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VIRTUAL PATH LAYOUT IN ATM PATH WITH GIVEN HOP COUNT

Sébastien Choplin

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LABORATOIRE I3S: Les Algorithmes / Euclide B – 2000 route des Lucioles – B.P. 121 –
06903 Sophia-Antipolis Cedex, France – Tél. (33) 492 942 701 – Télécopie : (33) 492 942 898
http://www.i3s.unice.fr/I3S/FR/

RÉSUMÉ :

MOTS CLÉS : ATM, Dimensionnement de réseau, Diamètre, Chemin

ABSTRACT: Motivated by Asynchronous Transfer Mode (ATM) in telecommunication networks, we investigate the problem of designing a virtual topology on a physical topology, which consists of finding a set of virtual paths (VPs) satisfying some constraints in terms of load (the number of VPs sharing a physical link) and hop count (the number of VPs used to establish a connection). For particular network: paths, we give tight bounds on the network capacity (the maximum load of a physical link) as a function of the virtual diameter (the maximum hop count for each connection).

KEY WORDS : ATM, Virtual Path Layout, Network Design and Dimensioning, Network Optimization

Virtual Path Layout in ATM Path with given hop count

Sébastien Choplin

Projet MASCOTTE
I3S-CNRS/INRIA/Université De Nice-Sophia Antipolis,
BP 93,F-06902 Sophia Antipolis Cedex, France.
Tel : (+33) (0) 4 92 38 78 80
Fax : (+33) (0) 4 92 38 79 71
Sebastien.Choplin@sophia.inria.fr

Abstract. Motivated by Asynchronous Transfer Mode (ATM) in telecommunication networks, we investigate the problem of designing a virtual topology on a physical topology, which consists of finding a set of *virtual paths (VPs)* satisfying some constraints in terms of *load* (the number of VPs sharing a physical link) and *hop count* (the number of VPs used to establish a connection). For particular network: paths, we give tight bounds on the *network capacity* (the maximum load of a physical link) as a function of the *virtual diameter* (the maximum hop count for each connection).

1 Introduction

The advent of fiber optic media has dramatically changed the classical views on the role and structure of digital communication networks. Specifically, the sharp distinction between telephone networks, cable television networks, and computer networks, has been replaced by a unified approach.

One of the most prevalent solutions for this new network challenge is called *Asynchronous Transfer Mode* (ATM for short), and is thoroughly described in the literature [KG98,Pri95]. The transfer of data in ATM is based on packets of fixed length, called *cells*. Each cell is routed independently, based on two routing fields at the cell header, called *virtual channel identifier* (VCI) and *virtual path identifier* (VPI). This method effectively creates two types of predetermined simple routes in the network, namely routes which are based on VPIs (called *virtual paths* or VPs) and routes based on VCIs and VPIs (called *virtual channels* or VCs). VCs are used for connecting network users (e.g., a telephone call); VPs are used for simplifying network management - routing of VCs in particular. Thus the route of a VC may be viewed as a concatenation of complete VPs. A major problem in this framework consists in defining the set of VPs in such a way that some good properties are achieved.

A capacity (or bandwidth) is assigned to each VP. The sum of the capacities of the VPs that share a physical link constitutes the *load* of this link. Naturally, this load must not exceed the link capacity, i.e., the amount of data it can carry.

The maximum load of the links (called the load of the physical network for this set of VPs) is a major component in the cost of the network, and should be kept as low as possible.

The maximum number of VPs in a virtual channel, called *hop count* in the literature, should also be kept as low as possible so as to guarantee low set up times for the virtual channels and high data transfer rates. In its most general formulation, the *Virtual Path Layout (VPL)* problem is an optimization problem in which, given a certain communication demand between pairs of nodes and constraints on the maximum load and hop count, it is first required to design a system of VPs satisfying the constraints and then minimizing some given function of the load and hop count.

We employ a restricted model similar to the one presented in [GZ94]. In particular, we assume that all VPs have equal capacities, normalized to 1. Hence the load of a physical link is simply the number of VPs sharing this link and we don't focus on the number of VCs contained in a VP. Although links based on optical fibers and cables are directed, traditional research uses an undirected model. Indeed, this model imposes the requirement that if there exists a VP from u to v then there exists also a VP from v to u. In fact, that is the way ATM networks are implemented at the present time. Therefore, we use an undirected model. The directed model has been studied in [BMPP98].

We focus on the *all-to-all problem* (all pairs of nodes are equally likely to communicate). Thus, the resulting maximum hop count can be viewed as the *diameter* of the graph induced by the set of VPs. More formally, given a communication network, the VPs form a virtual graph on the top of the physical one, with the same set of vertices but with a different set of edges. Specifically, a VP between u and v is represented by an edge between u and v in the virtual graph. This virtual graph provides a VPL for the physical graph. Each VC can be viewed as a simple path in the virtual graph. Therefore, a central problem is to find a tradeoff between the maximum load and the diameter of the virtual graph.

Here we consider the following restricted problem: the physical graph will be a path and given the diameter of the virtual graph we want to minimize the maximum load of a VPL on the path. The figure 1 is an example of VPL on the path with hop count 2 and maximum load 4. For example, the VP between the node 0 and the node 4 uses the physical links 0-1, 1-2, 2-3 and 3-4. The VC used for the request between the nodes 0 and 6 is formed by the VPs 0-4 and 4-6, with an hop count of 2. The maximum load reached on the physical links 3-4 and 4-5 is equal to 4.

