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# BLIND SEPARATION OF CONVOLUTIVE MIXTURES, A PARTIAL JOINT DIAGONALIZATION ( P A J O D ) APPROACH

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MOTS CLÉS : Aveugle, Separation de sources, Egalisation, Tenseur, Cumulants, Indépendance statistique

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ABSTRACT: Blind Separation of convolutive mixtures and Blind Equalization of Multiple-Input Multiple-Output (MIMO) channels are two different ways of naming the same problem, which we address here. The numerical algorithm, subsequently presented in detail, is based on preliminary theoretical results on contrasts. The algorithm consists of Partial Approximate Joint Diagonalization (PAJOD) of several matrices, containing some values of output cumulant multi-correlations.

KEY WORDS : Blind, Source Separation, Equalization, Tensor, Cumulants, Statistical Independence

# Blind Separation of Convolutive Mixtures: A Partial Joint Diagonalization (PAJOD) Approach

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## Abstract

Blind Separation of convolutive mixtures and Blind Equalization of Multiple-Input Multiple-Output (MIMO) channels are two different ways of naming the same problem, which we address here. The numerical algorithm, subsequently presented in detail, is based on preliminary theoretical results on contrasts. The algorithm consists of Partial Approximate Joint Diagonalization (PAJOD) of several matrices, containing some values of output cumulant multi-correlations.

## Résumé

La séparation aveugle de mélanges convolutifs et l'égalisation aveugle de canaux à entrées et sorties multiples (MIMO) sont deux manières différentes de désigner le même problème. L'algorithme numérique, que nous présentons en détail, est basé sur des résultats théoriques préliminaires sur les contrastes. L'algorithme consiste en la Diagonalisation Approximative Partielle Conjointe (PAJOD) de plusieurs matrices, chacune contenant plusieurs valeurs des multi-corrélations des sorties.

# 1 Introduction

Blind equalization (that is, without observing the inputs) of linear time-invariant systems has been studied extensively during the last decade. Single Input Single Output (SISO) equalization often requires High-Order Statistics (HOS) [20] [13] [26]; this can be implicit through constant modulus [25] [14] or constant power [8] criteria. For multiple channels (SIMO or MIMO), HOS can be used [24] [19], but second-order statistics can suffice, provided mild identifiability conditions are satisfied, but a number of limitations have been identified [21] [1] [12] [23] [5] [22]; see [16] and references therein. Because of their robustness to hypotheses, HOS-based methods remain very attractive. However, on-line (time recursive) blind equalization algorithms require long data blocks to converge (typically from 10,000 to 100,000 samples); therefore, it is of interest to devise off-line algorithms able to converge much faster (typically 500 samples), in order to cope with channels with shorter stationarity duration.

The case of static mixtures (as opposed to convolutive) has also retained a lot of attention, because its simpler form allows a deeper treatment, but often requires resorting to HOS. Put in simple words, one can say that the problem can be viewed as diagonalizing a tensor [4] [6], but can be addressed by diagonalizing approximatively a subset of matrix slices [2]. The latter algorithms are efficient when applied to short data records (or fast varying channels) because they are of block type (*i.e.* off-line), but on-line algorithms have also been devised [11] [24].

Our main contribution consists of a block algorithm dedicated to blind MIMO equalization. This algorithm has been shown to maximize a well-defined *contrast* [9] [18], as pointed out in section 3. On-line versions of this algorithm would be easy to implement, and are not studied in this paper. Instead, we concentrate on the description of the block algorithm, aiming at reaching acceptable performances on very short data records (*e.g.*, 200 to 400 symbols). Performances actually obtained will be reported in a subsequent paper.

# 2 Problem and Notation

Consider the following linear time-invariant invertible system:

$$\mathbf{x}(n) = \sum_{k=-\infty}^{\infty} \mathbf{F}(k)\mathbf{s}(n-k) \tag{1}$$

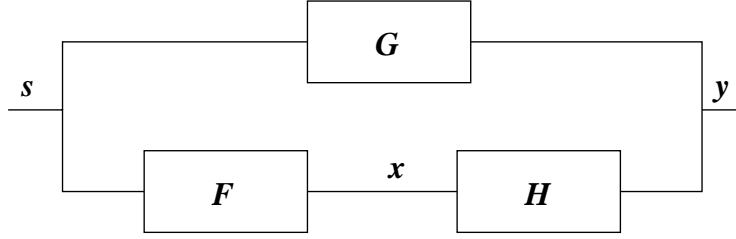


Figure 1: Observation  $\mathbf{x}$  is equalized by  $\mathbf{H}$ ; the global system is denoted  $\mathbf{G}$ .

where  $\mathbf{s}(n)$  denotes the  $N$ -dimensional source vector,  $\mathbf{x}(n)$  is the  $N$ -dimensional observation, and  $\{\mathbf{F}(n), n \in \mathbb{Z}\}$  denotes the  $N \times N$  impulse response matrix sequence. For convenience, vectors and matrices are denoted with bold lowercase and bold uppercase letters, respectively. Throughout the paper,  $(^\top)$  stands for transposition,  $(^H)$  for conjugate transposition, and  $(^*)$  for complex conjugation. Also denote by  $\mathbb{Z}$  the set of integers, and by  $\mathbb{N}$  the subset of positive integers, and by  $\check{\mathbf{G}}(z)$  the  $Z$ -transform of the time sequence  $\mathbf{G}(n)$ :

$$\check{\mathbf{G}}(z) \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} \mathbf{G}(k) z^{-k}$$

The problem consists of finding a filter  $\{\mathbf{H}(n), n \in \mathbb{Z}\}$  from the sole observation of the channel outputs,  $\mathbf{x}(n)$ , aiming at delivering an estimate  $\mathbf{y}(n)$  of the inputs  $\mathbf{s}(n)$ .

With this goal, the following hypotheses are assumed:

- A1.** Sources  $s_i(n)$ ,  $i \in \{1, \dots, N\}$  are mutually independent i.i.d. zero-mean processes, with unit variance.
- A2.**  $\mathbf{s}(n)$  is stationary up to the considered order,  $r$ ,  $r \leq 3$ , e.g.  $\forall i \in \{1, \dots, N\}$ , the order- $r$  marginal cumulants,

$$C_p^q[s_i] \stackrel{\text{def}}{=} \text{Cum}[\underbrace{s_i(n), \dots, s_i(n)}_{p \text{ terms}}, \underbrace{s_i^*(n), \dots, s_i^*(n)}_{q=r-p \text{ terms}}]$$

do not depend on  $n$ ; for simplicity,  $C_r^0[s_i]$  will be subsequently denoted  $C_r[s_i]$ , in accordance with the usual practice. For definitions of cumulants, refer to [17] and references therein.

**A3.** At most one source has a zero marginal cumulant of order  $r$ .

**A4.** The global transfer matrix,  $\check{\mathbf{G}}(z) = \check{\mathbf{H}}(z)\check{\mathbf{F}}(z)$ , satisfies the property

$$\check{\mathbf{G}}(z) \check{\mathbf{G}}^H(1/z^*) = \mathbf{I}$$

where  $\mathbf{I}$  denotes the  $N \times N$  identity matrix; in other words,  $\check{\mathbf{G}}(z)$  is *para-unitary*. An equivalent characterization in time domain is that

$$\sum_{\ell} \mathbf{G}(\ell) \mathbf{G}^H(\ell + \tau) = \mathbf{I} \delta(\tau) \quad (2)$$

where  $\delta(\tau)$  denotes the Dirac delta function ( $\delta(\tau) = 0$  if and only if  $\tau = 0$ ).

**Remark 1.** More generally, if sources are non i.i.d. linear processes, our approach holds valid. It suffices to still assume **A1** in a first stage in order to equalize the channel, and to extract the original sources in a second stage by linear regression between each equalizer output and the observations. In fact, the equalizer outputs are in that case the driving processes of the sources.

