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PANCYCLIC ARCS AND CONNECTIVITY IN TOURNAMENTS

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Pancyclic arcs and connectivity in tournaments

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Abstract

A *tournament* is an orientation of the edges of a complete graph. An arc is *pancyclic* in a tournament T if it is contained in a cycle of length l, for every $3 \leq l \leq |T|$. Let $p(T)$ denote the number of pancyclic arcs in a tournament T. In [4], Moon showed that for every non-trivial strong tournament T, $p(T) \geq 3$. Actually he proved a somewhat stronger result : for any non-trivial strong tournament $h(T) \geq 3$ where $h(T)$ is the maximum number of pancyclic arcs contained in the same hamiltonian cycle of T. Moreover Moon characterized the tournaments with $h(T) = 3$. All these tournaments are not 2-strong. In this paper, we investigate relationship between the functions $p(T)$ and $h(T)$ and the connectivity of the tournament T. Let $p_k(n) := \min\{p(T), T$ k-strong tournament of order $n\}$ and $h_k(n) := \min\{h(T), T$ k-strong tournament of order $n\}$. We conjecture that (for $k \geq 2$) there exists a constant $\alpha_k > 0$ such that $p_k(n) \geq \alpha_k n$ and $h_k(n) \geq 2k + 1$. In this paper, we establish the later conjecture when $k = 2$. We then characterized the tournaments with $h(T) = 4$ and those with $p(T) = 4$. We also prove that for $k \geq 2$, $p_k(n) \geq 2k+3$. At last, we characterize the tournaments having exactly five pancyclic arcs.

1 Introduction

A *tournament* is an orientation of the arcs of a complete graph. Paths and cycles are always directed. An l-cycle is a cycle of length l.

An arc or a vertex is *pancyclic* in a digraph D if, for every $3 \leq l \leq |D|$, it is contained in an l-cycle.

A tournament is *strong* (or *strongly connected*) if for any two vertices x and y there exists a path beginning in x and terminating at y. A nonstrong tournament is said to be *reducible*. A tournament is k*-strong*, if $T - Y$ is strong for any set Y of $k-1$ vertices. A tournament is $(=k)$*-strong* or *exactly* k*-strong*, if it is k-strong and not $(k+1)$-strong.

To contain a pancyclic arc or vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to the well known theorem of Camion [2]: *A tournament has a hamiltonian cycle if and only if it is strong.* Moon [3] gave an alternative proof of Camion's theorem by proving that every vertex of a strong tournament is pancyclic.

Analogously, one may wonder whether there are pancyclic arcs in tournament and how many. Moon [4] showed that every strong tournament contains at least three pancyclic arcs. Actually, he proved a somewhat stronger result : indeed, instead of considering the number $p(T)$ of pancyclic arcs in the tournament T, he proved that $h(T)$ the maximum number of pancyclic arcs contained in some hamiltonian cycle of T is at least 3.

Theorem 1 (Moon, [4]) *Let T be a strong tournament with $n \geq 3$ vertices.*

$$h(T) \geq 3$$

with equality holding only if $T \in \mathcal{P}_3$.

A tournament is in $\mathcal{P}_3$ if there is a vertex v such that $T - v$ is the transitive tournament $TT[t_1, t_2, \ldots, t_m]$ ((t_i, t_j) is an arc if and only if $i < j$), and an integer $1 < i_1 \leq m$ such that $v \rightarrow t_j$ if and only if $1 \leq j < i_1$. One can also describe the tournament of $\mathcal{P}_3$ as those obtained from the 3-cycle (v, s_1, s_2, v) by blowing up s_1 and s_2 into two transitive subtournaments S_1 and S_2.

Let $p_k(n)$ be minimum number of pancyclic arcs in a k-strong tournament of order n and let $h_k(n) := \min\{h(T); T \ k\text{-strong tournament of order } n\}$.

Because the tournaments of $\mathcal{P}_3$ are $(=1)$-strong, we have $p_1(n) = h_1(n) = 3$ and if $k \geq 2$, $p_k(n) \geq h_k(n) \geq 4$. However we think that this lower bound 4 is far to be tight.

In this paper, we show Section 3 sufficient conditions for an arc to be pancyclic in a tournament. Using these conditions, we give an easy alternative proof of Theorem 1. Moreover our method allow us to go further. In Section 4, we prove that for $k \geq 2$, $h_k(n) \geq 5$ and characterize the tournaments with $h(T) = 4$ and those with $p(T) = 4$. We prove Section 5 that $p_k(n) \geq 2k + 3$. Finally, we characterize the tournaments with exactly five pancyclic arcs.

However, our lower bounds for h_k and p_k seem to be still far from the exact value. We conjecture that for $k \geq 2$, $p_k(n)$ tends linearly to infinity :

Conjecture 1 For $k \geq 2$, there exists a constant $\alpha_k > 0$ such that $p_k(n) \geq \alpha_k n$.

We cannot expect to have more pancyclic arcs since there are k-strong tournaments having less than $2kn$ pancyclic arcs.

Proposition 1 $p_k(n) \leq 2kn - 2k^2 - k$

Proof. Let T_n be the k-strong tournament obtained from the rotative tournament on $2k+1$ vertices by blowing up a vertex with a transitive tournament TT of order $n-2k$. Every arc in TT is not pancyclic in T since it is contained in no 3-cycle. Thus $(n-2k)(n-2k-1)/2$ arcs are not pancyclic. ■

Proposition 2

$$h_k(n) \leq 3k$$

Proof. If $n \leq 3k$, we have trivially the answer. Suppose now that $n > 3k$. Consider the k-strong tournament T_n obtained from two transitive tournaments of order k, A and B, and one transitive tournament of order $n-2k$, C, such that $A \rightarrow B \rightarrow C \rightarrow A$. It is easy to see that every arc contained in one of the three subtournaments A, B and C is not pancyclic because it is contained in no 3-cycle. It follows that $h(T_n) \leq 3k$. ■

The bound $3k$ is not tight if the tournament is small that is of order $2k+1 \leq n < 3k$. However, we think that if n is large enough, the above example are extremal.

Conjecture 2 i) $h_k(n) \geq 2k+1$

ii) For n sufficiently large , $h_k(n) = 3k$.

Alspach [1] showed that every arc of a regular tournament is pancyclic. This implies that $h_k(2k+1) = 2k+1$.

2 Definition and preliminaries

In all this paper, we consider a tournament T with order n. Let x and y be two vertices of T. If (x, y) is an arc of T, we write $x \rightarrow y$ and say x *dominates* y. Likewise, let X and Y be two subdigraphs of T. We write $X \rightarrow Y$ if $x \rightarrow y$ for all pairs $(x, y) \in V(X) \times V(Y)$.

Let $A_1, A_2, \ldots, A_k$ be a family of subdigraphs of T. We denote by $T[A_1, A_2, \ldots, A_k]$ the subtournament induced by T on the set of vertices $\bigcup_{1 \leq i \leq k} V(A_i)$ and by $T - [A_1, A_2, \ldots, A_k]$ the subtournament induced by T on the set of vertices $V(T) \backslash \bigcup_{1 \leq i \leq k} V(A_i)$.

$A(X, Y)$ denotes the set of arc (x, y) with $x \in X$ and $y \in Y$. $A^+(X)$ is the set of arcs outgoing from X, that is $A(X, T - X)$ and $A^-(X)$ is the set of arcs ingoing into X, that is $A(T - X, X)$.

The *outneighbourhood* of a vertex v in a subtournament S of T (even not containing v) is the subtournament $N_S^+(v)$ induced by the vertices of S dominated by v; the *outdegree* $d_S^+(v)$ of v in S is the order of $N_S^+(v)$. Dually, the *inneighbourhood* of v in S is the subtournament $N_S^-(v)$ induced by the vertices of S dominating v and $d_S^-(v) = |N_S^-(v)|$. Often, we omit the subscript T in the above notations when we consider the out- or inneighbourhood or out- or indegree of a vertex in T. The minimun outneighbourhood in T, that is $\min\{d^+(v), v \in V(T)\}$, is denoted by $\delta^+(T)$, or δ^+ if T is clearly understood. Analogously, the minimun inneighbourhood in T is denoted by $\delta^-(T)$ or δ^-.

The *dual* of a digraph D is the digraph $\overline{D}$ on the same set of vertices such that $x \rightarrow y$ is an arc of $\overline{D}$ if and only if $y \rightarrow x$ is an arc of D. Obviously, $p(T) = p(\overline{T})$ and $h(T) = h(\overline{T})$.

An (x, y)-path is a path that begins in x and terminates at y.

One can easily show the following result (whose proof is left to the reader) that allows to extend an (x, y) path to a longer one.

Proposition 3 *Let $P = (v_1, v_2, \ldots, v_m)$ be a path in a tournament T and x a vertex of $T - P$.*

If there exist $1 \leq i < j \leq m$ such that $v_i \rightarrow x \rightarrow v_j$, then in T there is a path with $m + 1$ vertices starting in v_1 and ending in v_m.

A non-strong tournament T is said to be *reducible*. It admits a *reduction* into two subtournaments T_1 and T_2 such that $V(T_1) \cup V(T_2) = V(T)$ and $T_1 \rightarrow T_2$; in this case, we write $T = T_1 \rightarrow T_2$.

A (strong) *component* of T is a strong subtournament of T which is maximal by inclusion. Let $T_1, T_2, \ldots, T_m$ be the components of T. Then $(V(T_1), V(T_2), \ldots, V(T_m))$ is a partition of $V(T)$ and without loss of generality, we may suppose that $T_i \rightarrow T_j$ whenever $i < j$. In this case we say that $T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_m$ is the *decomposition* of T. The component T_1 (resp. T_m) is called the *outsection* (resp. *insection*) of T, denoted by $Out(T)$ (resp. $In(T)$); its order is denoted by $out(T)$ (resp. $in(T)$) and its vertices are called the *outgenerators* (resp. *ingenerators*) of T.

