geq 4$ by choosing a point of maximum weight at every stage of the game.

For the projective plane of order 3, recall that there are 13 lines with 4 points on each line, and 4 lines through each point. We calculate $w(L_3)$ after providing the strategy for Ophelia's first two moves. Suppose X_1 and O_1 are placed arbitrarily. Xeno places X_2 anywhere. If O_1 is already on the line containing X_1 and X_2, then O_2 should not be placed on this line. If O_1 is not on the line containing X_1 and X_2, then O_2 should be placed on this line. In either case, the configuration before move X_3 is represented by FIGURE 7.

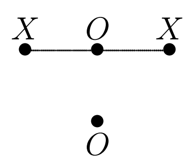

FIGURE 7: Configuration of $[(X_1, X_2), (O_1, O_2)]$ on Π_3

Xeno can place X_3 anywhere, leaving only four possible configurations of points, as represented in FIGURE 8. As long as $w(L_3) < 1$ in each case, then Ophelia has forced a draw.

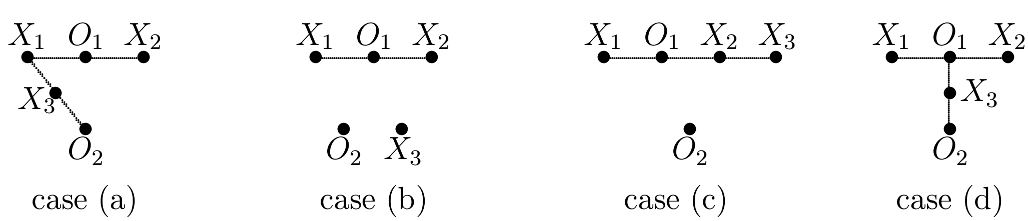

FIGURE 8: Possible configurations for $[(X_1, X_2, X_3), (O_1, O_2)]$ on Π_3

To calculate $w(L_3)$, in all four cases we start by eliminating the four lines containing O_1 and the remaining three lines containing O_2. Once these seven lines are eliminated from consideration, there are only six lines remaining to be included in the weight function.

Case (a): There are two partial lines through X_1 of cardinality 3. There is one partial line of cardinality 3 through X_2, and one partial line through X_2 and X_3 of cardinality 2. Through X_3 there is one remaining line of cardinality 3. Since only 12 out of 13 lines have been considered, there is one line of cardinality 4 remaining. This gives $w(L_3) = 2^{-2} + 4 \cdot 2^{-3} + 2^{-4} = \frac{13}{16}$.

Case (b): There is one partial line through X_1 of cardinality 3, and one partial line through X_1 and X_3 of cardinality 2. The same holds for X_2. All lines through X_3 have been considered. Since only 11 of the 13 lines have been considered, there are two lines of cardinality 4 remaining. This gives $w(L_3) = 2 \cdot 2^{-2} + 2 \cdot 2^{-3} + 2 \cdot 2^{-4} = \frac{14}{16}$.

Case (c): Using similar reasoning, we can show $w(L_3) = 6 \cdot 2^{-3} = \frac{6}{8}$.

Case (d): Likewise, we have $w(L_3) = 2 \cdot 2^{-2} + 3 \cdot 2^{-3} + 2^{-4} = \frac{15}{16}$.

In all possible cases we have $w(L_3) < 1$. Thus, Ophelia can force a draw on Π_3. □

Draws on affine planes

Using the same technique, we can give the drawing strategy for Ophelia on infinitely many affine planes. For an affine plane of order n, L_1 consists of $n+1$ partial lines of cardinality $n-1$ (once X_1 is removed) and $(n^2+n)-(n+1)$ lines of cardinality n. So, we have

$$w(L_1) = \sum_{i=1}^{n+1} 2^{-(n-1)} + \sum_{i=1}^{n^2-1} 2^{-n} = \frac{n^2+2n+1}{2^n}.$$

We see that $w(L_1) < 1$ when $n \geq 6$. Following the same argument as given in the previous proof, we see that Ophelia can force a draw on the affine planes of order $n \geq 7$ (since there is no such plane of order 6).

The only affine planes remaining are π_4 and π_5. It is interesting to note that we found greater difficulty determining the outcome of play on Π_3, π_4 and π_5 than on planes of higher order. While we were able to determine the outcome of play on Π_3 by performing calculations for all possible outcomes by hand, the unsuspected complexity of play on π_4 and π_5 lent itself to analysis by computer.