**Remark 2.** The Hypothesis **A4** is not restrictive. Indeed, one can always whiten the observations (in a non unique manner), by using a filter that factorizes the second-order power spectrum.

### 3 Contrasts

The results stated in this section show how contrast-based blind MIMO equalization can be posed in terms of a Partial Joint Approximate Diagonalization (PAJOD) of a set of cumulant matrices. For the sake of clarity, we shall subsequently consider only cumulants of order  $r = 4$ . But principles hold for orders 3 and higher [9].

#### 3.1 Definitions

To start with, denote the following cumulants:

$$C_4^{0,\mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}] = \text{Cum}[y_i(n), y_i(n), y_{j_1}(n - \ell_1), y_{j_2}(n - \ell_2)] \quad (3)$$

$$C_3^{1,\mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}] = \text{Cum}[y_i(n), y_i(n)^*, y_{j_1}(n - \ell_1), y_{j_2}(n - \ell_2)] \quad (4)$$

and

$$C_2^{2,\mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}] = \text{Cum}[y_i(n), y_i(n)^*, y_{j_1}(n - \ell_1), y_{j_2}(n - \ell_2)^*] \quad (5)$$

where  $\mathbf{j} = (j_1, j_2)$ , and  $\boldsymbol{\ell} = (\ell_1, \ell_2)$ . Also define  $\mathbf{J} = \{1, 2, \dots, N\}^2$ , and  $\mathbf{L}$  a subset of  $\mathbf{Z}^2$ ; unless otherwise specified,  $\mathbf{L} = \mathbf{Z}^2$ .

**Trivial filters.** Clearly, the blind equalization problem we have stated contains inherent indeterminacies. In fact, the set  $\mathcal{S}$  of source processes is characterized by assumptions, such as **A1**. One defines the set  $\mathcal{T}$  of *trivial filters*, as containing all filters that do not affect these assumptions. In other words,  $\mathcal{S}$  is stable by the operation of  $\mathcal{T}$ . For instance, filters of the form  $\boldsymbol{\Lambda}(z) \cdot \mathbf{P}$ , where  $\mathbf{P}$  is a permutation matrix, and  $\boldsymbol{\Lambda}(z)$  a diagonal filter, do not affect mutual independence between components of  $\mathbf{s}(n)$ . If in addition  $\mathbf{s}(n)$  is an i.i.d. non Gaussian process,  $\boldsymbol{\Lambda}(z)$  should contain only pure delays, integer multiples of the sampling period, and pure scale factors; in other words, the entries of  $\boldsymbol{\Lambda}(z)$  are of the form  $\lambda z^k$ ,  $k \in \mathbf{Z}$ .

**Contrasts.** Let  $\mathcal{H}$  be a set of filters, and denote  $\mathcal{H} \cdot \mathcal{S}$  the set of processes obtained by operation of filters of  $\mathcal{H}$  on processes of  $\mathcal{S}$ . An optimization criterion,  $\Upsilon(\mathbf{H}; \mathbf{x})$ , will be referred to as a contrast defined on  $\mathbf{H} \in \mathcal{H}$ ,  $\mathbf{x} \in \mathcal{H} \cdot \mathcal{S}$ , if it satisfies the three properties below [5]:

- P1. Invariance:** The contrast should not change within the set of acceptable solutions, which means that  $\Upsilon(\mathbf{H}; \mathbf{x}) = \Upsilon(\mathbf{I}; \mathbf{x}), \forall \mathbf{H} \in \mathcal{T}, \forall \mathbf{x} \in \mathcal{H} \cdot \mathcal{S}$ .
- P2. Domination:** If sources are already separated, any filter should decrease the contrast. In other words,  $\forall \mathbf{x} \in \mathcal{S}, \forall \mathbf{H} \in \mathcal{H}$ , then  $\Upsilon(\mathbf{H}; \mathbf{x}) \leq \Upsilon(\mathbf{I}; \mathbf{x})$ .
- P3. Discrimination:** The maximum contrast should be reached only for filters linked to each other via trivial filters:  $\forall \mathbf{x} \in \mathcal{S}, \Upsilon(\mathbf{H}; \mathbf{x}) = \Upsilon(\mathbf{I}; \mathbf{x}) \Rightarrow \mathbf{H} \in \mathcal{T}$ .

In the remaining, and in accordance with assumptions **A1** thru **A4**,  $\mathcal{H}$  will denote the set of para-unitary filters, and  $\mathcal{S}$  the set of i.i.d. processes with mutually independent components. As a consequence,  $\mathcal{H} \cdot \mathcal{S}$  is the set of standardized linear processes (*i.e.*, second-order white with unit covariance). Lastly, trivial filters of  $\mathcal{T}$  are of the form  $\boldsymbol{\Lambda}(z) \cdot \mathbf{P}$ , where  $\mathbf{P}$  is a permutation, and  $\boldsymbol{\Lambda}(z)$  a diagonal filter, whose entries are of the form  $\lambda z^k$ , with  $k \in \mathbf{Z}$  and  $|\lambda| = 1$ .

### 3.2 Particular contrasts proposed

We are now in a position to prove the proposition below [9] [23]:

**Proposition 1** *The functional*

$$\mathcal{J}_4^0(\mathbf{H}; \mathbf{x}) = \sum_{i=1}^N \sum_{j \in \mathcal{J}} \sum_{\ell \in \mathcal{L}} |\mathcal{C}_4^{0, \mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}]|^2 \quad (6)$$

is a contrast over  $\mathbf{x} \in \mathcal{H} \cdot \mathcal{S}$  and  $\mathbf{H} \in \mathcal{H}$ ; in other words, when observations  $\mathbf{x}(n)$ , and hence the outputs  $\mathbf{y}(n)$  of the para-unitary equalizer, are standardized.

*Proof.* Let us then prove proposition 1. The input-output relation of the global system is

$$y_i(n) = \sum_{q, m} G_{iq}(m) s_q(n - m).$$

Thus, using the multilinearity of cumulants and the definition of  $\mathcal{J}_4^0$ , we get:

$$\mathcal{J}_4^0 = \sum_i \sum_{j_1, j_2} \sum_{\ell_1, \ell_2} \left| \sum_{q, m} \sum_{q', m'} \sum_{k_1, p_1} \sum_{k_2, p_2} G_{iq}(m) G_{i'q'}(m') G_{j_1 k_1}(p_1) G_{j_2 k_2}(p_2) \right. \\ \left. \text{Cum}[s_q(n - m), s_{q'}(n - m'), s_{k_1}(n - \ell_1 - p_1), s_{k_2}(n - \ell_2 - p_2)] \right|^2$$

Since  $s_i(n)$  are i.i.d. (assumption **A1**), the only non-zero cumulants are obtained for  $m = m' = \ell_1 + p_1 = \ell_2 + p_2$ . next, since  $s_i(n)$  are mutually independent, non-zero terms also need that  $q = q' = k_1 = k_2$ . Deleting the null terms, and expanding the squared modulus yields:

$$\mathcal{J}_4^0 = \sum_i \sum_{j_1, j_2} \sum_{\ell_1, \ell_2} \sum_{q, m} \sum_{q', m'} G_{iq}^2(m) G_{i'q'}^{2*}(m') \\ G_{j_1 q}(m - \ell_1) G_{j_1 q'}^*(m' - \ell_1) G_{j_2 q}(m - \ell_2) G_{j_2 q'}^*(m' - \ell_2) \mathcal{C}_4[s_q] \mathcal{C}_4[s_{q'}]^*$$

Yet, since  $\mathbf{G} \in \mathcal{H} \cdot \mathcal{S}$  is standardized, it satisfies (2), and in particular, its columns are orthogonal and of unit modulus [5], which means:

$$\sum_{j, \ell} G_{jq}(k - \ell) G_{j'q'}^*(k' - \ell) = \delta_{qq'} \delta_{kk'} \quad (7)$$