Proposition 4 *In a tournament, a vertex is the beginning of a hamiltonian path if and only if it is an outgenerator.*

Proof. Let T be a tournament. Then $T = Out(T) \rightarrow T - Out(T)$ (with $T - Out(T)$ empty when T is strong). $Out(T)$ is strong, and thus by Camion's theorem admits a hamiltonian cycle C. Then every outgenerator v is the beginning of a hamiltonian path P of $Out(T)$. And $T - Out(T)$ has a hamiltonian path Q. So (P, Q) is hamiltonian path of T beginning in v. ■

Proposition 5 *Let T be a reducible tournament of order n. Let u be an ingenerator and t an outgenerator of T. For any $1 \leq l \leq n - 1$, there is a (t, u)-path of length l.*

Proof. Let $T_1 \to T_2$ be a reduction of T and for $i = 1, 2$, let n_i be the order of T_i. Clearly, t is an ingenerator of T_2 and u an outgenerator of T_1. Therefore, by Proposition 4, t is the terminus of a hamiltonian path $(t_{n_2-1}, t_{n_2-2}, \dots, t_1, t)$ of T_2 and u is the beginning of a hamiltonian path $(u, u_1, \dots, u_{n_1-2}, u_{n_1-1})$ of T_1. Now, for any $1 \leq l \leq n-1$, pick $0 \leq l_1 \leq n_1 - 1$ and $0 \leq l_2 \leq n_2 - 1$ such that $l_1 + l_2 = l - 1$. Then $(u, u_1, \dots, u_{l_1}, t_{l_2}, t_{l_2-1}, \dots, t)$ is a path of length l. ■

A *reductor* of a tournament is a smallest subtournament X such that $T - X$ is reducible. If T is $(= k)$-strong then every reductor has k vertices.

Proposition 6 *Let X be a reductor of a tournament T. Every element of X dominates an outgenerator of $T - X$ and is dominated by an ingenerator of $T - X$.*

Proof. Let x be an element of X. Let Y be the set of vertices that are not outgenerator of $T - X$. If $Out(T - X) \to x$, then $T - [X \setminus x]$ is reducible with reduction $Out(T - X) \to T[Y, x]$. This contradicts that X is a reductor.

Analogously, we prove that x is dominated by an ingenerator of $T - X$. ■

Proposition 7 *Let x and y be two vertices of a a reductor X of a 2-strong tournament T. If $in(T - X) \geq 3$, there are two distinct vertices z_x and z_y of $In(T - X)$ such that $z_x \to x$ and $z_y \to y$.*

Proof. Suppose that two such vertices do not exist, then by Proposition 6, there is a vertex $u \in In(T - X)$ such that $u \to \{x, y\}$ and $In(T - X) \setminus u \leftarrow \{x, y\}$. Then $T[X - [x, y], u]$ is a reductor of T, which is a contradiction. ■

3 Lower bounds for $p(T)$ and $h(T)$

3.1 Sufficient conditions for an arc to be pancyclic

Lemma 1 *Let X be a subtournament such that $T - X$ is reducible and every vertex of X dominates an outgenerator of $T - X$. Let v be an outgenerator of X and u an ingenerator of $T - X$. If $u \to v$, then the arc (u, v) is pancyclic.*

Proof. By Proposition 4, $v = v_0$ is the origin of a hamiltonian path $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$ of X. For $0 \leq i \leq k-1$, let t_i be an outgenerator of $T - X$ dominated by v_i. For $3 \leq l \leq n$, take $0 \leq k' \leq k - 1$ and $1 \leq l \leq n - k - 1$ such that $k' + l + 2 = l$. Then by Proposition 5, in $T - X$, there is a $(t_{k'}, u)$-path Q of length l. Thus $(v_0, v_1, \dots, v_{k'}, Q, v_0)$ is a cycle of length l going through $(u, v) = (u, v_0)$. ■

Corollary 1 *Let X be a reductor of a tournament T, v an outgenerator of X and u an ingenerator of $T - X$. If $u \to v$, then the arc (u, v) is pancyclic.*

Corollary 2 *Let T be a strong tournament.*

$$p(T) \geq h(T) \geq 2$$

Proof. Let X be a reductor of T. Let P be a hamiltonian path of X with beginning v and end w. Clearly, v is an outgenerator of X and w is an ingenerator of X. By Proposition 6, v is dominated by an ingenerator u of $T-X$ and t dominates an outgenerator t of $T-X$. Then by Corollary 1 (and its dual), (u,v) and (w,t) are pancyclic. And by Proposition 5, there is a (t,u)-path Q that is hamiltonian in $T-X$, thus (P,Q,v) is a hamiltonian cycle containing (u,v) and (w,t). ■

Let T be a strong tournament T with reductor X. Let $X_1 \to X_2 \to \cdots \to X_l$ be a decompositon of X and $T_1 \to T_2 \to \cdots \to T_m$ be a decomposition of $T-X$.

In the remaining of this section, we examine the number of pancyclic arcs in the different subtournaments of T.

Lemma 2 *For $1 \leq i \leq l$, if an arc is pancyclic in X_i, then it is also pancyclic in T.*

Proof. Let e be a pancyclic arc in one of the X_i. Then e is contained in a 3-cycle. It is also contained in a hamiltonian cycle and then a hamiltonian path of X_i. This hamiltonian path may be extended to a hamiltonian path $(v_0, v_1, \ldots, v_{k-1})$ of X using hamiltonian paths of the $X_{i'}$, for $i' \neq i$. Let j be the index such that $e = (v_j, v_{j+1})$. Let $4 \leq l \leq n$. Choose $0 \leq l_1 \leq j$, $0 \leq l_2 \leq k-j-2$ and $1 \leq l_3 \leq n-k-1$ such that $l_1+l_2+l_3+3 = l$. By Proposition 6, there is an ingenerator u of $T-X$ dominating v_{j-l_1} and an outgenerator t of $T-X$ dominated by v_{j+1+l_2}. Then by Proposition 5, in $T-X$ there is a (t,u)-path of length l_3. Hence $(v_{j-l_1}, v_{j-l_1+1}, \ldots, v_{j+1+l_2}, P, v_{j-l_1})$ is an l-cycle containing e. ■

Lemma 3 *Suppose that $T-X$ is the transitive tournament $TT[t_1, t_2, \ldots, t_m]$.*

For every $1 \leq i \leq m-1$, the arc (t_i, t_{i+1}) is pancyclic if and only if it is contained in a 3-cycle.

Proof. By Proposition 6, $X \to t_1$ and $t_m \to X$. Let $(v_0, v_1, \ldots, v_{k-2}, v_{k-1})$ be a hamiltonian path of X.

Suppose first that $i=1$. Let $4 \leq l \leq n$. Take $0 \leq k' < k$, and $1 \leq l' < m-2$ such that $k'+l'+3 = l$. By Proposition 5, there is a (t_2, t_m)-path P of length l' in $T[t_2, \ldots, t_m]$. Then $(t_1, P, v_0, v_1, \ldots, v_{k'}, t_1)$ is a cycle of length l. Thus if (t_1, t_2) is contained in a 3-cycle then it is pancyclic.

By duality, we have the result if $i = m-1$.

Suppose now that $1 < i < m-1$. Let v be the vertex such that (t_i, t_{i+1}, v, t_i) is a 3-cycle. Necessarily, v is in X. Then $(t_i, t_{i+1}, v, t_1, t_i)$ is a 4-cycle. Let $5 \leq l \leq n$. The arc (t_i, t_{i+1}) is contained in a cycle of length l. Indeed, take $0 \leq k' < k$, $0 < l_1 < i$ and

$0 \le l_2 < m - i$ such that $k' + l_1 + l_2 + 3 = l$. By Proposition 5, there is a (t_1, t_i)-path P_1 of length l_1 in $T[t_1, \ldots, t_i]$, and a (t_{i+1}, t_m)-path P_2 of length l_2 in $T[t_{i+1}, \ldots, t_m]$. Then $(t_i, P_2, v_0, v_1, \ldots, v_{k'}, P_1)$ is an l-cycle containing (t_i, t_{i+1}). Thus (t_i, t_{i+1}) is pancyclic. ■

Lemma 4 *For $1 < i < m$, if $|T_i| \ge 4$, then every T_i-pancyclic arc is pancyclic in T.*

Proof. Let $(v_0, v_1, \ldots, v_{k-2}, v_{k-1})$ be a hamiltonian path of X.

Let (a_1, a_2) be a pancyclic arc in T_i. It is contained in a hamiltonian cycle of T_i, $C_i = (a_1, a_2, \ldots, a_{n_i}, a_1)$.

Let us prove that (a_1, a_2) is contained in a cycle of length l for all $5 \le l \le n$. Let $S_1 = T[T_1, T_2, \ldots, T_{i-1}, a_1]$ and $S_2 = T - [X, S_1]$. Clearly, a_1 is an ingenerator of S_1 and a_2 an outgenerator of S_2. There exist three integers $1 \le l_1 \le |S_1| - 1$, $1 \le l_1 \le |S_2| - 1$ and $0 \le k' \le k - 1$ such that $l_1 + l_2 + k' = l - 3$. By Proposition 6, there is an ingenerator u_0 of $T - X$ and then also of S_2 which dominates v_0 and an outgenerator $t_{k'}$ of S_1 which is dominated by $v_{k'}$. By Proposition 5, there is a $(t_{k'}, a_1)$-path P_1 of length l_1 in S_1 and a (a_2, u_0)-path P_2 of length l_2 in S_2. Then $(P_1, P_2, v_0, v_1, \ldots v_{k'}, t_{k'})$ is the desired l-cycle.

And because it is pancyclic in T_i, (a_1, a_2) is contained in an l-cycle, for $3 \le l \le n_i$. ■

Lemma 5 *For any $1 < i < m$, if T_i is a 3-cycle, then two arcs of T_i are pancyclic in T.*

Proof. Let $(v_0, v_1, \ldots, v_{k-2}, v_{k-1})$ be a hamiltonian path of X. The component T_i is the 3-cycle (a_1, a_2, a_3, a_1).

In the same way as in the proof of Lemma 4 the arc (a_1, a_2), is contained in cycle of length l for all $5 \le l \le n$. Analogously, (a_2, a_3) and (a_3, a_1) are contained in cycle of length l for all $5 \le l \le n$.

Hence, it suffice to prove that two arcs of T_i are contained in a 4-cycle. Without loss of generality, we may assume that $\{a_1, a_2\} \leftarrow v_0$ or $\{a_1, a_2\} \rightarrow v_0$. If $\{a_1, a_2\} \leftarrow v_0$, then $(a_1, a_2, u_0, v_0, a_1)$ and $(a_2, a_3, u_0, v_0, a_2)$ are 4-cycles. If $\{a_1, a_2\} \rightarrow v_0$, then $(a_3, a_1, v_0, t_0, a_3)$ and $(a_1, a_2, v_0, t_0, a_1)$ are 4-cycles. ■

Lemma 6 *Let $C = (a_1, a_2, \ldots, a_{n_1}, a_1)$ be a hamiltonian cycle of T_1 such that a_1 is dominated by an ingenerator of X.*

If (a_j, a_{j+1}) is pancyclic in T_1 and $1 \le j \le n_1 - 2$ then (a_j, a_{j+1}) is pancyclic in T.