Ophelia's drawing strategy for π_5 is the same as that given for Π_3. The initial configurations are identical to the cases shown in FIGURE 8, and the weight functions for each case can be calculated as demonstrated in the previous proof. However, in π_5 some of these cases produced too many subcases to be calculated by hand, and a computer was used to verify that $w(L_i)$ was eventually less than 1. The following theorem summarizes these results.

DRAW THEOREM FOR π_n. *Ophelia can force a draw on every affine plane of order n with $n \geq 5$.*

There is only one plane left to consider. What happens on the affine plane of order 4? The following game shows that draws exist on π_4.

X	O	X	X
O	X	O	O
X	O	X	X
O	X	O	O

Since we had no examples of a plane for which a winning strategy and draws coexisted, it was natural to expect that Ophelia could force a draw. To our surprise, three independent computer algorithms show that Xeno has a winning strategy on this plane. The first two programs, written by students J. Yazinski and A. Insogna (University of Scranton), use a tree searching algorithm. The third program, written by I. Wanless (Oxford University), checks all possible games up to isomorphism. Thus, the affine plane of order 4 is the only plane for which Xeno has a winning strategy, and yet, draws exist. Finally, we have answered the two questions that we posed after initially introducing the game.

Answer 1: Xeno has a winning strategy on π_2, Π_2, π_3 and π_4.

Answer 2: Draws exist on π_n where $n \geq 4$, and on Π_n where $n \geq 3$.

Blocking configurations

Suppose you are playing as Ophelia on one of the infinitely many planes for which there is a drawing strategy. The algorithm given in the previous section may guarantee a draw, but it requires computations of Eulerian proportion in order to pick a point of maximum weight at each move. Since any opponent would surely cry foul were you to consult a computer, it could take *hours* to finish a game using this algorithm! The geometry of these planes suggests a more practical solution. We will relate Ophelia's strategy to this geometry in order to demonstrate some configurations (which Ophelia would like to construct) that can produce a draw with very few points. The desired set of points is called a blocking set since every line intersects the set but no line is contained in the set. More information on similar configurations can be found in recent survey articles [6, 13] and the references therein.

First, let us consider the projective plane of order 3. Since it is easier to understand Π_3 by describing P and L rather than giving its graph, take the elements of the following array on the left as the point set of this plane, and the right array as a possible game.

$$\begin{pmatrix} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ & 10 & 11 & 12 & \\ & & 13 & & \end{pmatrix} \qquad \begin{pmatrix} & & X & & \\ & X & X & X & \\ O & O & X & O & O \\ & X & O & X & \\ & & O & & \end{pmatrix}$$

We have simply taken the standard form of π_3 and added $1, 5, 9$ and 13 as the points at infinity. The lines are given by

$$\{2,3,4,13\}, \{6,7,8,13\}, \{10,11,12,13\}, \{2,6,10,1\}, \{3,7,11,1\}, \{4,8,12,1\}, \{2,7,12,9\},$$

$$\{3,8,10,9\}, \{4,6,11,9\}, \{4,7,10,5\}, \{3,6,12,5\}, \{2,8,11,5\}, \text{ and } \{1,5,9,13\}.$$

It is easily checked that the game shown on the right above is a draw. The set of points marked with X, $\{1,2,3,4,7,10,12\}$, and those marked with an O, $\{5,6,8,9,11,13\}$, are both blocking sets. Further inspection shows that these sets have a specific configuration in common. We will focus on Ophelia's blocking set. The line $\ell = \{1,5,9,13\}$ has all but one point labelled with an O. Through at least one of the points on ℓ marked with an O, say 5, there is a line $m = \{2,5,8,11\}$ that also has all but one point marked with an O. We also see that lines ℓ and m contain five of the six points that compose Ophelia's blocking set. The sixth point lies on the line through points 1 and 2, the two points marked with an

X on lines ℓ and m. We can find the same configuration in Xeno's blocking set by taking $\ell' = \{2, 3, 4, 13\}$ and $m' = \{2, 7, 12, 9\}$, which makes 5 the sixth point since it lies on the line through points 9 and 13. Of course, 10 is an extraneous point of the set for Xeno, included for the sake of presenting a complete game.