Applying this property to the pairs of indices  $(j_1, \ell_1)$  and  $(j_2, \ell_2)$ , we get:

$$\mathcal{J}_4^0 = \sum_i \sum_{q, m} |G_{iq}^2(m)|^2 |\mathcal{C}_4[s_q]|^2$$

Last, from (7), we have in particular [5] [23]:  $\sum_{k,i} |G_{ij}(k)|^4 \leq 1$  which eventually yields  $\mathcal{J}_4^0 \leq \sum_i |C_4[s_i]|^2$  which proves that  $\mathcal{J}_4^0(\mathbf{H}; \mathbf{x}) \leq \mathcal{J}_4^0(\mathbf{I}; \mathbf{x})$ , for any  $\mathbf{G} \in \mathcal{H}$  and  $\mathbf{s} \in \mathcal{S}$ . Equality holds if and only if  $\sum_{k,i} |G_{ij}(k)|^4 = 1$ , which is possible only for trivial filters.  $\diamond$

Now denote the cumulant tensor of observations:

$$\mathbf{T}_{\mathbf{a},\mathbf{b}}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} \text{Cum}[x_{a_1}(n - \alpha_1), x_{a_2}(n - \alpha_2), x_{b_1}(n - \beta_1), x_{b_2}(n - \beta_2)] \quad (8)$$

where  $\mathbf{a}$ ,  $\boldsymbol{\alpha}$ ,  $\mathbf{b}$  and  $\boldsymbol{\beta}$  are vectors of size 2. The entries of  $\mathbf{a}$  and  $\mathbf{b}$  belong to  $\{1, \dots, N\}$ , by construction.

Consider a FIR equalizer  $\{\mathbf{H}(n), 0 \leq n \leq L - 1\}$ . The range of variation of  $\boldsymbol{\beta}$  is left unspecified for the moment, whereas that of  $\boldsymbol{\alpha}$  is set to  $\{0, 1, \dots, L - 1\}^2$ . The reasons for this choice will become clear in the proof of proposition 2. This tensor can be stored in a set of  $NL \times NL$  matrices, denoted  $\mathbf{M}(\mathbf{b}, \boldsymbol{\beta})$ . In fact, for any fixed  $(\mathbf{b}, \boldsymbol{\beta})$ , the entries of these matrices are given by:

$$\mathbf{M}_{\eta\mu}(\mathbf{b}, \boldsymbol{\beta}) = \mathbf{T}_{\mathbf{a},\mathbf{b}}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad (9)$$

with

$$\eta = \alpha_1 N + a_1, \quad \mu = \alpha_2 N + a_2$$

In short, we shall denote this matrix storage by

$$\mathbf{M}(\mathbf{b}, \boldsymbol{\beta}) \equiv \mathbf{T}_{\mathbf{a},\mathbf{b}}^0(\boldsymbol{\alpha}, \boldsymbol{\beta})$$

in the sequel. Defining  $\|\mathbf{Diag}\{\mathbf{A}\}\|^2 = \sum_i |A_{ii}|^2$ , we have the following

**Proposition 2** *The contrast  $\mathcal{J}_4^0$  can be rewritten as a PAJOD criterion of a set of  $NL \times NL$  matrices:*

$$\mathcal{J}_4^0(\mathbf{H}; \mathbf{x}) = \sum_{\mathbf{b}} \sum_{\boldsymbol{\gamma}} \|\mathbf{Diag}\{\mathbb{H} \mathbf{M}(\mathbf{b}, \boldsymbol{\gamma}) \mathbb{H}^T\}\|^2 \quad (10)$$

where  $\mathbb{H}$  is  $N \times NL$  semi-unitary, i.e., satisfies  $\mathbb{H}\mathbb{H}^H = \mathbf{I}$ , and  $\mathbf{M}(\mathbf{b}, \boldsymbol{\gamma})$  is defined as in (9). Here,  $\mathbf{b}$  varies in  $\mathbf{J} = \{1, \dots, N\}^2$  and  $\boldsymbol{\gamma}$  in  $\mathbf{Z}^2$ .

*Proof.* The relations between equalizer inputs and outputs can be written as:

$$\begin{aligned} C_4^{0,\mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}] &= \sum_{\mathbf{a}, \mathbf{b}} \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} H_{ia_1}(\alpha_1) H_{ia_2}(\alpha_2) H_{j_1 b_1}(\beta_1) \\ &\quad H_{j_q b_2}(\beta_2) T_{\mathbf{a}, \mathbf{b}}(\boldsymbol{\alpha}, \boldsymbol{\beta} + \boldsymbol{\ell}). \end{aligned} \quad (11)$$

Yet, the para-unitary condition **A4** on  $\check{\mathbf{G}}(z)$  yields that  $\check{\mathbf{H}}(z)$  is itself para-unitary, which yields the same orthogonality property as (7):

$$\sum_{j\ell} H_{jr}^*(\tau + \ell) H_{jr'}(\tau' + \ell) = \delta_{rr'} \delta_{\tau\tau'}. \quad (12)$$

Thus, taking the square modulus of (11), making the change of variables  $\gamma_k = \beta_k + \ell_k$ , and eliminating the unuseful indices leads to

$$\begin{aligned} \mathcal{J}_4^0 &= \sum_{iaa'\alpha\alpha'bb'\gamma\gamma'} H_{ia_1}(\alpha_1) H_{ia_2}(\alpha_2) H_{ia'_1}^*(\alpha'_1) H_{ia'_2}^*(\alpha'_2) \\ &\quad \cdot T_{\mathbf{a}, \mathbf{b}}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) T_{\mathbf{a}', \mathbf{b}'}^*(\boldsymbol{\alpha}', \boldsymbol{\gamma}') \delta(\mathbf{b} - \mathbf{b}') \delta(\boldsymbol{\gamma} - \boldsymbol{\gamma}'), \end{aligned} \quad (13)$$

which can be rearranged into

$$\mathcal{J}_4^0(\mathbf{H}; \mathbf{x}) = \sum_{ib\boldsymbol{\gamma}} \left| \sum_{a\boldsymbol{\alpha}} H_{ia_1}(\alpha_1) H_{ia_2}(\alpha_2) \cdot T_{\mathbf{a}, \mathbf{b}}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) \right|^2.$$

Lastly, grouping indices  $a_j$  and  $\alpha_j$  together in a single index  $\mathbf{p}_j$ , one can remark that the  $L$  matrices  $\mathbf{H}(\alpha)$  can be stored in a  $N \times NL$  matrix,  $\mathbb{H}$

$$\mathbb{H} \stackrel{\text{def}}{=} [\mathbf{H}(0), \mathbf{H}(1), \dots, \mathbf{H}(L-1)],$$

with full compatibility with (9), so that eventually  $\mathcal{J}_4^0 = \sum_{ib\boldsymbol{\gamma}} \left| \sum_{\mathbf{p}_1 \mathbf{p}_2} \mathbb{H}_{i\mathbf{p}_1} \mathbb{H}_{i\mathbf{p}_2} M_{\mathbf{p}_1 \mathbf{p}_2}(\mathbf{b}, \boldsymbol{\gamma}) \right|^2$ . Here the para-unitary property of  $\mathbf{H}(\tau)$  implies that  $\mathbb{H}\mathbb{H}^H = \mathbf{I}_N$ .  $\diamond$

**Remark 3.** The para-unitarity of  $\check{\mathbf{H}}(z)$  implies that  $\mathbb{H}$  is semi-unitary, but the reverse is not true. In other words, only part of the information is exploited.