Proof. By T_1-pancyclicity, (a_j, a_{j+1}) is contained in a cycle of length l for $3 \le l \le n_1$.

Let now l be an integer of $[n_1 + 1, n]$.

Let v_{k-1} be an ingenerator of X dominating a_1. By Proposition 4, there exists a hamiltonian path $(v_0, v_1, \ldots, v_{k-2}, v_{k-1})$ of X. By Proposition 6, for every $0 \le i \le k - 1$, there is an ingenerator u_i of $T - X$ dominating v_i.

Let $1 \le l_1 \le n - k - j - 1$ and $0 \le k' \le k - 1$ such that $k' + l_1 = l - j - 2$. Obviously, a_{j+1} is an outgenerator of $T' = T - [X, a_1, a_2, \ldots, a_j]$. And $u_{k-1-k'}$ is an ingenerator of T'. Thus, by Proposition 5, there is an $(a_{j+1}, u_{k-1-k'})$ path P of length l_1. Hence $(P, v_{k-1-k'}, v_{k-1-k'}, \ldots, v_{k-1}, a_1, a_2, \ldots, a_{j+1})$ is a cycle of length l. ■

Lemma 7 *Let $C = (a_1, a_2, \ldots, a_{n_1}, a_1)$ be a hamiltonian cycle of T_1 and v an ingenerator of X. If $v \to a_1$ and $a_{n_1} \to v$ then arc of $(a_1, a_2, \ldots, a_{n_1})$ that is T_1-pancyclic is also T-pancyclic.*

Proof. Let e be an arc of P that is T_1-pancyclic.

If $e \neq (a_{n_1-1}, a_{n_1})$, then by Lemma 6, e is pancyclic in T,

If $e = (a_{n_1-1}, a_{n_1})$ let us prove that e is pancyclic in T. By pancyclicity in T_1, e is contained in a cycle of length l for $3 \leq l \leq n_1$. And $(a_1, a_2, \ldots, a_{n_1}, v, a_1)$ is an (n_1+1)-cycle containing e.

Let now l be an integer of $[n_1+2, n]$. By Proposition 4, there is a hamiltonian path such that $(v_0, v_1, \ldots, v_{k-2}, v_{k-1})$ be a hamiltonian path of X. By Proposition 6, for every $0 \leq i \leq k-1$, there is an ingenerator u_i of $T - X$ dominating v_i.
Let $1 \leq l_1 \leq n-k-n_1-1$ and $0 \leq k' \leq k-1$ such that $k' + l_1 = l - n_1 - 1$. Obviously, a_{n_1} is an outgenerator of $T' = T - [X, Out(X) - a_{n_1}]$. And $u_{k-1-k'}$ is an ingenerator of T'. Thus, by Proposition 5, there is an $(a_{n_1}, u_{k-1-k'})$-path P of length l_1. Hence $(P, v_{k-1-k'}, v_{k-1-k'}, \ldots, v_{k-1}, a_1, a_2, \ldots, a_{n_1})$ is a cycle of length l. ■

Lemma 8 *There are at least $h(T_1) - 1$ pancyclic arcs in T_1.*

Proof. Let $C = (a_1, a_2, \ldots, a_{n_1}, a_1)$ be a hamiltonian cycle of T_1 containing $h(T_1)$ T_1-pancyclic arcs. Let v be an ingenerator of X.

If $v \to T_1$, by Lemma 6, every T_1-pancyclic arc of T_1 is also pancyclic in T.

Hence without loss of generality, we may assume that $v \to a_1$ and $a_{n_1} \to v$. The path $(a_1, a_2, \ldots, a_{n_1})$ contains at least $h(T_1) - 1$ T_1-pancyclic arcs and by Lemma 7 these arcs are T-pancyclic. ■

3.2 The lower bounds

Using the above lemmas, we derive lower bounds for $p(T)$ and $h(T)$.

Definition 1 Let k be an integer and T a tournament. Let us define $\epsilon(k, T)$ as 1 if $|T| \geq k$ and 0 otherwise.

Theorem 2 *If $T - X$ is transitive then*

$$p(T) \geq |A(In(T-X); Out(X))| + |A(In(X); Out(T-X))| + \sum_{j=1}^{l} p(X_j) + 1 \qquad (1)$$

Otherwise

$$p(T) \geq |A(In(T-X); Out(X))| + |A(In(X); Out(T-X))| + \sum_{j=1}^{l} p(X_j)$$

$$+\sum_{i=2}^{m-1}\{2\epsilon(3,T_i)+(p(T_i)-2)\epsilon(4,T_i)\}$$
$$+\epsilon(3,T_1)(h(T_1)-1)+\epsilon(3,T_m)(h(T_m)-1) \qquad (2)$$

Proof. Since every outgenerator of $T-X$ is dominated by an ingenerator of $T-X$, according to Corrolary 1, every arc of $A(In(T-X);Out(X))$ is pancyclic. By duality, every arc of $A(In(X);Out(T-X))$ is pancyclic. According to Lemma 2, there are at least $\sum_{j=1}^{l} p(X_j)$ T-pancyclic arcs in X.

Suppose now that $T-X$ is a transitive tournment. Then since $v_0 \rightarrow t_1$ and $v_0 \leftarrow t_m$ then there is an index i such that (t_i, t_{i+1}, v_0, t_i) is a 3-cycle. So by Lemma 3, this arc is pancyclic. So we obtain Equation 1.

To obtain Equation 2, let us now count the number of pancyclic arcs contained in each T_i such that $|T_i| \geq 3$. If $i=1$ or $i=m$, by Lemma 8 (or its dual), $h(T_i)-1$ arc are T-pancyclic.

If $1<i<m$, then, if T_i is a 3-cycle then by Lemma 5, 2 arcs of T_i are pancyclic in T; and if $|T_i| \geq 4$, by Lemma 4, each T_i-pancyclic arc is pancyclic in T. ■

Note that in the proof of Theorem 2, we also show the following :

Proposition 8 *Let X be a reductor of a strong tournament. There is at least one T-pancyclic arc in $T-X$.*

Theorem 3 *If $T-X$ is transitive then*

$$h(T) \geq 3+\sum_{j=1}^{l}\epsilon(3,X_j).\min\{h(X_j);|X_j|-1\} \qquad (3)$$

otherwise

$$\begin{aligned} h(T) \;\geq\; & 2+\sum_{j=1}^{l}\epsilon(3,X_j).\min\{h(X_j);|X_j|-1\}+\sum_{i=2}^{m-1}\epsilon(3,T_i).\min\{h(T_i);|T_i|-1\} \\ & +\epsilon(3,T_1)(h(T_1)-1)+\epsilon(3,T_m)(h(T_m)-1) \qquad (4) \end{aligned}$$

Proof. For $1 \leq j \leq l$, let P_j be a hamiltonian path of X_j defined as follows :

- If X_j is reduced to a single vertex x_j then $P_j=(x_j)$.
- if $|X_j| \leq 3$, let P_j is obtained from a hamiltonian cycle of X_j containing $h(X_j)$ X_j-pancyclic arcs by removing a non X_j-pancyclic arc if $h(X_j)<|X_j|$ or any arc if $h(X_j)=|X_j|$.

By Lemma 2, each P_j contains $\epsilon(3, X_j). \min\{h(X_j); |X_j| - 1\}$ T-pancyclic arcs. Let v be the beginning of P_1 and w the end of P_l.

Suppose first that $T - X$ is the transitive tournament $TT[t_1, t_2, \dots, t_m]$. By Lemma 3, there exists i such that (t_i, t_{i+1}) is pancyclic and by Corollary 1, (w, t_1) and (t_m, v) are pancyclic. Thus, the hamiltonian cycle $(t_1, t_2, \dots, t_m, P_1, P_2, \dots, P_l, t_1)$ contains $3 + \sum_{j=1}^{l} \epsilon(3, X_j). \min\{h(X_j); |X_j| - 1\}$ pancyclic arcs.

Suppose now that $T - X$ is not transitive.

For $1 < i < m$, let Q_i be a hamiltonian path of T_i defined as follows :

- If T_i is reduced to a single vertex t_i then $Q_i = (t_i)$.
- If $|T_i| = 3$, then Q_i is a path formed by two arcs of T_i that are pancyclic in T. (Such arcs exists according to Lemma 5.)
- If $|T_i| \leq 4$, then Q_i is the path obtained from a hamiltonian cycle of T_i containing $h(T_i)$ T_i-pancyclic arcs by removing a non T_i-pancyclic arc if $h(T_i) < |T_i|$ or any arc if $h(T_i) = |T_i|$.

By Lemma 4, each Q_i contains $\epsilon(3, T_i). \min\{h(T_i); |T_i| - 1\}$ T-pancyclic arcs.

Let $C_1 = (a_1, a_2, \dots, a_{n_1}, a_1)$ be a hamiltonian path of T_1 containing $h(T_1)$ T_1-pancyclic arcs. Then set $Q_1 := (a_1, a_2, \dots, a_{n_1})$. Analogously define Q_m. By (the proof of) Lemma 6 Q_1 (resp. Q_m) contains at least $h(T_1) - 1$ (resp. $h(T_m) - 1$) pancyclic arcs in T.

Then the hamiltonian cycle $(w, Q_1, Q_2, \dots, t_m, P_1, P_2, \dots, P_l)$ gives Equation 4. ■

3.3 Tournaments with $h(T) = 3$

From the lower bounds, one can easily derive Theorem 1 :

Proof of Theorem 1 Let us prove the result by induction on the order n of T. If $n = 3$, it is obviously true.

Suppose now that it is true for strong tournaments of order less than n. Let X be a reductor of T and $T_1 \to T_2 \to \cdots \to T_m$ be a decomposition of $T - X$.

If $T - X$ is transitive, Equation 3 yields $h(T) \geq 3$ and if it is not then Equation 4 gives $h(T) \geq 4$.

Suppose now T is a tournament such that $h(T) = 3$. Let X be a reductor of T. By Equation 4 then X and $T - X$ are a transitive tournaments. Set $X = TT[v_0, v_1, \dots, v_{k-1}]$ and $T - X = TT[t_1, t_2, \dots, t_{n-k}]$. By Corollary 1, (t_{n-k}, v_0) and (v_{k-1}, t_1) are pancyclic. Therefore there is at most one pancyclic arc on the hamiltonian path of $T - X$. Thus by Lemma 3, there is an index i such that $X \to t_j$ if and only if $j \leq i$. And (t_i, t_{i+1}) is pancyclic.