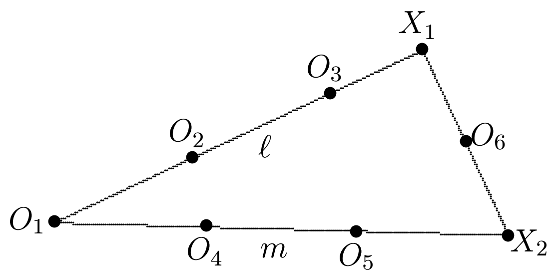

FIGURE 9: Configuration of a draw on Π_3

Interestingly, every draw on Π_3 displays such a configuration, as depicted in FIGURE 9. To show this, assume Ophelia has a produced a draw on Π_3 with the points in the set $A = \{O_1, \ldots, O_6\}$. If no three points of A are on a line, then through O_1 there is a line containing each O_i, $2 \cdot\ i \cdot\ 6$, and these five lines are distinct. However, this is impossible since there cannot be five lines through O_1 on Π_3. (Note also that no four points of A are on a line because then A would contain a line.) Hence, some three points of A are on a line. Without loss of generality, assume that we now have line ℓ as shown in FIGURE 9. X_1 has three lines through it other than ℓ. Each of these lines must have a point claimed by Ophelia since the set A has a point on every line. Hence, each of the remaining three points of A must be incident with exactly one of these lines, and a simple check shows that we have the configuration given above. This blocking configuration is not unique to Π_3. It can be generalized to projective planes of higher order as shown by the following theorem.

BLOCKING SETS ON Π_n THEOREM. *On any projective plane of order n with $n \geq 3$, there exists a blocking set of $2n$ points.*

Proof. This purely geometric result is shown within the game structure by constructing the blocking set. Let ℓ be a line in a projective plane of order $n \geq 3$, with points $q_1, \ldots, q_{n+1}$.

Suppose Ophelia has accumulated $O_i = q_i$ for $i = 1, \ldots, n$. On Π_n, each of these points is incident with $n + 1$ lines. Hence $n^2 + 1$ lines now have an O on them.

Assume Xeno claims q_{n+1}, otherwise Ophelia wins. There are n lines through q_{n+1} other than ℓ. Label these lines $\ell_1, \ldots, \ell_n$, as in FIGURE 10. Choose a line $m \neq \ell$ through q_1 and let O_{n+i} be the intersection of m and ℓ_i for $i = 1, \ldots, n-1$. Let O_{2n} be any point on ℓ_n other than the intersection of m and ℓ_n, otherwise Ophelia wins. Since $n > 2$, we are guaranteed that such a point exists. Finally, we have $\{O_1, O_2, \ldots, O_{2n}\}$ as the required set of $2n$ points since each of the n^2+n+1 lines are incident with a point in this set, and no line is contained in this set. □