Next, define the functionals

$$\mathcal{J}_3^1(\mathbf{H}; \mathbf{x}) = \sum_{i=1}^N \sum_{\mathbf{j} \in \mathcal{J}} \sum_{\boldsymbol{\ell} \in \mathcal{L}} |C_3^{1,\mathbf{y}}[i, \mathbf{j}, \boldsymbol{\ell}]|^2$$

and

$$\mathcal{J}_2^2(\mathbf{H}; \mathbf{x}) = \sum_{i=1}^N \sum_{j \in \mathcal{J}} \sum_{\ell \in \mathcal{L}} |\mathcal{C}_2^{2,\mathbf{y}}[i, \mathbf{j}, \ell]|^2$$

where  $\mathcal{C}_p^{q,\mathbf{y}}[i, \mathbf{j}, \ell]$  are defined in (4) and (5). Then it can be proved that  $\mathcal{J}_3^1$  and  $\mathcal{J}_2^2$  are contrasts, and that they are also both of PAJOD type:

**Proposition 3** *The contrasts  $\mathcal{J}_3^1$  and  $\mathcal{J}_2^2$  can be rewritten as PAJOD criteria:*

$$\mathcal{J}_3^1(\mathbf{H}; \mathbf{x}) = \sum_{\mathbf{b}} \sum_{\gamma} \|\mathbf{Diag}\{\mathbb{H} \mathbf{M}(\mathbf{b}, \gamma) \mathbb{H}^{\mathbb{H}}\}\|^2 \quad (14)$$

where  $\mathbf{M}(\mathbf{b}, \gamma)$  is defined as, with the same storage as in (9):

$$\mathbf{M}(\mathbf{b}, \gamma) \equiv \mathbf{T}_{\mathbf{a}, \mathbf{b}}^1(\boldsymbol{\alpha}, \gamma) \stackrel{\text{def}}{=} \text{Cum}[x_{a_1}(n - \alpha_1), x_{a_2}(n - \alpha_2)^*, x_{b_1}(n - \gamma_1), x_{b_2}(n - \gamma_2)]$$

and

$$\mathcal{J}_2^2(\mathbf{H}; \mathbf{x}) = \sum_{\mathbf{b}} \sum_{\gamma} \|\mathbf{Diag}\{\mathbb{H} \mathbf{N}(\mathbf{b}, \gamma) \mathbb{H}^{\mathbb{H}}\}\|^2 \quad (15)$$

with

$$\mathbf{N}(\mathbf{b}, \gamma) \equiv \mathbf{T}_{\mathbf{a}, \mathbf{b}}^2(\boldsymbol{\alpha}, \gamma) \stackrel{\text{def}}{=} \text{Cum}[x_{a_1}(n - \alpha_1), x_{a_2}(n - \alpha_2)^*, x_{b_1}(n - \gamma_1), x_{b_2}(n - \gamma_2)^*]$$

where  $\mathbb{H}$  is semi-unitary,  $\mathbf{b}$  varies in  $\{1, \dots, N\}^2$ , and  $\gamma$  in  $\mathbb{Z}^2$ .

The proof goes along the same lines as that of proposition 2.

**Remark 4.** The above criterion  $\mathcal{J}_2^2$  differs from that proposed in [2] in several respects: (i) the matrices  $\mathbf{N}(\mathbf{b}, \gamma)$  are built completely differently, because of the convolutive model, (ii) the matrix sought for is not square unitary but rectangular, which involves quite different calculations, as will be subsequently seen.

**Proposition 4** *If the equalizer is of finite length  $L$ , and the channel of finite length  $M$ , then contrasts  $\mathcal{J}_{4-q}^q$ , defined in propositions 2 and 3, can be rewritten as a PAJOD criteria of a finite set of at most  $(2M + L - 2)^2 N^2$  matrices, where  $\mathbb{H}$  is semi-unitary,  $\mathbf{b}$  varies in  $\{1, \dots, N\}^2$ , and  $\gamma$  in  $\{-M + 1, \dots, M + L - 2\}^2$ .*

**Lemma 5** *If channel and equalizer are both of finite length  $M$  and  $L$ , respectively, then the cumulant tensors  $\mathbf{T}_{\mathbf{a}, \mathbf{b}}^q(\boldsymbol{\alpha}, \gamma)$ ,  $q \in \{0, 1, 2\}$ , are null whenever an entry  $\gamma_k$  of  $\gamma$  falls outside the interval  $\{-M + 1, \dots, M + L - 2\}$ .*

*Proof.* In fact, propositions 2 or 3 still apply. Consider proposition 2 for instance ( $q = 0$ ), and let's prove the lemma. From definition (8) and input-output channel equations  $x_i(n) = \sum_{qm} F_{iq}(m) s_q(n - m)$ , we get by multi-linearity of cumulants:

$$\mathbf{T}_{\mathbf{a}, \mathbf{b}}^0(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = \sum_{i,j=0}^{M-1} \sum_{\boldsymbol{\ell}} \sum_{u,v=1}^N \sum_{\mathbf{w} \in \mathcal{J}} F_{a_1 u}(i) F_{a_2 v}(j) F_{b_1 w_1}(\ell_1) F_{b_2 w_2}(\ell_2)$$

$$\text{Cum}[s_u(t - \alpha_1 - i), s_v(t - \alpha_2 - j), s_{w_1}(t - \gamma_1 - \ell_1), s_{w_2}(t - \gamma_2 - \ell_2)],$$

with  $\boldsymbol{\ell} \in \{0, M - 1\}^2$ . Yet, from **A1**,  $s_u(n)$  are i.i.d. processes, and the expression is null unless  $\alpha_1 + i = \alpha_2 + j = \gamma_1 + \ell_1 = \gamma_2 + \ell_2$ . Next from **A1**,  $s_u(n)$  are mutually independent, so that the expression is also null unless  $u = v = w_1 = w_2$ . This yields

$$\mathbf{T}_{\mathbf{a}, \mathbf{b}}^0(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = \sum_{i=0}^{M-1} \sum_{u=1}^N F_{a_1 u}(i) F_{a_2 u}(i + \alpha_1 - \alpha_2) F_{b_1 u}(i + \alpha_1 - \gamma_1)$$

$$F_{b_2 u}(i + \alpha_1 - \gamma_2) \mathbf{C}_4[s_u]$$

since the support of  $\mathbf{F}(\cdot)$  is  $\{0, 1, \dots, M - 1\}$ , the above quantity is null outside the intervals  $0 \leq i + \alpha_1 - \gamma_k \leq M - 1, \forall k, 1 \leq k \leq 2$ . The fact that  $0 \leq \alpha_1 \leq L - 1$  proves eventually the lemma. The proof is very similar in the case of proposition 3 ( $q = 1$  or  $q = 2$ ). The proposition 4 then directly follows.  $\diamond$

**Remark 5.** In practice, it is sufficient to vary the entries  $\gamma_k$  in the central third of the set  $\{-M + 1, \dots, M + L - 2\}$ , namely  $\{0, 1, \dots, L - 1\}$ .

## 4 Numerical algorithms

The goal of this section is to demonstrate that the computation of the equalizer can be carried out within a limited (polynomial) number of operations.

From now on, we shall assume that (i) the channel length  $M$  is known, (ii) the equalizer has the same length  $L = M$ , and (iii)  $\mathcal{L} = \{0, 1, \dots, L - 1\}^2$ . The robustness with respect to these assumptions will be investigated in a companion paper.

The propositions of the previous section teach us that a semi-unitary matrix,  $\mathbb{H}$ , of size  $N \times NL$ , must be found, which should diagonalize approximately and jointly the set of  $N^2 L^2$  matrices,  $\mathbf{M}(b_1, b_2, \gamma_1, \gamma_2)$ . Each of these matrices is of size  $NL \times NL$ . The goal is to maximize the sum of the squared moduli of the  $N$  first diagonal entries of the  $N^2 L^2$  matrices, as depicted in figure 2.