Suppose that T is 2-strong. Then $T - t_1$ and $T - t_{n-k}$ are strong, so $2 \leq i \leq n - k - 2$. By Lemma 1, (t_{n-k-1}, v_0) is pancyclic in $T - t_{n-k}$ and (t_{n-k}, v_1) is pancyclic in $T - v_0$. Thus two arcs are T-pancyclic because they are contained in the n-cycle $C_3 = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_3, \dots, t_{n-k-2}, t_{n-k}, v_1, v_2, \dots, v_{k-2}, t_1, t_{n-k-1})$ if $k \geq 3$ or $C_2 =$

$(t_{n-3}, v_0, t_1, t_{n-2}, v_1, t_2, t_3, \ldots, t_{n-3})$ if $k = 2$. Analogously, (v_{k-1}, t_2) and (v_{k-2}, t_1) are also T-pancyclic. Hence the cycle C_2 or C_3 contains four pancyclic arcs. This is a contradiction.

Thus T is $(=1)-$ *strong*. Then by Lemma 3 it is in $\mathcal{P}_3$. ■

Proposition 9 *If $T \in \mathcal{P}_3$, then T has a unique hamiltonian cycle C and every arc of C is contained in an l-cycle for $4 \leq l \leq |T|$.*

4 Tournaments with $h(T) = 4$

4.1 Useful lemmas

We now prove a generalization of Lemma 3.

Lemma 9 *Let T be a strong tournament, X a reductor of T and $T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_m$ be a decomposition of $T - X$.*

i) For $1 < i < m-1$, if there exist $t_i \in T_i$ and $t_{i+1} \in T_{i+1}$ such that (t_i, t_{i+1}) is in a 3-cycle then (t_i, t_{i+1}) is pancyclic.

ii) If $m \geq 3$ and $T_1 = \{t_1\}$ and there is a vertex $t_2 \in T_2$ such that (t_1, t_2) is in a 3-cycle then (t_1, t_2) is pancyclic.

Proof. Let $(v_0, v_1, \ldots, v_{k-1})$ be a hamiltonian path of X and for $1 \leq j \leq m$ and set $n_j = |T_j|$.

i) Let v be the vertex of X such that (t_i, t_{i+1}, v, t_i) is a 3-cycle. Then v belongs to X and then by Proposition 6, dominates a vertex $t \in T_1$. Then $(t_i, t_{i+1}, v, t, t_i)$ is a 4-cycle. Let $5 \leq l \leq n$. Take $0 \leq k' < k$, $0 < l_1 < -1 + \sum_{j=1}^{i} n_j$ and $0 < l_2 \leq \sum_{j=i+1}^{m} n_j$ such that $k' + l_1 + l_2 + 3 = l$. Let t_1 be an element of T_1 that is dominated by $v_{k'}$ and t_m be an element of T_m dominating v_0. By Proposition 5, there is a (t_1, t_i)-path P_1 of length l_1 in $T[T_1, \ldots, T_i]$, and a (t_{i+1}, t_m)-path P_2 of length l_2 in $T[T_{i+1}, \ldots, T_m]$. Then $(t_i, P_2, v_0, v_1, \ldots, v_{k'}, P_1)$ is an l-cycle containing (t_i, t_{i+1}). Hence (t_i, t_{i+1}) is pancyclic.

ii) Let $4 \leq l \leq n$. Let us prove that (t_1, t_2) is contained in an l-cycle. By Proposition 6, $v_{k-1} \rightarrow t_1$. Let t_m be an element of T_m dominating v_0. Since $m \geq 3$, $T - [X, t_1]$ is reducible and t_2 is one of its outgenerators and t_m one of its ingenerators. Therefore, by Proposition 5, there is a (t_2, t_m)-path P of length $l-3$ in $T - [X, t_1]$. Then $(P, v_0, v_1, \ldots, v_{k-1}, t_1, t_2)$ is the desired l-cycle. ■

Lemma 10 *Let X be the reductor of a 2-strong tournament. Suppose that X is the transitive tournament $TT[v_0, v_1, \ldots, v_{k-1}]$.*

For every $0 \leq j \leq k-2$, the arc (v_j, v_{j+1}) is pancyclic if and only if it is contained in a 3-cycle.

Proof. Let $0 \leq j \leq k-2$ and $4 \leq l \leq n$. Let us prove that (v_j, v_{j+1}) is contained in an l-cycle. There exist $0 \leq l_1 \leq j$, $0 \leq l_2 \leq k-j-2$ and $1 \leq l_3 \leq n-k$ such that $l_1 + l_2 + l_3 + 3 = l$. By Propositon 6, there is an ingenerator u of $T - X$ dominating v_{j-l_1} and an outegenerator t of $T - X$ dominated by v_{j+1+l_2}. And by Proposition 5, in $T - X$, there is a (t, u)-path P of length l_3. Therefore, $(P, v_{j-l_1}, v_{j-l_1+1}, \dots, v_{j+1+l_2}, t)$ is the desired l-cycle. ■

4.2 $h_2(n) \geq 5$

Theorem 4 *For every a 2-strong tournament T, $h(T) \geq 5$.*

Proof. Let T be a 2-strong tournament. Let X be a reductor of T and $T_1 \to T_2 \to \dots \to T_m$ be a decomposition of $T - X$. By Equations 3 and 4, we may assume that X is a transitive tournament, say $TT[v_0, v_1, \dots, v_{k-1}]$, and that at most one of the T_i is not reduced to a single vertex t_i.

I) Suppose that for some $2 \leq i \leq m-1$, T_i is not reduced to a single vertex.
Then by Equation 4, we may assume that T_i is a 3-cycle (a, b, c, a). Without loss of generality, we may assume that both a and b dominate a vertex in X. Then by Lemma 9, (t_{i-1}, a) or one arc of $\{(t_j, t_{j+1}), 1 \leq j \leq i-2\}$ is pancyclic and by (the proof of) Lemma 5, the arcs (a, b) and (c, a) are pancyclic. If c dominates a vertex in X then (b, c) is pancyclic, and if c is dominated a vertex in X then by Lemma (c, t_{i+1}) or one arc of $\{(t_j, t_{j+1}), i+1 \leq j \leq m-1\}$ is pancyclic. In any case, the hamiltonian cycle $(t_1, t_2, \dots, t_{i-1}, a, b, c, t_{i+1}, \dots, t_m, v_0, v_1, \dots, v_{k-1}, t_1)$ contains five pancyclic arcs.

II) Suppose that $|T_1| \geq 3$.

By Equation 4, we may assume that $h(T_1) = 3$. So $T_1 \in \mathcal{P}_3$ according to Theorem 1. Let w be the reductor of T_1 such that $T_1 - w$ is the transitive tournament. Let C_1 be the hamiltonian cycle of T_1; it contains three T_1-pancyclics arcs. For any vertex $r \in T_1$, let r^- (resp. r^+) be the vertex dominating (resp. dominated by) r in C_1.

If for some $2 \leq j < m$, t_j is dominated by a vertex of X, then by Lemma 9 (dual), there is a pancyclic arc in the path $(t_j, t_{j+1}, \dots, t_m)$. Let P be the hamiltonian path of T_1 along C_1 beginning at an outneighbour of v_{k-1}. The hamiltonian cycle $C = (v_{k-1}, P, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_m, v_0, v_1, \dots, v_{k-1})$ contains five pancyclic arcs. Hence we may assume that $X \leftarrow t_j$, for $2 \leq j \leq m$.

If there is a pancyclic arc in the path $(v_0, v_1, \dots, v_{k-1})$, the cycle C contains five pancyclic arcs. So, by Lemma 10, we may assume that $N^+_{T_1}(v_{k-1}) \subseteq N^+_{T_1}(v_{k-2}) \subseteq \dots \subseteq N^+_{T_1}(v_0)$. There are two distinct vertices r_1 and r_2 of T_1 that are dominated by v_{k-1} and so by X.

Let P_1 (resp. P_2) be the subpaths of C_1 with beginning r_1 (resp. r_2) and terminus r_2^- (resp. r_1^-).

1) Suppose first that $m \geq 3$.

Let C' be the cycle defined as follows:

- If $k \geq 3$, then $C' := (v_0, v_{k-1}, P_1, t_m, v_1, v_2, \ldots, v_{k-2}, P_2, t_2, t_3, \ldots, t_{m-1}, v_0)$;
- if $k = 2$, then $C' := (v_0, P_2, t_m, v_1, P_1, t_2, t_3, \ldots, t_{m-1}, v_0)$.

Since C_1 contains three T_1-pancyclic arcs, one of them, say e is contained in P_1 or P_2. So, by Lemma 6, e is pancyclic in T. Moreover C' contains the four arcs (t_m, v_1), (t_{m-1}, v_0), (v_{k-2}, r_2) and (v_{k-1}, r_1). By Lemma 1, these arcs are pancyclic in respectively in $T - [v_0]$, $T - [t_m]$, $T - [v_{k-1}]$ and T. Hence because they are contained in C', these four arcs are T-pancyclic. Thus C' contains five pancyclic arcs.

2) Suppose now that $m = 2$. Then v_0 (and then every v_j, $0 \leq j \leq k-1$) has an inneighbour s_2^- in T_1 which then dominates X. Without loss of generality we may suppose that this inneighbour s_2^- is between r_1 and r_2 in C_1 and that $s_2 \to v_0$. Let Q_1 be the (r_1, s_2^-)-path and Q_2 the (s_2, r_1^-)-path along C_1.

Set $C_0 := (v_0, Q_2, t_2, v_1, v_2, \ldots, v_{k-1}, Q_1, v_0)$. It is easy to check that (v_0, s_2) and (t_2, v_1) and (v_{k-1}, r_1) are pancyclic in T. If $P_1 \cup P_2$ contains two T_1-pancyclic arcs then these two are also pancyclic in T by Lemma 6. So C_0 contains five pancyclic arcs.

Hence we may assume that both (r_1^-, r_1) and (s_2^-, s_2) are T_1-pancyclic. Let e be the third T_1-pancyclic arc. Let x be a vertex of Q_1 such that $v_0 \to x$ and $x^+ \to v_0$. Such a vertex exists since $v_0 \to r_1$ and $s_2^- \to v_0$.

