Problèmes d'optimisation

- Introduction
- Existence
- Unicité
- Calcul d'une décomposition orthogonale d'un tenseur symétrique
- Calcul d'une décomposition inversible
- Calcul d'une décomposition générale
- Conclusions

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Définitions

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- $\mathbf{T}$ est *Cubique* quand toutes les dimensions $K_\ell = K$ sont égales
- $\mathbf{T}$ est *Symétrique* s'il est cubique et que ses composantes ne changent pas par permutation *quelconque* des indices

Une fonction objectif très simple

- On veut minimiser l'erreur de modèle:

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- Autre écriture:

$$||\mathbf{T} - \sum_p \lambda_p \, \boldsymbol{a}(p) \otimes \boldsymbol{b}(p) \otimes .. \otimes \boldsymbol{c}(p)||^2$$

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Tensor rank

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- This yields the *Canonical Decomposition* (CanD), sometimes referred to as *Parafac*, or *Outer Product decomposition*.

Connection between symmetric tensors and homogeneous polynomials

- For instance cubics in n variables:

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- **Waring problem** A polynomial of degre d can be decomposed into a sum of r dth-powers of linear forms:

$$p(\mathbf{x}) = \sum_{p=1}^{r} L_p(\mathbf{x})^d$$

Topologie

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Topologie

- L'ensemble des tenseurs de rang au plus r n'est fermé que pour $r \leq 1$ et r maximal (tout l'espace)
- Le rang maximal des tenseurs d'ordre et de dimensions données est très mal connu
- Il faudrait connaitre le rang exact de **T** pouir le décomposer (éviter l'approximation), mais on ne sait pas faire

A sequence of rank-2 tensors converging towards a rank-4

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$$\mathbf{T}_0 = \boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{v} + \boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{v}\otimes\boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{u}$$

which is proportional to the rank-4 symmetric tensor:

$$3\mathbf{T}_0 = 8\,(\boldsymbol{u}+\boldsymbol{v})^{\otimes 4} - 8(\boldsymbol{u}-\boldsymbol{v})^{\otimes 4} - (\boldsymbol{u}+2\boldsymbol{v})^{\otimes 4} + (\boldsymbol{u}-2\boldsymbol{v})^{\otimes 4} \quad (1)$$

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- This is the *maximal rank* of 4th order tensors of dimension 2.

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- In the complex field, all typical ranks are generic
  Tensors with *entries randomly drawn* according to a continuous pdf are generic.
- Importance in practice: noisy measurements

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- Cas des tenseurs non symétriques: rang générique?

Orthogonal decomposition of a symmetric tensor

If **Q** orthogonal, the two problems are equivalent:

1. Direct:

$$\min_{\mathbf{Q},\mathbf{\Lambda}} ||\mathcal{C}_{ijk\ell} - \sum_{p=1}^{P} Q_{ip}\, Q_{jp}\, Q_{kp}\, \Lambda_{ppp}||^2$$

2. Inverse:

$$\min_{\mathbf{Q},\mathbf{\Lambda}} ||\sum_{ijk} Q_{ip}\, Q_{jq}\, Q_{kr}\, \mathcal{C}_{ijk} - \Lambda_{ppp}\, \delta_{pqr}||^2$$

Proof.
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- Existence is guaranteed

Pair sweeping (1)

- If we assume the inverse approach, then one can compute a sequence of best plane rotations,

$$\mathbf{G} \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+\theta^2}} \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

maximizing: $\Upsilon(\mathbf{Q}) \stackrel{\text{def}}{=} \sum_i T_{iii}^2$, with
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  $T_{pqr} = \sum_{ijk} Q_{ip}\, Q_{jq}\, Q_{kr}\, C_{ijk}$.
- At each iteration, we have a single unknown, θ, that can be imposed to lie in $(-1, 1]$.

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At every iteration, the absolute maximum wrt θ is obtained algebraically. Example for 3rd order:

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- Denote $\xi = \theta - 1/\theta$. stationary points are roots of a degre-2 trinomial:

$$\omega(\xi) = d_2\,\xi^2 + d_1\,\xi - 4\,d_2$$

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- θ is obtained from ξ by rooting a 2nd degree trinomial.

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- **Bonne nouvelle:** la solution en θ reste calculable par radicaux pour des tenseurs d'ordre 4 (racines d'un polynôme de degré 4, puis d'un autre de degré 2).

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- **Bonne nouvelle:** la solution en θ reste calculable par radicaux pour des tenseurs d'ordre 4 (racines d'un polynôme de degré 4, puis d'un autre de degré 2).
- **Mauvaise nouvelle:** marche bien en pratique, maus convergence vers le maximum *absolu* n'est toujours pas théoriquement prouvée.