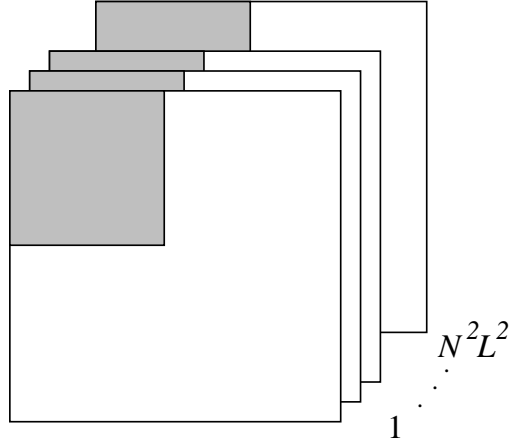


Figure 2: The semi-unitary matrix  $\mathbb{H}$  aims at diagonalizing jointly the  $N \times N$  leading sub-matrices.

#### 4.1 Jacobi sweeping

In order to reach this goal, one looks for a  $NL \times NL$  unitary matrix,  $\mathbf{U}$ , whose  $\mathbb{H}$  will be the the leading  $N \times NL$  submatrix. This unitary matrix can be built by accumulating Givens rotations, as proposed in the Jacobi algorithm [4] [15]:

$$\mathbf{U} = \prod_{1 \leq i < j \leq NL} \Theta[i, j],$$

where  $\Theta[i, j]$  coincides with the identity matrix except for 4 entries, namely:

$$\Theta[i, j]_{ii} = \Theta[i, j]_{jj} = \cos(\theta[i, j])$$

and

$$\Theta[i, j]_{ji} = -\Theta[i, j]_{ij}^* = \sin(\theta[i, j]) e^{j\psi[i, j]}.$$

This rotation can indeed always be imposed to have a real cosine [15] [4]. The cosine,  $c$ , and sine,  $s$ , must be determined so as to maximize, successively for every pair  $[i, j]$ :

$$\mathcal{J}^1 = \sum_{\mathbf{b}, \boldsymbol{\beta}} \sum_{k=1}^N \left| \sum_{\eta, \mu=1}^{NL} \Theta_{\eta k}^*[i, j] \Theta_{\mu k}[i, j] M_{\eta\mu}(\mathbf{b}, \boldsymbol{\beta}) \right|^2 \quad (16)$$

or, depending on the contrast chosen:

$$\mathcal{J}^0 = \sum_{\mathbf{b}, \boldsymbol{\beta}} \sum_{k=1}^N \left| \sum_{\eta, \mu=1}^{NL} \Theta_{\eta k}[i, j] \Theta_{\mu k}[i, j] M_{\eta\mu}(\mathbf{b}, \boldsymbol{\beta}) \right|^2 \quad (17)$$

## 4.2 Processing every pair

As depicted by the grey areas in figure 2, indices  $[i, j]$  do not describe all possible pairs from the set  $\{1, \dots, NL\}^2$ . In fact, since  $k \leq N$ , it suffices that  $i \leq N$ ; in addition, we also have that  $i < j$ . As a consequence, two cases must be distinguished, depending on the fact that  $j \leq N$  or not.

In the two cases, we have to find the roots of polynomials (stationary points of a contrast, a rational function in the unknown). Denote in this section  $c = \cos(\theta[i, j])$  and  $s = \sin(\theta[i, j]) e^{j\psi[i, j]}$ :

$$\Theta[i, j] = \begin{pmatrix} c & -s^* \\ s & c \end{pmatrix}$$

and drop provisionally the dependence of  $M_{ij}$  on  $(\mathbf{b}, \boldsymbol{\beta})$  for the sake of convenience.

- **Case  $j \leq N$ :** One maximizes the sum of the 2 diagonal terms on which one has some action. For  $\mathcal{J}_3^1$  or  $\mathcal{J}_2^2$ , this is a classical expression [3]:

$$\mathcal{J}^1 = \sum_{\mathbf{b}, \boldsymbol{\beta}} |c^2 M_{ii} + cs^* M_{ji} + cs M_{ij} + ss^* M_{jj}|^2 + |ss^* M_{ii} - cs^* M_{ji} - cs M_{ij} + c^2 M_{jj}|^2,$$

whereas for  $\mathcal{J}_4^0$ , the criterion to be maximized takes the form:

$$\mathcal{J}^0 = \sum_{\mathbf{b}, \boldsymbol{\beta}} |c^2 M_{ii} + cs M_{ji} + cs M_{ij} + s^2 M_{jj}|^2 + |s^{*2} M_{ii} - cs^* M_{ji} - cs^* M_{ij} + c^2 M_{jj}|^2.$$

- **Case  $j > N$ :** here only the first diagonal term should be maximized, so that:

$$\mathcal{J}^1 = \sum_{\mathbf{b}, \boldsymbol{\beta}} |c^2 M_{ii} + cs^* M_{ji} + cs M_{ij} + ss^* M_{jj}|^2,$$

and

$$\mathcal{J}^0 = \sum_{\mathbf{b}, \boldsymbol{\beta}} |c^2 M_{ii} + cs M_{ji} + cs M_{ij} + s^2 M_{jj}|^2,$$

with appropriate definitions of matrices  $\mathbf{M}(\mathbf{b}, \boldsymbol{\beta})$ .

### 4.2.1 Real framework

One considers in this section real data, channel, and equalizer. In this situation, all three criteria  $\mathcal{J}^q, q \in \{0, 1, 2\}$ , coincide. In the first case ( $j \leq N$ ), with the help of a change of variables, this rooting can be converted into the solving of two trinomials of degree 2, as in [3]. This transformation is not possible in the second case ( $j > N$ ), and the (still analytical) rooting of the fourth degree polynomial is mandatory.

The contrast criterion for a pair contains two parts,  $\Phi_1$  and  $\Phi_2$ , which can be expressed as a rational function of  $t = \tan \theta$ :

$$\Phi_1 = (1 + t^2)^{-2} \sum_{i=0}^4 d_i t^i, \quad \Phi_2 = (1 + t^2)^{-2} \sum_{i=0}^4 d_{4-i} (-t)^i$$

with

$$\begin{aligned} d_0 &= \sum M_{ii}^2 \\ d_1 &= 2 \sum M_{ii}(M_{ij} + M_{ji}) \\ d_2 &= 2 \sum M_{ii}M_{jj} + \sum (M_{ij} + M_{ji})^2 \\ d_3 &= 2 \sum (M_{ij} + M_{ji})M_{jj} \\ d_4 &= \sum M_{jj}^2 \end{aligned}$$

**Case  $j > N$ :** We maximize  $\Phi_1$  with respect to  $t$ . This amounts to solving the degree-4 polynomial:

$$d_3 t^4 + (2 d_2 - 4 d_4) t^3 + 3(d_1 - d_3) t^2 + (4 d_0 - 2 d_2) t - d_1 = 0$$

The selection of the best angle among the four candidates can be done by simply calculating the contrast value  $\Phi_1(t)$  at these four points.

**Case  $j \leq N$ :** Now we maximize  $\mathcal{J} = \Phi_1 + \Phi_2$ . One can show that stationary values in  $t = \tan \theta$  are generically the roots of a simpler degree-4 polynomial:

$$t^4 + \rho t^3 - 6t^2 - \rho t + 1 = 0$$

where

$$\rho = \frac{4(d_0 + d_4 - d_2)}{d_1 - d_3} = 2 \frac{\sum_{\mathbf{b}\beta} (M_{ii} - M_{jj})^2 - (M_{ij} + M_{ji})^2}{\sum_{\mathbf{b}\beta} (M_{ii} - M_{jj})(M_{ij} + M_{ji})}$$

Because of symmetries in the coefficients, one can instead root the trinomial in variable  $Z = \tan 2\theta$ :

$$Z^2 + \frac{\rho}{2} Z + 1 = 0$$

and go back to the angle tangent by rooting in a second stage  $Zt^2 + 2t - Z = 0$ .