The arc (x, x^+) is contained in the 3-cycle (x, x^+, v_0, x). For $4 \leq l \leq n_1$, according to Proposition 9 it is contained in an l-cycle because it belongs to C_1. The arc (x, x^+) is also in the n_1+1-cycle (v_0, Q_2, Q_1, v_0) and the n_1+2-cycle $(v_0, Q_2, Q_1, t_2, v_0)$. For $n_1 + 3 \leq l \leq n$, it is contained in the l-cycle $C(l) = (v_0, Q_2, t_2, v_1, v_2, \ldots v_{k+l-n-1}, Q_1, v_0)$. Hence (x, x^+) is pancyclic. If $e \neq (x, x^+)$ then $C(n)$ contains five pancyclic arcs. Thus, we may assume that $e = (x, x^+)$, so $e \in Q_1$.

Three cases may arise:

- a) Suppose that $w = s_2 = r_1^-$. Then $T_1 - [s_2^-, s_2] \to s_2^-$, thus (s_2^-, v_0, r_1, s_2^-) is a 3-cycle. For $4 \leq l \leq n_1 + 1$, (s_2^-, v_0) is contained in the l-cycle obtained by replacing the arc (s_2^-, s_2) by (s_2^-, v_0, s_2) in the $(l-1)$-cycle in T_1. In particular, it is contained in the (n_1+1)-cycle $D = (v_0, Q_2, Q_1, v_0)$. Now since $v_0 \to X - v_0$ and $X \to r_1$, by Proposition 3 (applied to k' vertices of $X - v_0$ one after another), (v_0, Q_2) may be extended into a (v_0, s_2^-)-path of length $|Q_2| + k'$, for $1 \leq k' \leq k-1$. So D may be extended into an l-cycle containing (s_2^-, v_0) for $n_1 + 2 \leq l \leq n-1$. Moreover (s_2^-, v_0) is contained in C_0. Hence, (s_2^-, v_0) is pancyclic and C_0 contains five pancyclic arcs.

- b) Suppose that $w = r_1$. Then $r_1 \to T_1 - Q_2$ and $Q_2 \to r_1$. Then (s_2^-, v_0, r_1, s_2^-) is a 3-cycle. For $4 \le l \le n_1 + 1$, pick Q_1' a subpath of Q_1 terminating at s_2^- of length $l_1 < |Q_1| - 2$ and Q_2' a subpath of Q_2 beginning in s_2 of length $l_2 < |Q_2| - 2$ such that $l_1 + l_2 + 4 = l$, then $(s_2^-, v_0, Q_2', r_1, Q_1')$ is an l-cycle. Analogously to the end of Case a), we obtain that (s_2^-, v_0) is pancyclic and C_0 contains five pancyclic arcs.
- Suppose that $w = s_2^- = x^+$. Then $r_1 \to w$, so (s_2^-, v_0, r_1, s_2^-) is a 3-cycle. And $T_1 - s_2^-$ is a transitive tournament with outgenerator s_2 and ingenerator x. Thus for $4 \le l \le n_1 + 1$, in $T_1 - s_2^-$, there is an (s_2, x)-path P of length $l - 2$. Hence, (Q, s_2^-, v_0, s_2) is an l-cycle. Again, analogously to the end of Case a), we obtain that (s_2^-, v_0) is pancyclic and C_0 contains five pancyclic arcs.

III) Suppose now that $T - X$ is a transitive tournament.

If $v_0 \to t_{n-k-1}$, then $X' = T[t_{n-k}, v_1, v_2, \ldots, v_{k-1}]$ is a reductor of T and $T - X'$ is not transitive. Indeed v_0 is dominated by some vertex t_i and then (t_i, v_0, t_1, t_i) is a 3-cycle. So we have the result by one of the previous cases.

Analogously, we obtain the result if $v_{k-1} \leftarrow t_2$. Thus we may assume that $v_0 \leftarrow t_{n-k-1}$ and $v_{k-1} \to t_2$.

By Lemma 1, (t_{n-k-1}, v_0) is pancyclic in $T - t_{n-k}$ and (t_{n-k}, v_1) is pancyclic in $T - v_0$. These two arcs are also T-pancyclic because they are contained in the n-cycle $C_3 = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_3, \ldots, t_{n-k-2}, t_{n-k}, v_1, v_2, \ldots, v_{k-2}, t_1, t_{n-k-1})$ if $k \ge 3$ or $C_2 = (t_{n-3}, v_0, t_1, t_{n-2}, v_1, t_2, t_3, \ldots, t_{n-3})$ if $k = 2$. Analogously, (v_{k-1}, t_2) and (v_{k-2}, t_1) are T-pancyclic.

Suppose that $k \ge 3$, then the four pancyclic arcs (t_{n-k-1}, v_0), (t_{n-k}, v_1), (v_{k-1}, t_2) and (v_{k-2}, t_1) are contained in the two n-cycles C_3 and $C_3' = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_{n-k}, v_1, v_2, \ldots, v_{k-2}, t_1, t_3, t_4, \ldots, t_{n-k-1})$. If an arc in $\{(t_i, t_{i+1}), 2 \le i \le n - k - 2\}$ is T pancyclic then C_3 or C_3' contains five pancyclic arcs. Therefore we may assume that no arcs in $\{(t_i, t_{i+1}), 2 \le i \le n - k - 2\}$ is T pancyclic. By Lemma 3, $t_i \to v_0$ for $2 \le i \le n - k$ and (t_1, t_2) is pancyclic, and $v_{k-1} \to t_i$ for $1 \le i \le n-k-1$ and (t_{n-k-1}, t_{n-k}) is pancyclic. Thus at least one of the arcs (v_i, v_{i+1}) is in a 3-cycle and so is pancyclic according to Lemma 10. Hence the cycle $C = (v_0, v_1, \ldots, v_{k-1}, t_1, t_2, \ldots, t_{n-k}, v_0)$ contains five pancyclic arcs.

Suppose now that $k = 2$. We may assume that there is no pancyclic arcs in $\{(t_i, t_{i+1}), 2 \le i \le n - 4\}$, otherwise C_2 contains five pancyclic arcs. Then by Lemma 3, $t_i \to v_0$ for $2 \le i \le n - 2$ so (t_1, t_2) is pancyclic, and $v_1 \to t_i$ for $1 \le i \le n - 3$ so (t_{n-3}, t_{n-2}) is pancyclic. Moreover (v_0, v_1) is in the 3-cycle (v_0, v_1, t_2, v_0), so by Lemma 10, it is pancyclic. Hence the cycle $C = (v_0, v_1, t_1, t_2, \ldots, t_{n-2}, v_0)$ contains five pancyclic arcs. ■

This result is best possible since the regular tournament R_5 on five vertices is 2-strong and obviously satisfies $h(R_5) = 5$.

4.3 Tournaments such that $h(T) = 4$

Definition 2 *A tournament is in $\mathcal{P}_4$ if there is a vertex v such that $T - v$ is the transitive tournament $TT[t_1, t_2, \dots, t_m]$ and three integers $1 < i_1 < i_2 < i_3 \le m$ such that $v \to t_j$ if and only if $1 \le j < i_1$ or $i_2 \le j < i_3$. Note that if $T \in \mathcal{P}_4$ then $\overline{T} \in \mathcal{P}_4$.*

Proposition 10 *If $T \in \mathcal{P}_4$ then $p(T) = h(T) = 4$.*

Proof. Let T be a tournament of $\mathcal{P}_4$. The only possible pancyclic arcs are those contained in the unique hamiltonian cycle $C = (v, t_1, t_2, \dots, t_m, v)$ of T. By Corollary 1 (v, t_1) and (t_m, v) are pancyclic and by Lemma 3, (t_{i_1-1}, t_{i_1}) and (t_{i_3-1}, t_{i_3}) are pancyclic. The other arcs of C are not pancyclic because they are in no 3-cycle. ■

Definition 3 The tournament H_4 is the tournament depicted Figure 1 with vertex set $\{v, w, u, s, t\}$ such that (w, u, s) is a 3-cycle dominating t and $v \to \{w, u\}$ and $v \leftarrow \{s, t\}$.

The tournament I_4 is the tournament depicted Figure 1.

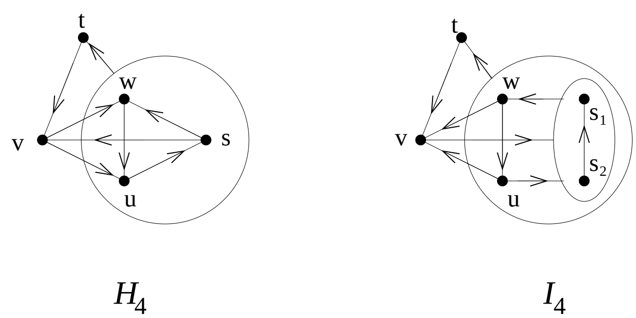

Figure 1: The tournaments H_4 and I_4

A tournament is in $\mathcal{H}_4$ if it is obtained from H_4 by blowing up the vertex s into a transitive tournament S.

A tournament is in $\mathcal{H}'_4$ if it is obtained from H_4 by blowing up the vertex t into a transitive tournament S.

A tournament T is in $\overline{\mathcal{H}_4}$ (resp. $\overline{\mathcal{H}'_4}$) if its dual $\overline{T}$ is in $\mathcal{H}_4$. (resp. $\mathcal{H}'_4$).

Proposition 11 *Let T be a tournament of $\mathcal{H}_4$. Then $h(T) = 4$. If $T \neq H_4$ then $p(T) = 6$ and $p(H_4) = 5$.*

Proof. Let T be a tournament of $\mathcal{H}_4$. Let $P = (s_1, s_2, \dots, s_{n-4})$ be the (unique) hamiltonian path of S. T has only two hamiltonian cycles, $C = (v, u, P, w, t, v)$ and $C' = (v, w, u, P, t, v)$ whose union contains all the T-pancyclic arcs. Since v is a reductor, by Corollary 1, (t, v), (v, w) and (w, u) are T-pancyclic and, by Lemma 6, (w, u) and

(u, s_1) are T-pancyclic. Moreover, (w, t) is not pancyclic because it is in no 4-cycle, and each arc of (P, t) is not pancyclic because it is in no 3-cycle.