Two different formulations:

1. Direct: look for **A**, **B**, **C**:

$$\min_{\mathbf{A},\mathbf{B},\mathbf{C}} ||T_{ijk} - \sum_{p=1}^{P} A_{ip}\, B_{jp}\, C_{kp}||^2$$

   i.e. decompose **T** into a sum of P rank-one terms

2. Inverse: look for **A**, **B**, **C**:

$$\min_{\mathbf{A},\mathbf{B},\mathbf{C}} \sum_{mnp \neq ppp} |\sum_{ijk} A_{mi}\, B_{nj}\, C_{pk}\ T_{ijk}|^2$$

   i.e. try to diagonalize **T** by linear transforms

Un problème mal posé

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- Approche directe: La triangularisation conjointe permet d'estimer les facteurs orthogonal et triangulaire d'une matrice inversible, l'un après l'autre

Joint triangularization of matrix slices

1. From $\mathbf{T}$, determine a collection of matrices (e.g. matrix slices), $\mathbf{T}[k]$, satisfying $\mathbf{T}[k] = \mathbf{A}\,\mathbf{D}[k]\,\mathbf{B}^{\mathsf{T}}$, $\mathbf{D}[k] \stackrel{\text{def}}{=} \mathbf{diag}\{C_{k,:}\}$.

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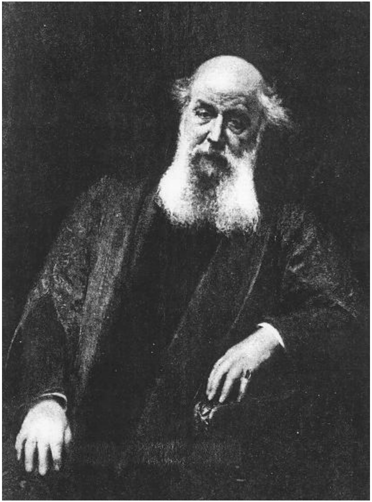
5. Compute matrix $\mathbf{C}$ from $\mathbf{T}$, $\mathbf{A}$ and $\mathbf{B}$ by solving the over-determined linear system $\mathbf{C}\cdot\{(\mathbf{B}\odot\mathbf{A})^\mathsf{T}\} = \mathbf{T}_{K\times JI}$

Binary case (1)



James Joseph Sylvester **(1814–1897)**

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Sylvester's theorem in $\mathbb{R}$

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$$p(x, y) = \sum_{j=1}^{r} \lambda_j \, (\alpha_j \, x + \beta_j \, y)^d \text{ if and only if:}$$

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  2. $q_c(x,y)$ has distinct real roots.
- Valid even in non generic cases.

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Sylvester's theorem in $\mathbb{C}$

- A binary quantic $p(x, y) = \sum_{i=0}^{d} c(i)\, \gamma_i\, x^i\, y^{d-i}$ can be written as a sum of dth powers of r distinct linear forms:

  $p(x, y) = \sum_{j=1}^{r} \lambda_j\, (\alpha_j\, x + \beta_j\, y)^d$ if and only if:

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Sylvester's theorem in $\mathbb{C}$

- A binary quantic $p(x, y) = \sum_{i=0}^{d} c(i)\, \gamma_i\, x^i\, y^{d-i}$ can be written as a sum of dth powers of r distinct linear forms:
  $p(x, y) = \sum_{j=1}^{r} \lambda_j\, (\alpha_j\, x + \beta_j\, y)^d$ if and only if:
  1. there exists a vector $\boldsymbol{g}$ of dimension $r + 1$, with components g_ℓ, such that

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r \\ \vdots & & & \vdots \\ \gamma_{d-r} & \cdots & \gamma_{d-1} & \gamma_d \end{bmatrix} \boldsymbol{g}^* = \mathbf{0}. \qquad (2)$$

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- Valid even in non generic cases.

Algorithm for rth order symmetric tensors of dimension 2

Start with $r = 1$ ($d \times 2$ matrix) and increase r until it looses its column rank

1	2
2	3
3	4
4	5
5	6
6	7
7	8

$\longrightarrow$

1	2	3
2	3	4
3	4	5
4	5	6
5	6	7
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1	2	3	4
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Iterative algorithms for under-determined mixtures

Take 3rd order case to illustrate the reasoning. Define

$$\boldsymbol{p} \stackrel{\text{def}}{=} \begin{bmatrix} \text{vec}\{\mathbf{A}^{\mathsf{T}}\} \\ \text{vec}\{\mathbf{B}^{\mathsf{T}}\} \\ \text{vec}\{\mathbf{C}^{\mathsf{T}}\} \end{bmatrix}, \quad \text{and the gradient } \boldsymbol{g} = \begin{bmatrix} \boldsymbol{g}_{\mathbf{A}} \\ \boldsymbol{g}_{\mathbf{B}} \\ \boldsymbol{g}_{\mathbf{C}} \end{bmatrix}$$