### 4.2.2 Complex framework

In the complex framework, stationary points are defined by two polynomial equations in two (real) variables, which makes the solution a little more complicated. Again, criteria  $\mathcal{J}^1$  and  $\mathcal{J}^2$  can be treated together; only the definition of the matrices  $\mathbf{M}(\mathbf{b}, \boldsymbol{\beta})$  differs. On the other hand, criterion  $\mathcal{J}^0$  also differs in the expression of the transformation to apply.

Here, we consider a set  $2 \times 2$  sub-matrices, say  $\mathbf{M}(k)$ , and a plane rotation  $\Theta$ , that we decide to parametrize by the tangent of its angle,  $\rho$  and its complex phase,  $\psi$ :

$$\mathbf{M}(k) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \quad \text{and} \quad \Theta = \frac{1}{\sqrt{1+\rho^2}} \begin{pmatrix} 1 & -\rho e^{-j\psi} \\ \rho e^{j\psi} & 1 \end{pmatrix}$$

with  $\rho \stackrel{\text{def}}{=} \tan \theta$ .

#### a) Criteria $\mathcal{J}^1$ and $\mathcal{J}^2$ .

Here, the transformed matrices are expressed as  $\Theta^H \mathbf{M}(k) \Theta$ . Again, define  $\Phi_1$  (resp.  $\Phi_2$ ) as the sum of the squared moduli of the first (resp. second) diagonal entries of all transformed matrices. Then we have:

$$\begin{aligned} \Phi_1 &= \frac{1}{(1+\rho^2)^2} \sum_k \left| \alpha_k + \rho e^{-j\psi} \gamma_k + \rho e^{j\psi} \beta_k + \rho^2 \delta_k \right|^2 \\ \Phi_2 &= \frac{1}{(1+\rho^2)^2} \sum_k \left| \delta_k - \rho e^{-j\psi} \beta_k - \rho e^{j\psi} \gamma_k + \rho^2 \alpha_k \right|^2 \end{aligned}$$

Of course, by construction,  $\Phi_2(\rho, \psi) = \Phi_1(\frac{1}{\rho}, -\psi + \pi)$ .

**Case  $j > N$ :** here, the unknowns  $\rho$  and  $\psi$  should be found so as to maximize  $\Phi_1$ . For this purpose, the variable  $t \stackrel{\text{def}}{=} \tan \psi/2$  is introduced. Then,  $(1+\rho^2)^2(1+t^2)^2 \Phi_1$  is a polynomial in  $t$  and  $\rho$ . Its exact expression as a function of  $\alpha_k, \beta_k, \gamma_k$  and  $\delta_k$  is given in appendix. Stationary values in  $\rho$  and  $t$  exactly cancel both the polynomials below:

$$\begin{aligned} P(\rho, t) &\stackrel{\text{def}}{=} (1+\rho^2)^3(1+t^2)^2 \frac{\partial \Phi_1}{\partial \rho} \\ Q(\rho, t) &\stackrel{\text{def}}{=} (1+\rho^2)^2(1+t^2)^3 \frac{\partial \Phi_1}{\partial t} \end{aligned} \tag{18}$$

$P(\rho, t)$  contains 22 monomials, whose leading one is  $\rho^4 t^4$ , whereas  $Q(\rho, t)$  contains 13 monomials, whose leading one is  $\rho^2 t^4$ . We note that the second one is much simpler, and that it is of degree 2 in  $\rho$ .

Considered as polynomials in  $\rho$ ,  $P$  and  $Q$  admit a common solution if and only if their resultant (determinant of the Sylvester matrix) is null, which yields:

$$\begin{vmatrix} Q_4 & 0 & P_2 & 0 & 0 & 0 \\ Q_3 & Q_4 & P_1 & P_2 & 0 & 0 \\ Q_2 & Q_3 & P_0 & P_1 & P_2 & 0 \\ Q_1 & Q_2 & 0 & P_0 & P_1 & P_2 \\ Q_0 & Q_1 & 0 & 0 & P_0 & P_1 \\ 0 & Q_0 & 0 & 0 & 0 & P_0 \end{vmatrix} = 0, \quad (19)$$

where  $Q_i(t)$  (resp.  $P_j(t)$ ) denote the coefficients of  $\rho^i$ ,  $0 \leq i \leq 4$  in  $Q(\rho, t)$  (resp. of  $\rho^j$ ,  $0 \leq j \leq 2$  in  $P(\rho, t)$ ). This determinant is a polynomial in  $t$  only, and its roots contain all the roots of system (18). It turns out that this polynomial is of degree 24, and that it generally admits no more than 8 real roots, which is consistent with Bézout theorem, stating that the maximal number of solutions should be  $4^2$ . Plugging back these real roots in  $Q(\rho, t)$  allows to compute two candidates for  $\rho$  associated with each candidate for  $t$ . The best solution  $(\rho, t)$  (*i.e.* leading to the global maximum) is then selected by computing the value of the rational function  $\Phi_1(\rho, t)$ .

**Case  $j \leq N$ :** now, the optimization criterion is  $\mathcal{J} = \Phi_1 + \Phi_2$ . Because of symmetries, this criterion is much simpler to maximize [2] [7]. In fact, define

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \stackrel{\text{def}}{=} \Theta^H \mathbf{M}(k) \Theta.$$

Then, one can first notice that

$$\mathcal{J}_2 = \sum_k |a_k|^2 + |d_k|^2 = \frac{1}{2} \sum_k \left\{ |a_k - d_k|^2 + |a_k + d_k|^2 \right\}$$

and next, that  $a_k + d_k = \alpha_k + \delta_k$ , which is thus constant with respect to  $\Theta$ . The maximization of  $\mathcal{J}_2$  is consequently equivalent to that of  $\sum_k |a_k - d_k|^2$ .

Yet, if  $\rho = \tan \theta$ , one can check out that

$$a_k - d_k = (\alpha_k + \delta_k) \cos \theta + (\beta_k + \gamma_k) \sin \theta \cos \psi + j (\beta_k - \gamma_k) \sin \theta \sin \psi$$

Then, it is easy to show that  $\mathcal{J}_2$  can be expressed as a quadratic form:

$$\mathbf{w}^T \Re[\mathbf{B}\mathbf{B}^H] \mathbf{w} + \text{constant}, \quad (20)$$

where

$$\mathbf{w} = [\cos 2\theta, \sin 2\theta \cos \psi, \sin 2\theta \sin \psi]^\top$$

and where the  $k$ -th column of  $\mathbf{B}$  is  $[\alpha_k + \delta_k, \beta_k + \gamma_k, j(\beta_k - \gamma_k)]^\top$ .

As a consequence, finding the maxima of  $\mathcal{J}$  amounts to maximizing a real quadratic form in 3 variables.

It has been possible to arrange criterion  $\mathcal{J}$  in a quadratic form because some terms in  $\Phi_1$  and  $\Phi_2$  have cancelled each other, in particular those involving:  $\sin^2 \theta$ ,  $\cos \theta$ ,  $\sin \theta \sin \psi$ , and  $\sin \theta \cos \psi$ , which are not present in (20).

**b) Criterion  $\mathcal{J}^0$ .**

In this subsection, one considers another kind of congruent transformation, in which the unitary plane rotation is not conjugated. In other words, the following matrices are wished to be partially approximately diagonal:  $\Theta^\top \mathbf{M}(k) \Theta$ . The sum of the squared moduli of the first diagonal entries of these matrices can be written as:

$$\Phi_1 = \frac{1}{(1 + \rho^2)^2} \sum_k \left| \alpha_k + \rho e^{j\psi} \gamma_k + \rho e^{j\psi} \beta_k + \rho^2 e^{2j\psi} \delta_k \right|^2$$

with similar notations as before. Remark that now,  $\beta_k$  and  $\gamma_k$  enter the expression only through their sum, which we denote  $\sigma_k$ . The same remark holds true for the second diagonal entries:

$$\Phi_2 = \frac{1}{(1 + \rho^2)^2} \sum_k \left| \delta_k - \rho e^{-j\psi} \beta_k - \rho e^{-j\psi} \gamma_k + \rho^2 \alpha_k e^{-2j\psi} \right|^2$$

Now by construction,  $\Phi_2(\rho, \psi) = \Phi_1(\frac{1}{\rho}, \psi + \pi)$ .