$T[w, u, S]$ is in $\mathcal{P}_3$ and thus (s_{n-4}, w) is pancyclic in $T[w, u, S]$, so it is contained in an l-cycle for all length $3 \leq l \leq n-2$. And for $5 \leq l \leq n$, the arc (s_{n-4}, w) is contained in the l-cycle $(s_{n-4}, w, t, v, , u, s_{n+1-l}, \ldots, s_{n-4})$. So if $n \geq 6$, then (s_{n-4}, w) is pancyclic. It is easy to check that if $n = 5$ ($T = H_4$), the arc (s_{n-4}, w) is contained in no 4-cycle. ■

Proposition 12 *Let T be a tournament of $\mathcal{H}'_4$. Then $h(T) = 4$. And if $T \neq H_4$ then $p(T) = 6$.*

Proof.

Let $P = (t_1, t_2, \ldots, t_{n-4})$ be the hamiltonian path of S. v is a reductor of T.

By Corollary 1, (t, v), (v, w) and (v, u) are pancyclic. And By Lemma 6, (w, u) and (u, s) are pancyclic. T has only two hamiltonian cycles, $C = (v, u, s, w, P, v)$ and $C' = (v, w, u, s, P, v)$ whose union contains all the T-pancyclic arcs.

Now an arc in (s, P) is not pancyclic in T because it is contained in no 3-cycle ; (s, w) is not pancyclic because it is contained in no 4-cycle.

$T[v, w, S]$ is in $\mathcal{P}_3$ and thus (w, t_1) is pancyclic in $T[v, w, S]$, so it is contained in a l-cycle for all length $3 \leq l \leq n-2$. And for $5 \leq l \leq n$, the arc (w, t_1) is contained in the l-cycle $(v, u, s, w, t_1, t_2, \ldots, t_{l-5}, v)$. So if $T \neq H_4$, then $n \geq 6$, so (w, t_1) is pancyclic. ■

Proposition 13 $h(I_4) = 4$ *and* $p(I_4) = 6$.

Proof. It is easy to check that the pancyclic arcs in I_4 are (t, v), (v, s_1), (v, s_2), (s_1, w), (s_2, w) and (w, u). ■

Proposition 14 *Let T be a $(= 1)$-strong tournament with reductor v, $T_1 \to T_2 \to \cdots \to T_m$ a decomposition of $T - v$ such that $|T_1| \geq 3$ and (b, b^+) be an arc of a hamiltonian cycle C_1 of T_1.*

If $v \to b$ and $b^+ \to v$, then (b, b^+) is pancyclic.

Proof. Since T is strong v is dominated by a vertex t_m of T_m. (b, b^+, v, b) is a 3-cycle. Let $4 \leq l \leq n$. By Proposition 5, there is an (b^+, t_m)-path of length $l-3$ in $T - [v, b]$. So (P, v, b, b^+) is an l-cycle. ■

Theorem 5 *Let T be a tournament. $h(T) = 4$ if and only if $T \in \mathcal{P}_4 \cup \mathcal{H}_4 \cup \mathcal{H}'_4 \cup \overline{\mathcal{H}_4} \cup \overline{\mathcal{H}'_4} \cup \{I_4, \overline{I_4}\}$.*

Proof. Suppose that $h(T) = 4$. By Theorem 4, T is $(= 1)$-strong. Let v be a reductor of T and $T_1 \to T_2 \to \cdots \to T_m$ be a decomposition of $T - v$. By Equation 4, at most one of the T_i is not reduced to a single vertex t_i.

- Suppose that $T - v$ is transitive, then it has a unique hamiltonian cycle and then by Lemma 3, $T \in \mathcal{P}_4$.

- Suppose now that there is some $2 \leq i \leq m-1$ such that $|T_i| \geq 3$. Then we have the result in the same way as in Theorem 4.

- Suppose now that $|T_1| \geq 3$. If $d^+(v) = 1$, then its outneighbour v_1 is also a reductor and all the components of $T - v_1$ are reduced to a vertex except possibly the insection. Hence we may assume that $d^+(v) \geq 2$.

  It follows from Equation 4 that $c(T_1) = 3$. So $T_1 \in \mathcal{P}_3$. Let w be the reductor of T_1 and $T_1 - w = TT[u_1, u_2, \dots, u_{n_1}]$, and let i_1 be the integer such that $w \to u_i$ if and only if $i \geq i_1$. Let $C_1 = (w, u_1, u_2, \dots u_{n_1}, w)$ be the unique hamiltonian cycle of T_1.

  The three arcs $e_1 = (u_{n_1}, w)$, $e_2 = (w, u_1)$ and $e_3 = (u_{i_1-1}, u_{i_1})$ are T_1-pancyclic. And by the proof Lemma 8, two of them are T-pancyclic and they are contained is a path P_1 starting at an outneighbour of v and that is hamiltonian in T_1.

  $T - [v, T_1] \to v$ otherwise by Lemma 9, there is a pancyclic arcs in the path $(t_2, t_3, \dots t_m)$. And thus $(v, P_1, t_2, t_3, \dots t_m, v)$ contains five pancyclic arcs.

  If $v \to T_1$, then by Lemma 6, e_1, e_2 and e_3 are T-pancyclic. And for $x \in T_1$, the arc (x, t_2) is pancyclic. Indeed, for any $3 \leq l \leq n$, there exist $0 \leq l_1 \leq n_1$ and $0 \leq l_2 \leq m-2$ such that $l_1 + l_2 + 3 = l$. By Lemma 5, in T_1, there is path Q_1 of length l_1 terminating in x. So $(v, Q_1, t_2, t_3, \dots, t_{l_1+2}, v)$ is an l-cycle. Thus $h(T) \geq 5$.

  So we may suppose that $A = \{(a, a^+) \in C_1, a \to v \to a^+\}$ and $B = \{(b, b^+) \in C_1, b^+ \to v \to b\}$ are not empty.

  We have $A \subset \{e_1, e_2, e_3\}$. Otherwise let $(a, a^+) \in A \setminus \{e_1, e_2, e_3\}$. The path P_{a^+} obtained from C_1 by removing the arc (a, a^+) contains e_1 e_2 and e_3 that are T-pancyclic by Lemma 7. Hence $(v, P_{a^+}, t_2, t_3, \dots, t_m, v)$ contains five pancyclic arcs.

  Now $B \subset \{e_1, e_2, e_3\}$. Otherwise let (a, a^+) be an arc of A. The path P_{a^+} obtained from C_1 by removing the arc (a, a^+) contains two arcs of $\{e_1, e_2, e_3\}$ that are T-pancyclic by Lemma 7 and an arc of $B \setminus \{e_1, e_2, e_3\}$ that is also T-pancyclic by Proposition 14. Hence $(v, P_{a^+}, t_2, t_3, \dots, t_m, v)$ contains five pancyclic arcs.

  So, we have $|A| = |B| = 1$.

  Five cases may arise

  i) If $A = \{e_1\}$ and $B = \{e_2\}$, then $T_1 - w \to v$. So $T - w$ is transitive and $T \in \mathcal{P}_4$.

  ii) If $A = \{e_1\}$ and $B = \{e_3\}$, then e_2 and e_3 are T-pancyclic.

  Suppose that $i_1 \geq 3$. The arc e_1 is contained in an l-cycle for $3 \leq l \leq n_1 + 1$ because it is T_1-pancyclic. And for $5 \leq l \leq n$, there exist $1 \leq l_1 \leq n_1 - 2$ and $0 \leq l_2 \leq m-2$ such that $l_1 + l_2 + 4 = l$. By Proposition 5, in $T_1 - [w, u_1]$, there is a (u_2, u_{n_1})-path Q_1 of length l_1. Then $(v, Q_1, w, t_2, t_3, \ldots, t_{2+l_2}, v)$ is an l-cycle. Hence if $n_1 \geq 3$, e_1 is also pancyclic and $(v, u_2, u_3, \ldots u_{n_1}, w, u_1, t_2, t_3, \ldots, t_m, v)$ contains five pancyclic arcs. So $n_1 = 2$, that is $|T_1| = 3$. Then $T \in \mathcal{H}'_4$.

Suppose that $i_1 = 2$. If $m \geq 3$, then the arc (w, t_2) is pancyclic. Indeed, it is in the 3-cycle (w, t_2, v, w) and the 4-cycle (w, t_2, t_3, v, w). And for $5 \leq l \leq n$, choose $2 \leq l_1 \leq n_1$ and $2 \leq l_2 \leq m$ such that $l_1 + l_2 + 1 = l$. Then $(v, u_1, u_2, \ldots, u_{l_1}, w, t_2, t_3, \ldots, t_m, v)$ is an l-cycle. Thus $h(T) \geq 5$. So $m = 2$ and $T \in \mathcal{H}_4$.

iii) If $A = \{e_2\}$ and $B = \{e_3\}$, then e_1 and e_3 are T-pancyclic. If $i_1 = 2$ then $T - u_1$ is a transitive tournament, so $T \in \mathcal{P}_4$. Suppose now that $i_1 \geq 3$. Let us consider the arc $e = (u_{i_1-1}, t_2)$. For $3 \leq l \leq m + i_1 - 1$, let $0 \leq l_1 \leq i_1$ and $0 \leq l_2 \leq m - 2$, such that $l_1 + l_2 + 3 = l$. Then e is in the l-cycle $(v, u_{i_1-l_1}, u_{i_1-l_1+1}, \ldots, u_{i_1}, t_2, t_3, \ldots t_{2+l_2}, v)$. For $6 \leq l \leq n$, there exists $1 \leq l_3 \leq n_1 - 2$ and $0 \leq l_4 \leq m - 2$ such that $l_3 + l_4 + 5 = l$. By Proposition 5, there is a (u_1, u_{n_1})-path of length l_3 in $T_1 - [w, u_{i_1-1}]$. Then e is in the l-cycle $(v, P_1, w, u_{i_1-1}, t_2, t_3, \ldots t_{2+l_4}, v)$. Hence if $m + i_1 - 1 \geq 5$, e is T-pancyclic and $h(T) \geq 5$. Thus we may assume $m = 2$ and $i_1 = 3$, so $n_1 = n - 3$.

Suppose that $n_1 \geq 4$, then e_2 is pancyclic. Indeed it is contained in an l-cycle for every length $3 \leq l \leq n - 2$, because it is T_1-pancyclic, and it is in the n-cycle $C = (v, u_2, u_3, \ldots, u_{n_1}, w, u_1, t_2, v)$ and the $(n-1)$-cycle $(v, u_2, u_3, \ldots, u_{n_1-1}, w, u_1, t_2, v)$. Thus C contains five pancyclic arcs which is a conradiction.