1. Alternate Least Squares (ALS)
2. Gradient update: $\boldsymbol{p}(k+1) = \boldsymbol{p}(k) - \mu(k)\,\boldsymbol{g}(k)$
3. Newton update: $\boldsymbol{p}(k+1) = \boldsymbol{p}(k) - \mathbf{H}(k)^{-1}\,\boldsymbol{g}(k)$
4. LM update: $\boldsymbol{p}(k+1) = \boldsymbol{p}(k) - [\mathbf{J}(k)\mathbf{J}(k)^{\mathsf{H}} + \lambda(k)\mathbf{I}]^{-1}\,\boldsymbol{g}(k)$
5. BFGS update: $\boldsymbol{p}(k+1) = \boldsymbol{p}(k) - [\mathbf{J}(k)\mathbf{J}(k)^{\mathsf{H}} + \mathbf{M}(k)]^{-1}\,\boldsymbol{g}(k)$
6. Conjugate gradient...

ALS (1)

$$\varepsilon = ||\mathbf{T}_{I\times KJ} - \mathbf{A}(\mathbf{C}\odot\mathbf{B})^{\mathsf{T}}||^2 \qquad (3)$$

Advantage: compact writing of the best matrix **A**, for fixed **B** and **C**, since (3) in quadratic in **A** [?]:

$$\hat{\mathbf{A}} = \mathbf{T}_{I\times KJ}\cdot\{(\mathbf{C}\odot\mathbf{B})^{\mathsf{T}}\}^{\dagger}$$

where † denotes pseudo-inverse.

Similarly:

$$\begin{aligned} ||\mathbf{T}_{J\times IK} - \mathbf{B}(\mathbf{A}\odot\mathbf{C})^{\mathsf{T}}||^2 &\rightarrow \hat{\mathbf{B}} = \mathbf{T}_{J\times IK}\cdot\{(\mathbf{A}\odot\mathbf{C})^{\mathsf{T}}\}^{\dagger} \\ ||\mathbf{T}_{K\times JI} - \mathbf{C}(\mathbf{B}\odot\mathbf{A})^{\mathsf{T}}||^2 &\rightarrow \hat{\mathbf{C}} = \mathbf{T}_{K\times JI}\cdot\{(\mathbf{B}\odot\mathbf{A})^{\mathsf{T}}\}^{\dagger} \end{aligned}$$

ALS (2)

Start with arbitrary $\mathbf{B}(0)$ and $\mathbf{C}(0)$
For $k = 1...kmax$,

- $\mathbf{A}(k+1) = \mathbf{T}_{I\times KJ} \cdot \{(\mathbf{C}(k) \odot \mathbf{B}(k))^\mathsf{T}\}^\dagger$
- $\mathbf{B}(k+1) = \mathbf{T}_{J\times IK} \cdot \{(\mathbf{A}(k+1) \odot \mathbf{C}(k))^\mathsf{T}\}^\dagger$
- $\mathbf{C}(k+1) = \mathbf{T}_{K\times JI} \cdot \{(\mathbf{B}(k+1) \odot \mathbf{A}(k+1))^\mathsf{T}\}^\dagger$

Hence the ALS algorithm also needs that:

$$R \leq \min(JK, IK, IJ)$$

According to Kruskal [?], this inequality is always satisfied.

ALS drawbacks

1. Fairly slow convergence when reaching plateaux
2. May be stuck about local minima

Optimal Line Search: reduction to univariate polynomial

The same principle applies to any iterative algorithm [?]:

- Compute a search direction $[\delta\mathbf{A}, \delta\mathbf{B}, \delta\mathbf{C}]$, which can be the gradient $\boldsymbol{g}$, a direction $\mathbf{H}^{-1}\boldsymbol{g}$, or a difference $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}] - [\mathbf{A}(k), \mathbf{B}(k), \mathbf{C}(k)]$...

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- Compute the 6 first coefficients of the 6th degree polynomial $\varepsilon(\mu)$, defined by replacing $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ by $[\mathbf{A} + \mu\,\delta\mathbf{A}, \mathbf{B} + \mu\,\delta\mathbf{B}, \mathbf{C} + \mu\,\delta\mathbf{C}]$

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- Can be executed at every iteration, or less often. Idem for higher orders. Often allows to escape from local minima

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- Il existe des algorithmes pointus, moins coûteux. Mais leur conditions d'utilisation sont plus restrictives (rang notamment).
- Les algorithmes d'optimisation standard les plus bêtes ne peuvent pas être mauvais avec une fonction aussi sympatique qu'un polynôme, mais le pb est la présence de minima locaux.

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