**Case  $j > N$ :** It is shown in appendix that

$$\Phi_1 = \frac{1}{(1 + \rho^2)^2} \sum_{k=0}^4 e_k \rho^k,$$

where  $e_0$  and  $e_4$  are constants,  $e_1$  and  $e_3$  are linear in  $\cos \psi$  and  $\sin \psi$ , and  $e_2$  is of the form  $u + v \cos 2\psi + w \sin 2\psi$ . Thus,  $\Phi_1$  takes formally the same form as for  $\mathcal{J}^2$  and  $\mathcal{J}^1$ . The maximization of  $\Phi_1$  can be carried out with the help of the same tools.

**Case  $j \leq N$ :** This time, define

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \stackrel{\text{def}}{=} \Theta^\top \mathbf{M}(k) \Theta.$$

Now,  $a_k - d_k$  is again independent of  $\sigma_k$ , but is not constant anymore. On the other hand,  $b_k - c_k = \beta_k - \gamma_k$ , which is a constant, but this is of little usefulness.

However, criterion  $\mathcal{J}^0$  takes a form similar to  $\Phi_1$ :

$$\Phi_1 = \frac{1}{(1 + \rho^2)^2} \sum_{k=0}^4 f_k \rho^k,$$

where  $f_0$  and  $f_4$  are constants,  $f_1$  and  $f_3$  are linear in  $\cos \psi$  and  $\sin \psi$ , but where this time  $f_2$  is linear in  $\cos 2\psi$  and  $\sin 2\psi$  (there is no constant term). See appendix for details. It turns out that some terms of  $\Phi_1$  and  $\Phi_2$  cancel each other, and that a quadratic form can again be defined:

$$\mathcal{J}^0 = \mathbf{w}^\top \mathbf{Q} \mathbf{w}, \quad \mathbf{w} \stackrel{\text{def}}{=} [\cos 2\theta, \sin 2\theta \cos \psi, \sin 2\theta \sin \psi]^\top$$

See appendix for its exact expression.

## 5 Performances

### 5.1 Working example

One considers a Finite Impulse Response (FIR) real mixture of length  $L = 3$  of  $N = 2$  real white processes. Thus, there are  $N^2 L^2 = 36$  square matrices, each of size  $NL = 6$ , and the goal is to jointly and approximately diagonalize their  $2 \times 2$  leading matrix by congruent transform. With this goal, a real orthogonal  $6 \times 6$  matrix,  $\mathbf{U}$  is estimated. Matrix  $\mathbb{H}$  corresponds to the two first rows of  $\mathbf{U}$ .

The channel is para-unitary, to preserve second-order whiteness as explained in section 2. It has been generated as follows:

$$\check{\mathbf{F}}(z) = \mathbf{R}(\phi_1) \cdot \mathbf{Z} \cdot \mathbf{R}(\phi_2) \cdot \mathbf{Z} \cdot \mathbf{R}(\phi_3),$$

where

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix},$$

and

$$\mathbf{R}(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

Note the absence of phase terms, because real channels are desired. Because of the 3 free parameters above, we have some control on the location of zeros of the 4

length-3 SISO channels. Here, we have chosen  $\phi_1 = 52^\circ$ ,  $\phi_2 = 53^\circ$ , and  $\phi_3 = 70^\circ$ . As a result, the zeros of  $\check{F}_{12}(z)$  are inside the unit disk, and those of  $\check{F}_{21}(z)$  are outside;  $\check{F}_{11}(z)$  and  $\check{F}_{22}(z)$  each have one zero inside and one outside, as shown in figure 3. So,  $\check{\mathbf{F}}(z)$  has a stable inverse, whereas the components  $\check{F}_{12}(z)$  and  $\check{F}_{21}(z)$  have not.

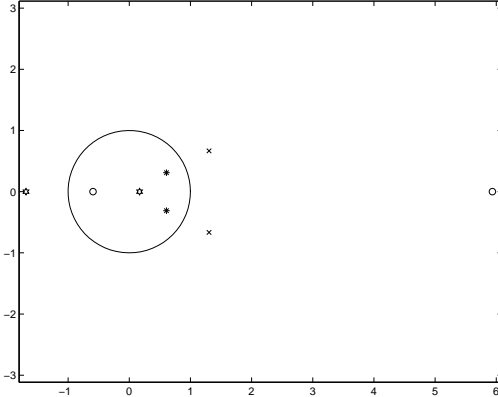


Figure 3: Zeros of the 4 channels in  $\mathbf{F}(z)$ .

## 5.2 Performance criteria

When evaluating performances of MIMO equalizers, a difficulty to overcome stems from inherent indeterminacies. In fact, equalizer  $\mathbf{H}(z)$ , and hence global filter  $\mathbf{G}(z)$ , can be estimated only up to a multiplicative matrix of the form  $\mathbf{D}(z) = \mathbf{\Lambda}(z)\mathbf{P}$ , as defined in section 2.

### 5.2.1 Distance criterion

Let the global transfer function

$$\mathbf{G}(z) = \sum_{n=0}^{2L-1} \mathbf{G}(n)z^{-n}.$$

One can decide to store matrices  $\mathbf{G}(n)$  in a  $N \times N(2L-1)$  array,  $\mathbb{G}$ , by merely stacking the matrices one after the other. Then finding the best matrix  $\mathbf{D}(z)$  amounts to searching every row of  $\mathbb{G}$  for the entry of largest modulus, under the constraint that their column index are different modulo  $N$ .

Let us now explain how this is done in our example where  $N = 2$  and  $L = 3$ :

- Case 1: search for column  $j_1$  (resp.  $j_2$ ) containing the entry of largest modulus,  $\mathbb{G}_{1,j_1}$  (resp.  $\mathbb{G}_{2,j_2}$ ), among the entries of row 1 (resp. 2) of odd column index (resp. even). Normalize row 1 (resp. 2) of  $\mathbb{G}$  by  $\mathbb{G}_{1,j_1}$  (resp.  $\mathbb{G}_{2,j_2}$ ). Compute the Froebenius distance between matrix  $\tilde{\mathbb{G}}$  obtained this way and matrix  $\mathbb{D}$ , of same size, containing two 1's at locations  $(1, j_1)$  and  $(2, j_2)$  and zero elsewhere.
- Case 2: search for column  $k_1$  (resp.  $k_2$ ) containing the entry of largest modulus,  $\mathbb{G}_{2,k_1}$  (resp.  $\mathbb{G}_{1,k_2}$ ), among the entries of row 1 (resp. 2) of even column index (resp. odd). Repeat the same normalizing operations and distance calculations as in case 1.
- Choose the case leading to the minimal distance,  $\epsilon(\mathbb{G}) = \|\tilde{\mathbb{G}} - \mathbb{D}\|$ , which actually corresponds to:

$$\epsilon(\mathbb{G}) \stackrel{\text{def}}{=} \underset{\mathbf{P}, \nu_1, \alpha_1, \nu_2, \alpha_2}{\text{Min}} \|\mathbb{G} \mathbf{P} \text{Diag}(\nu_1 z^{\alpha_1}, \nu_2 z^{\alpha_2})\| \quad (21)$$

This procedure can be easily extended to  $N \geq 2$ ; then,  $N!$  cases must be tested instead of 2.