Hence $n_1 = 3$, so $T = I4$.

iv) If $A = \{e_3\}$ and $B = \{e_1\}$, then e_1 and e_2 are T-pancyclic. If $i_1 \leq n_1 - 2$ then by Lemma 6, e_3 is also pancyclic, so $(v, u_{n_1}, w, u_1, u_2, \ldots u_{n_1-1},\ t_2, t_3, \ldots, t_m, v)$ contains five pancyclic arcs, which is a contradiction.

If $i_1 = n_1$ and $T - u_{n_1}$ is a transitive tournament, so $T \in \mathcal{P}_4$.

So we may assume $i_1 = n_1 - 1$. The tournament $T' = T[v, u_{n_1-1}, u_{n_1}, t_2, t_3, \ldots t_m]$ is in $\mathcal{P}_3$ and $e' = (u_{n_1}, t_2)$ is pancyclic in T'. Thus it is contained in an l-cycle for every length $3 \leq l \leq m + 2$. And $6 \leq l \leq n$, the arc e' is in the l-cycle $(v, u_{n_1-1}, w, u_1, \ldots, u_{l_1+1}, u_{n_1}, t_2, t_3, \ldots, t_m, v)$ where l_1 and l_2 are two integers such that $0 \leq l_1 \leq i_1 - 2$, $0 \leq l_2 \leq m - 2$ and $l_1 + l_2 + 6 = l$. So if $m \geq 3$, e' is pancyclis and $h(T) \geq 5$.

Hence we may assumme that $m = 2$. Now if $n_1 \geq 4$, then $e_3 = (u_{n_1-2}, u_{n_1-1})$ is pancyclic. Indeed it is T_1 pancyclic and contained int the $(n-1)$-cycle $(v, u_{n_1}, w, u_2, u_3, \ldots, u_{n_1-1}, t_2, t_3, \ldots, t_m, v)$ and the n-cycle $C = (v, u_{n_1}, w, u_1, u_2, \ldots, u_{n_1-1}, t_2, t_3, \ldots, t_m, v)$. Hence C contains five pancyclic arcs, which is a contradition.

So $n_1 = 3$ and $T = I_4$.

v) If $A = \{e_3\}$ and $B = \{e_2\}$, then e_1 and e_2 are T-pancyclic. If $i_1 < n_1$, then $n_1 \geq 3$. The arc e_3 is contained in any cycle of length $3 \leq l \leq n_1+1$ because it is T_1-pancyclic. Let $5 \leq l \leq n$. Pick $0 \leq l_1 \leq i_1-2$, $0 \leq l_2 \leq n_1-i_1$ and $0 \leq l_3 \leq m-2$. Then e_3 is in the l-cycle $(v, w, u_{i_1-1-l_1}, u_{i_1-l_1}, \dots, u_{i_1+l_2}, t_2, t_3, \dots, t_{2+l_3}, v)$. Thus, e_3 is pancyclic and $(v, u_{n_1}, w, u_1, u_2, \dots u_{n_1-1}, t_2, t_3, \dots, t_m, v)$ contains five pancyclic arcs which is a contradiction.

Hence $i_1 = n_1$. Then u_1 is a reductor of T_1 such that $T - u_1$ is transitive. So we are in Case ii).

- If $|T_m| \geq 3$, by duality, we have that $\overline{T} \in \mathcal{P}_4 \cup \mathcal{H}_4 \cup \{I4\}$. Thus $T \in \mathcal{P}_4 \cup \overline{\mathcal{H}_4} \cup \{\overline{I4}\}$.

■

Theorem 5 and Proposition 11, 12 and 13 yield immediatly the following result:

Corollary 3 *A tournament has exactly 4 pancyclic arcs if and only if it is in $\mathcal{P}_4$.*

5 Number of pancyclic arcs in k-strong tournaments

Theorem 6 (Yao, Guo and Zhang, [5]) *Every tournament contains a vertex x such that every outgoing arc is pancyclic.*

From this result, we derive lower bounds on the number of pancyclic arcs in a strong tournament :

Lemma 11

$$p(T) \geq h(T) + \delta^+ - 1 \quad (5)$$
$$p(T) \geq h(T) + \delta^+ + \delta^- - 3 \quad (6)$$

Proof. By Theorem 6 and its dual, there is a vertex x such that every outgoing arc is pancyclic and a vertex y such that every ingoing arc is pancyclic. Obviously, $|A^+(x) \cap A^-(y)| \leq 1$. There are $h(T)$ pancyclic arcs on a hamiltonian cycle. At most one of them is in $A^+(X)$ and at most two of them are in $A^+(x) \cup A^-(y)$. Hence, $p(T) \geq h(T) + d^+(x) - 1 \geq h(T) + \delta^+ - 1$, and

$$\begin{aligned} p(T) &\geq d^+(x) + d^-(y) - |A^+(x) \cap A^-(y)| + h(T) - 2 && (7) \\ &\geq d^+(x) + d^-(y) + h(T) - 3 && (8) \end{aligned}$$

■

It follows directly from Theorems 4 and Equation 6, that for $k \geq 2$, every k-strong tournament has at least $2k + 2$ pancyclics arcs.

Corollary 4 *Every k-strong tournament has at least $2k+2$ pancyclic arcs.*

We now prove a slighty better result.

Theorem 7 *Every k-strong $(k \geq 2)$ tournament has at least $2k+3$ pancyclic arcs.*

Proof. Let x and y be vertices such that the arcs of $A^+(x) \cup A^-(y)$ are pancyclic. By Equation 7, we have the result, if $d^+(x) \geq k+1$ or $d^-(y) \geq k+1$ or $A^+(x) \cap A^-(y) = \emptyset$. Thus we may assume that $x \to y$, and $d^+(x) = d^-(y) = k$.

Then $X = N^+(x)$ is a reductor of T containing y. We have $2k-1$ pancyclic arcs in $A^+(x) \cup A^-(y)$ and by Corollary 1, there is at least one pancyclic arc e_1 in $A(X, Out(T-X))$.

If X is not transitive, then according to Lemma 2, there are at least three pancyclic arcs in X, with at most one of them in $A^-(y)$. And by Proposition 8, there is at most one pancyclic arc in $T-X$. It follows that $p(T) \geq 2k+3$. Hence we may assume that X is the transitive tournament $TT[v_0, v_1, \ldots, v_{k-1}]$.

Let $T_1 \to T_2 \to \cdots \to T_m$ be a decomposition of $T-X$. By Lemmas 4, 5 and 6, at most one of the T_i is not reduced to the vertex t_i.

Suppose that for some $2 \leq i \leq m-1$, T_i is not reduced to a single vertex. If $|T_i| \geq 4$ then by Lemma 4, there are at least three pancyclic arcs in T_i. Thus $p(T) \geq 2k+3$. Asssume now that T_i is a 3-cycle. Two arcs of T_i are pancyclic by Lemma 5. Let t_i be a vertex of T_i. By Lemma 9, if $t_i \to v_0$, then (t_{i-1}, t_i) is pancyclic and if $t_i \leftarrow v_0$, then (t_i, t_{i+1}) is pancyclic. Hence, $p(T) \geq 2k+3$.

Suppose now that T_1 is not reduced to a single vertex. By Lemma 8, we have the result if $h(T_1) \geq 4$. So we may assume that $T_1 \in \mathcal{P}_3$. Also by Lemma 8, there are two pancyclic arcs in T_1. According to Proposition 7 (dual), there are two distinct vertices w_1 and w_2 of T_1 such that $v_{k-1} \to w_1$ and $v_{k-2} \to w_2$. Let C_1 be the hamiltonian cycle of T_1. For any vertex $r \in T_1$, let r^- (resp. r^+) be the vertex dominating (resp. dominating by) r in C_1.

- Suppose first that $m \geq 3$. If a vertex t_i with $2 \leq i \leq m-1$ dominates an element of X then, by Lemma 9, one of the arcs (t_j, t_{j+1}) with $i \leq j \leq m-1$ is pancyclic. Hence $p(T) \geq 2k+3$. So we may assume that $T - [T_1, X] \to X$. Let us prove that (v_{k-2}, w_2) is pancyclic : it is contained in the hamiltonian cycle $(P_2, t_m, v_{k-1}, P_1, t_2, t_3, \ldots, t_{m-1}, v_0, v_1, \ldots v_{k-2}, w_2)$, where P_1 (resp. P_2) is the (w_1, w_2^-)-path (resp. (w_2, w_1^-)-path) along C_1. For $3 \leq l \leq n-1$, let $0 \leq k' \leq k-2$, $0 \leq m' \leq m-2$ and $0 \leq l_1 \leq n_1 - 1$ such that $l_1 + k' + m' + 3 = l$. There is a path Q_1 of length l_1 beginning in w_2. Thus (v_{k-2}, w_2) is contained in the l-cycle $(v_{k-2}, Q_2, t_{m-m'}, t_{m-m'+1}, \ldots, t_m, v_{k-2-k'}, v_{k-1-k'}, \ldots, v_{k-2})$.

  Hence T has at least $2k+3$ pancyclic arcs.

- Suppose now that $m = 2$. Since $d^-_X(v_0) \leq k-2$, v_0 is dominated by at least one vertex in T_1. Hence there exists a vertex w_0 of T_1 dominated by v_0 such that $w_0^- \rightarrow v_0$.

  Because $d^+_X(v_{k-1}) \leq k-2$, v_{k-1} dominates at least two vertices of T_1. At least one of them, say w_1, is distinct of w_0.

  Let us now prove that (v_0, w_0) is pancyclic. For $3 \leq l \leq n_1 + 1$, let Q_0 be a path in T_1 starting at w_0 of length $l-3$. Then (v_0, Q_0, t_2, v_0) is an l-path. Let Q_1 (resp. Q_2) be the subpath of C_1 starting at w_1 (resp. w_0) and ending in w_0^- (resp. w_1^-) along C_1. Then for $n_1 + 2 \leq l \leq n$, $(Q_1, v_0, Q_2, t_2, v_{n+1-l}, v_{n-l}, \ldots, v_{k-1}, w_1)$ is an l-path containing (v_0, w_0).

  Hence T has at least $2k+3$ pancyclic arcs.

Suppose now that $T - X$ is the transitive tournament $TT[t_1, t_2, \ldots, t_{n-k}]$.
Let i_0 be the smallest integer $i > 1$ such that $v_{k-1} \rightarrow t_i$.