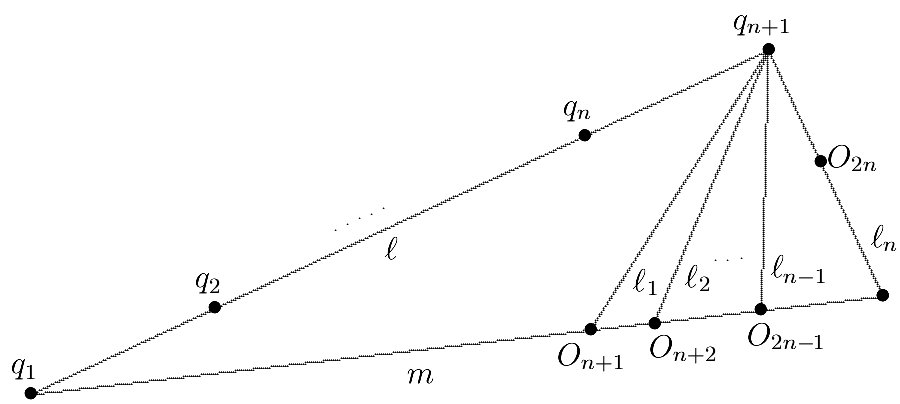

FIGURE 10: Configuration of blocking set on Π_n

The blocking set constructed in the proof translates to a drawing strategy for Ophelia that is free from computation. On a projective plane, Ophelia may attempt to acquire points that display such a configuration. At first consideration, the reader might find this strategy counterintuitive. If Ophelia's goal is to block every line, then how could it make sense to continue to place her marks on lines that are already blocked (ℓ and m)? The answer lies in the geometry of these planes. Since each point is incident with $n + 1$ lines, O_1 blocks $n + 1$ lines. Since there exists a line between O_1 and any other point, O_2 will block n lines regardless of its placement. O_3 will block n lines if it is placed on ℓ, but only $n-1$ lines if not placed on ℓ. By continuing to place her marks on ℓ, Ophelia is maximizing the number of lines blocked by each O_i. The points $q_1, q_2, \ldots, q_n$ claimed by Ophelia, as shown in FIGURE

10, block n^2+1 of the n^2+n+1 lines on Π_n. This is the largest number of lines she can block with n points.

The blocking set on an affine plane of order greater than 4 displays a similar configuration, consisting of $2n-1$ points. To show this, let $\ell_1, \ldots, \ell_n$ be the lines of a parallel class and suppose $\ell_1 = \{q_1, q_2, \ldots, q_n\}$, as in FIGURE 11. Let $O_i = q_i$ for $1, \ldots, n-1$, and assume Xeno claims q_n. Let $\ell_1, m_1, \ldots, m_n$ be the lines through q_n, and let O_{n-1+j} be the point of intersection of m_j and ℓ_{j+1} for $j = 1, \ldots, n-1$. Let O_{2n-1} be any point on m_n that is not collinear with $O_n, \ldots, O_{2n-2}$. This can be done because $n > 4$, that is, there are more than four points on a line.

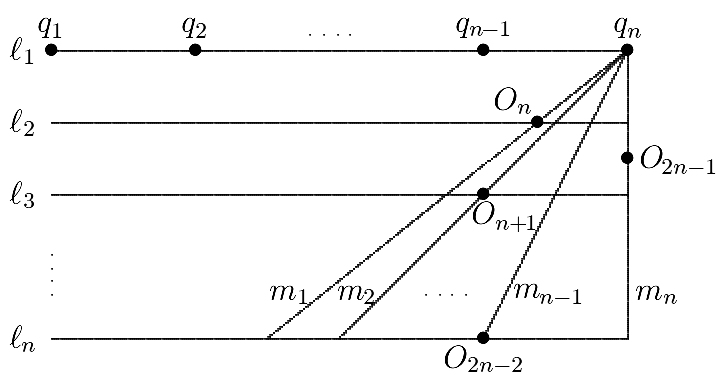

FIGURE 11: Configuration of blocking set on π_n

Notice that $O_n, O_{n+1}, \ldots, O_{2n-1}$ do not lie on a line since this line would have to interesect ℓ_1, which it does not. However it may be possible that $n-1$ of them lie on a line with O_i for $i \cdot \; n-1$. This can be avoided by simply changing the order of lines $\{m_i\}$, which is possible when $n > 4$. Thus, we have $\{O_1, O_2, \ldots, O_{2n-1}\}$ as the required set of $2n-1$ points. This work establishes the following result.

BLOCKING SETS ON π_n THEOREM. *On any affine plane of order n with $n \geq 5$, there exists a blocking set of $2n-1$ points.*

As we can see from these proofs, if Ophelia can place $2n$ or $2n-1$ marks (depending on the type of plane) in the required manner then the game will be a draw. While this offers the second player an easy algorithm to follow, it is not a drawing strategy since it is not guaranteed to produce a draw. If Xeno's best move happens to be a point on the line which was to be part of Ophelia's blocking configuration, then she must begin acquiring points on a different line.

Conclusion

At the University of Scranton we hold an annual single-elimination "Tic-tac-toe on π_4" tournament where students compete for the top spot (and prizes!). It is not uncommon to see the serious competitors practicing for weeks before the contest. We encourage the reader to play too, as we have found that these students gain not only an understanding of affine planes, but also develop an intuition for finite geometries that reveals properties and symmetries not easily seen by reading definitions in a geometry text. For practice on π_4, try playing against a computer [8]. We also welcome a proof of the existence of a winning strategy for Xeno on π_4 that does not rely on a computer. For further reading, surveys of other tic-tac-toe games can be found in Berlekamp, Conway, and Guy [4] and Beck [2].

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