### 5.2.2 Symbol Error Rate

Another criterion is focussed on source estimates instead of the channel estimate itself. As before, we have to get rid of the possible delay  $\alpha$ , the possible factor  $\nu = e^{j\psi}$ , and the possible permutation  $\sigma$ , that may be present in the estimate, and minimize  $\nu a_{\sigma(i)}(n - \alpha) - \hat{a}_i(n)$ .

This can be done in a similar manner as in the previous section, even if it turns out to be more computationally complex. In fact, under each of the two hypotheses, one must explore 10 different cases (5 possible delays for each row). In general, one must calculate the error rate of  $N!N(2L - 1)$  potential estimators (instead of 20 for  $N = 2$ ), which can become quite costly. In such cases, one may want to assume the matrix  $\mathbb{D}$  obtained via the distance criterion.

## 6 Concluding remarks

Based on the theoretical results obtained in [9], a first numerical algorithm has been developed in [10]. Even if it is not yet optimized, this algorithm demonstrates

that it is possible to equalize blindly FIR MIMO channels from short data records (typically 300 symbols). Nevertheless, we have demonstrated that the computation of the blind equalizer can be completed within a polynomial complexity, which can be attractive in a burst-mode transmissions, or as initialization of on-line algorithms.

Subjects that need to be further addressed include (i) the robustness to channel length misadjustment, (ii) improvements on accuracy or complexity of the algorithm, (iii) computer experiments for a wide variety of channels and sources.

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## 7 Appendix

Matrices  $\mathbf{M}(k)$  are denoted

$$\mathbf{M}(k) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}$$

and the unitary transform

$$\Theta = \frac{1}{\sqrt{1+\rho^2}} \begin{pmatrix} 1 & -\rho e^{-j\psi} \\ \rho e^{j\psi} & 1 \end{pmatrix}$$

### 7.1 Contrasts $\mathcal{J}_3^1$ and $\mathcal{J}_2^2$

$\Phi_1$  represents the squared modulus of the (1, 1) entry of the matrix  $\sum_k \Theta^H \mathbf{M}(k) \Theta$ . It can be shown that

$$\Phi_1 = \frac{1}{(1+\rho^2)^2} \sum_{k=0}^4 d_k \rho^k,$$

with

$$\begin{aligned} d_0 &= \sum_k |\alpha_k|^2 \\ d_1 &= 2\Re\{\sum_k \alpha_k^* \sigma_k\} \cos \psi - 2\Im\{\sum_k \alpha_k^* \Delta_k\} \sin \psi \\ d_2 &= \sum_k |\beta_k|^2 + \sum_k |\gamma_k|^2 + 2\Re\{\sum_k \alpha_k^* \delta_k\} \\ &\quad + 2\Re\{\sum_k \beta_k \gamma_k^*\} \cos 2\psi - 2\Im\{\sum_k \beta_k \gamma_k^*\} \sin 2\psi \\ d_3 &= 2\Re\{\sum_k \delta_k^* \sigma_k\} \cos \psi - 2\Im\{\sum_k \delta_k^* \Delta_k\} \sin \psi \\ d_4 &= \sum_k |\delta_k|^2 \end{aligned}$$

where  $\sigma_k \stackrel{\text{def}}{=} \beta_k + \gamma_k$  and  $\Delta_k \stackrel{\text{def}}{=} \beta_k - \gamma_k$ .

The contrast takes then the form  $\mathcal{J}^2 = \Phi_1(\rho, \psi) + \Phi_1(\frac{1}{\rho}, \pi - \psi)$ :

$$\begin{aligned} \mathcal{J}^2 &= \sum_k \left| \alpha_k \cos^2 \theta + \beta_k \cos \theta \sin \theta e^{j\psi} + \gamma_k \cos \theta \sin \theta e^{-j\psi} + \delta_k \sin^2 \theta \right|^2 \\ &\quad + \left| \alpha_k \sin^2 \theta - \beta_k \cos \theta \sin \theta e^{j\psi} - \gamma_k \cos \theta \sin \theta e^{-j\psi} + \delta_k \cos^2 \theta \right|^2 \end{aligned}$$

which can be rearranged in a quadratic form, as already pointed out.

## 7.2 Contrast $\mathcal{J}_4^0$

Now  $\Phi_1$  represents the squared modulus of the (1,1) entry of the matrix  $\sum_k \Theta^\top \mathbf{M}(k) \Theta$ . In the complex case, we have

$$\Phi_1 = \frac{1}{(1 + \rho^2)^2} \sum_{k=0}^4 e_k \rho^k,$$

with

$$\begin{aligned} e_0 &= \sum_k |\alpha_k|^2 \\ e_1 &= 2 \sum_k \Re\{\alpha_k^* \sigma_k e^{j\psi}\} = 2\Re\{\sum_k \alpha_k^* \sigma_k\} \cos \psi - 2\Im\{\sum_k \alpha_k^* \sigma_k\} \sin \psi \\ e_2 &= \sum_k |\sigma_k|^2 + 2 \sum_k \Re\{\alpha_k^* \delta_k e^{2j\psi}\} \\ &= \sum_k |\sigma_k|^2 + 2\Re\{\sum_k \alpha_k^* \delta_k\} \cos 2\psi - 2\Im\{\sum_k \alpha_k^* \delta_k\} \sin 2\psi \\ e_3 &= 2 \sum_k \Re\{\sigma_k^* \delta_k e^{j\psi}\} = 2\Re\{\sum_k \sigma_k^* \delta_k\} \cos \psi - 2\Im\{\sum_k \sigma_k^* \delta_k\} \sin \psi \\ e_4 &= \sum_k |\delta_k|^2 \end{aligned}$$

and where  $\sigma_k \stackrel{\text{def}}{=} \beta_k + \gamma_k$ .

Next, contrast  $\mathcal{J}^0 = \Phi_1(\rho, \psi) + \Phi_1(\frac{1}{\rho}, \psi + \pi)$  takes the form:

$$\begin{aligned} \mathcal{J}^0 &= \sum_k \left| \alpha_k \cos^2 \theta + \sigma_k \cos \theta \sin \theta e^{j\psi} + \delta_k \sin^2 \theta e^{2j\psi} \right|^2 \\ &\quad + \left| \alpha_k \sin^2 \theta - \sigma_k \cos \theta \sin \theta e^{j\psi} + \delta_k \cos^2 \theta e^{2j\psi} \right|^2 \\ &= f_0 [\cos^4 \theta + \sin^4 \theta] + \cos^2 \theta \sin^2 \theta [f_{20} + f_{2c} \cos 2\psi + f_{2s} \sin 2\psi] \\ &\quad + (f_{1c} \cos \psi + f_{1s} \sin \psi) [\cos^3 \theta \sin \theta - \cos \theta \sin^3 \theta] \end{aligned}$$

with

$$\begin{aligned} f_0 &= \sum_k |\alpha_k|^2 + |\delta_k|^2 \\ f_{20} &= 2|\sigma_k|^2 \\ f_{2c} &= 4\Re\{\alpha_k^* \delta_k\} \end{aligned}$$

$$\begin{aligned}
f_{2s} &= -4 \Im\{\alpha_k^* \delta_k\} \\
f_{1c} &= 2 \Re\{\sigma_k(\alpha_k - \delta_k)^*\} \\
f_{1s} &= -2 \Im\{\sigma_k(\alpha_k + \delta_k)^*\}
\end{aligned}$$

where we still denote  $\sigma_k = \beta_k + \gamma_k$ . As in the case of  $\mathcal{J}^2$ , the terms involving  $\sin 2\theta \sin \psi$ ,  $\sin 2\theta \cos \psi$ , or  $\cos 2\theta$  have in fact cancelled out.

As a consequence,  $\mathcal{J}^0$  is a quadratic form in

$$\mathbf{w} \stackrel{\text{def}}{=} [\cos 2\theta, \sin 2\theta \cos \psi, \sin 2\theta \sin \psi]^\top$$