- Suppose that $i_0 = 2$. Since T is 2-strong, v_0 is dominated by vertex t_{i_1} distinct from t_{n-k}. By Lemma 1, (v_{k-1}, t_2) and (v_{k-2}, t_1) are respectively $(T - t_1)$- and $(T - v_{k-1})$-pancyclic, thus they are contained in l-cycle for any $3 \leq l \leq n-1$. And they both are in the following hamiltonian cycle : $(v_{k-1}, t_2, t_3, \ldots, t_{i_1}, v_0, v_1, \ldots, v_{k-2}, t_1, t_{i_1+1}, t_{i_1+2}, \ldots t_{n-k}, v_{k-1})$. Thus they both are pancyclic in T. Moreover, by Proposition 8, there is a T-pancyclic arc in $T - X$. Hence, $p(T) \geq 2k+3$.

- Suppose now that $i_0 > 2$. Then $X' = T[X - v_{k-1}, t_1]$ is a reductor of X. And by Lemma 3, (t_1, t_2) is T-pancyclic.

  The subtournament $T' = T[t_{i_0}, t_{i_0+1}, \ldots t_{n-k}, v_{k-1}]$ is a strong component of $T - X'$. By Lemma 8, T' contains $h(T') - 1$ pancyclic arcs in T. And at most one of them is in $A^+(x)$. Thus, if $h(T') \geq 4$, we obtain $p(T) \geq 2k+3$. So we may assume that $h(T') = 3$, so $T' \in \mathcal{P}_3$. Since $T' - v_{k-1}$ is transitive, (t_{n-k}, v_{k-1}) and (v_{k-1}, t_{i_0}) are T'-pancyclic. Let e' be the third T'-pancyclic arc. By Lemma 6, e' is also T-pancyclic because $t_{n-k} = x$ dominates v_0.

  If v_0 is dominated by a vertex of $T' - [x, v_{k-1}]$ then by Lemma 6, (v_{k-1}, t_{i_0}) is pancyclic in T and so $p(T) \geq 2k+3$.

  If v_0 dominates $T' - [x, v_{k-1}]$, it must have an inneighbour t_{i_2} with $2 \leq i_2 \leq i_0 - 1$. Hence, by Proposition 9, there is a pancyclic arc (t_i, t_{i+1}) with $1 \leq i \leq i_0 - 2$. So, $p(T) \geq 2k+3$.

■

6 Tournaments with exactly five pancyclic arcs

Definition 4 A tournament is in $\mathcal{P}_5$ if there is a vertex v such that $T-v$ is the transitive tournament $TT[t_1, t_2, \ldots, t_m]$ and three integers $1 < i_1 < i_2 < i_3 < i_4 < i_5 \leq m$ such that $v \rightarrow t_j$ if and only if $1 \leq j < i_1$ or $i_2 \leq j < i_3$ or $i_4 \leq j < i_5$.

The tournament Q_5 depicted Figure 2 is constructed from the disjoint union of the transitive tournament $TT[t_1, t_2, t_3, t_4, t_5]$ and the arc (u, v) such that $N^-_{Q_5}(v) = \{u, t_2\}$ and $N^+_{Q_5}(u) = \{v, t_4\}$. A tournament is in $\mathcal{Q}_5$ if it is obtained from Q_5 by blowing up each

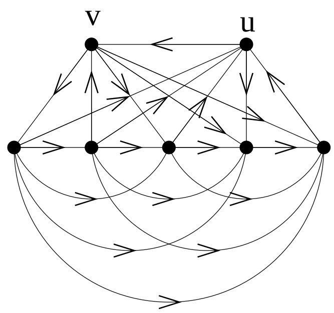

Figure 2: The tournament Q_5

vertex t_i in a transitive tournament T_i for $1 \leq i \leq 5$. Note that if $T \in \mathcal{Q}_5$ then $\overline{T} \in \mathcal{Q}_5$.

Proposition 15 *If $T \in \mathcal{Q}_5$, then $p(T) = h(T) = 5$.*

Proof. Let T be a tournament in $\mathcal{Q}_5$. For $1 \leq i \leq 5$, let P_i be the unique hamiltonian path of T_i and let s_i (resp. t_i) be the beginning (resp. terminus) of P_i. T has a unique hamiltonian cycle $(u, v, P_1, P_2, P_3, P_4, P_5, u)$. For $1 \leq i \leq 5$, any arc in P_i is not pancyclic since it is contained in no 3-cycle. Moreover, (t_2, s_3) and (s_3, t_4) are also in no 3-cycle. Hence the only possible pancyclic arcs are (u, v), (v, s_1), (t_1, s_2) (t_4, s_5) and (t_5, u). It is simple matter to check that these five arcs are pancyclic. So $p(T) = h(T) = 5$. ■

Theorem 8 *A tournament has exactly 5 pancyclic arcs if and only if it is in $\mathcal{P}_5 \cup \mathcal{Q}_5 \cup \{H_4, \overline{H_4}\}$.*

Proof. It is easy to check that every tournament of $\mathcal{P}_5$ has exactly five pancyclic arcs. And by Propositions 15 and 11 every tournament of $\mathcal{Q}_5 \cup \{H_4, \overline{H_4}\}$ has exactly five pancyclic arcs.

Let T be a tournament with exactly five pancyclic arcs. By Corollary 4, T is $(= 1)$-strong.

Let v be a reductor of T.

By Equation 2, at most one of the T_i, $1 \leq i \leq m$ is not reduced to a single vertex.

1. If $T - v$ is a transitive tournament. Then by Lemma 3, T is in $\mathcal{P}_5$.

2. Suppose that there exists $1 < i < m$, such that T_i is not reduced to a vertex. By Corollary 1, (t_m, v) and (v, t_1) are pancyclic. By directionnal duality, we may suppose that there is a vertex $t_i \in T_i$ such that $t_i \to v$. Then by Lemma 9, there is a pancyclic arc in $T[t_1, t_2, \dots, t_i]$. Thus, there are at most two pancyclic arcs in T_i. Then by Lemma 4 and the proof of Lemma 5, T_i is a 3-cycle containing two pancyclic arcs and there is a vertex $s_i \in T_i$ dominated by v. Then by Lemma 9, there is a pancyclic arc in $T[s_i, t_{i+1}, \dots, t_m]$. So $p(T) \geq 6$ which is a contradiction.

3. Suppose that T_m is not reduced to a vertex. Without loss of generality, we may suppose that v is a vertex such that $T - v$ has the smallest insection provided that all the other components are reduced to a vertex.

   By Equation 2, $d^-_{T_m}(v) \leq 2$.

   - Assume that $d^-_{T_m}(v) = 2$.

     Hence by Equation 5, $h(T) \leq 3$ thus by Theorem 1, $T_m \in \mathcal{P}_3$. Let w be a reductor of T_m such that $T_m - w$ is the transitive tournament $TT[a_1, \dots, a_p]$. Let i be the index such that $w \to a_i$ and $w \leftarrow a_{i+1}$.

     $v \to T_m - [a_i, w, a_p]$, otherwise by Lemma 6, $h(T) \geq 5$ which is a contradiction. Also $v \to T - [v, T_m]$ oterwise by Lemma 9, there is a pancyclic arc in $\{(t_j, t_{j+1}), 1 \leq j \leq m - 2\}$.

     - Suppose that $N^-_{T_m}(v) = T[a_i, w]$. If $i \neq 1$, then by Lemma 6 dual, the three arcs (w, a_1), (a_p, w) and (a_i, a_{i+1}) are pancyclic. So $p(T) \geq 6$ which is a contradiction. Thus $i = 1$. It is easy to see that (a_1, a_2) is contained in a cycle of any length l, for $5 \leq l \leq n$. It follows that $|T_m| = 3$. Then $T \in \overline{\mathcal{H}'_4}$ and by Proposition 12, $T = \overline{H_4}$.
     - If $N^-_{T_m}(v) = T[w, a_p]$, then by Lemma 6 dual, (a_p, w) and (a_i, a_{i+1}) are pancyclic. It is easy to see that (w, a_1) is contained in a cycle of any length l, for $5 \leq l \leq n$. And because (w, a_1) is pancyclic in T_m, it follows that $|T_m| = 3$. Thus $T \in \overline{\mathcal{H}'_4}$ and by Proposition 12, $T = \overline{H_4}$.
     - Suppose that $N^-_{T_m}(v) = T[a_i, a_p]$. If $i \neq p - 1$, then by Lemma 6 dual, the three arcs (w, a_1), (a_p, w) and (a_i, a_{i+1}) are pancyclic. So $p(T) \geq 6$ which is a contradiction. Thus $i = p - 1$. It is easy to see that (a_p, w) is contained in every cycle of length l, for $5 \leq l \leq n$. And because (a_p, w) is pancyclic in T_m, it follows that $|T_m| = 3$. So $T \in \overline{\mathcal{H}'_4}$ and by Proposition 12, $T = \overline{H_4}$.

   - Suppose that $d^-_{T_m}(v) = 1$, then let u be the inneighbour of v in T_m. Then u is a reductor. Let $U_1 \to U_2 \to \dots U_q$ be a decomposition of $T - u$. U_q is a strict subtournament of T_m. Thus, by definition of v, one of the U_j $1 \leq j < q$ is not reduced to a vertex. Since $v \to t_1$, it is necessarily U_1 and v is dominated by a vertex in $T - T_m$. Let s be the biggest index such that $t_i \to v$. $U_1 = T[v, t_1, t_2, \ldots, t_s]$. And by Equation 4, each U_j $j \geq 2$ is reduced to a single vertex u_j. If $U_1 \notin \mathcal{P}_3$, then there are two indices $i_1 < i_2 < s$ such that $(v, t_{i_1}, t_{i_1+1}, v)$ and $(v, t_{i_2}, t_{i_2+1}, v)$ are 3-cycle. Then, by Lemma 9, (t_{i_1}, t_{i_1+1}) and (t_{i_2}, t_{i_2+1}) are T-pancyclic. Hence, $p(T) \geq 6$, which is a contradiction. Therefore $U_1 \in \mathcal{P}_3$.

Analogously, $T_m - u$ is a transitive tournament so $T_m \in \mathcal{P}_3$. Then $T \in \mathcal{Q}_5$ with $T_1 = N^+_{U_1}(v)$, $T_2 = N^-_{U_1}(v)$, $T_4 = N^+_{T_m}(v)$, $T_5 = N^-_{T_m}(v)$ and $T_3 = T - [u, v, T_1, T_2, T_4, T_5]$.

4. If $|T_1| \geq 3$, then by directionnnal duality, $\overline{T} \in \mathcal{Q}_5$ or $\overline{T} = \overline{H_4}$ thus $T \in \mathcal{Q}_5$ or $T = H_4$.

■

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