2 Related Work

Many articles deal with the VPL problem defined above. Most of them consider the minimization of the diameter of the virtual graph given a load (or capacity) of the physical network, see [SV96,GZ94,GWZ95a,SV96,GCZ96a,KKP97,EFZ97] for the undirected case and [BMPP98] for the directed case. The dual problem

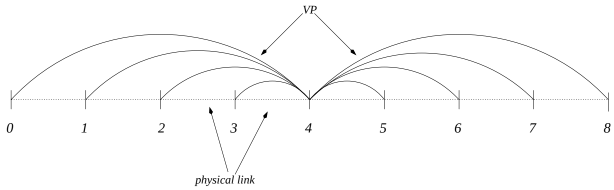


Fig. 1. Example of VPL on the path with hop count 2 and maximum load 4

we consider here has been less studied. Several bounds can be derived from the preceding problems but they are not tight ; a lower bound for planar graphs is given in [BG97]. The particular case of path with the one to many request has been studied in [FZ97,GWZ95a]. In that case, one wants to minimize the eccentricity of the sender in the virtual graph rather than the diameter. This work is sometimes used to give bound on the minimum diameter since it is at most twice the minimum eccentricity. The reader can find an excellent survey of results of the undirected model in [Zak97]. A related problem has been studied in [ABC$^+$97].

3 The model

We use the model described in [BMPP99,Cha98,GCZ96b]. The physical network is represented by an undirected graph $G = (V, E)$ with V the set of nodes of the network and E the set of physical links between them. Usually, we have a family of request $R \subset V \times V$. We are interested in a specific one: the *all-to-all* case (Gossipping). In this case a connection is required between all pairs of vertices; namely, R is formed by all $\binom{n}{2}$ couples of distinct elements of V. Therefore the maximal hop count corresponds to the diameter of the virtual graph. A VPL is a collection of simple paths in the network G, called VP.

Definition 1. A VPL(H,P) is defined by a virtual graph $H = (V, E')$ with the same vertices as G and a routing function P which associates a path $P(e')$ in G to each edge e' in E' .

Definition 2. Given a VPL(H,P), the *load* $l(e)$ of an edge $e \in E$ of the physical graph is the number of VP's that include e; namely, $l(e) = |\{e' \in E' | e \in P(e')\}|$.

Definition 3. Given a VPL(H,P), the *maximal edge load* $\pi(G, H, P) = \max_{e \in E} l(e)$.

Given a graph H, we are looking for a routing function P which minimizes $\pi(G, H, P)$.

Definition 4. The *minimal load* of G for H is $\pi(G, H) = \min_P \pi(G, H, P)$.

The problem stated in the introduction can be formulated as follow:

Given a path P_n with n vertices, a hop count h, what is the minimal load $\pi(P_n, h) = \min_H \{\pi(P_n, H) | D_H \leqslant h\}$ induced by a virtual graph of diameter D_H at most h on the path P_n ?

4 Results

The best bounds known were : $\pi(P_n,h) = \Omega(\frac{1}{2}\sqrt[h]{n^2})$ from [BG97] and $\pi(P_n,h) = O(h\sqrt[h]{n^2})$ from [BMPP98].
$\pi(P_n,1) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, in that case the parameter is called the edge-forwarding index studied in [MHS89],
$\pi(P_n,2) = \lfloor \frac{n}{2} \rfloor$,
$\pi(P_n,3) = \frac{1}{2}\sqrt[3]{\left(\frac{3n}{2}\right)^2} + O(\sqrt[9]{n^4})$.
We give here the leading term of $\pi(P_n,h)$ and the corresponding construction for a fixed hop count h :

$$\pi(P_n,h) = \frac{\sqrt[p]{p!}}{2}\sqrt[h]{n^2} + o(\sqrt[h]{n^2}) \text{ for } h = 2p$$
$$\pi(P_n,h) = \frac{\sqrt[h]{2}}{2\sqrt[h]{{M_p}^2}}\sqrt[h]{n^2} + o(\sqrt[h]{n^2}) \text{ for } h = 2p+1$$

with $M_p = \sup_{0 \leqslant \alpha \leqslant 2} \left\{ \alpha \sum_{j=0}^{p} \frac{j!(-\alpha^2)^j}{(p-j)!(2j+1)!} \right\}$. M_p can be easily computed for a given p, since for a fixed p, it is the maximum value of a bounded degree polynomial function.

Proof. All the proofs are in the Appendix.

5 Conclusion and future directions

In this paper, we have determined tight bounds of $\pi(P_n,h)$ for all n (the number of vertices) and fixed hop count h. We can derive from our results an upper bound for the directed model ; perhaps similar methods to those of this paper can provide good lower bounds. Using similar methods, we have also proved that $\pi(C_n,2) = \frac{n}{3} + O(1)$ where C_n is the cycle (or ring) with n vertices. It seems more difficult to determine the load $\pi(C_n,h)$ for $h \geqslant 3$. These results on paths and cycles can be used for designing general networks by partitioning the networks into subnetworks isomorphic to paths and cycles and optimizing the load on each subnetwork. We used a restricted set of requests (all-to-all problem) ; but we believe that it can be used in the case where the requests happen dynamically with fixed uniform probability.

A Appendix - Proofs

Definition 5. An *interval* I is a set of connected vertices in the physical graph P_n and we denote its complementary by $\overline{I} = V \backslash I$.

Definition 6. Let $n_{\mathcal{AA}}(\pi,h)$ be $\max\{n|\pi(P_n,h) \leqslant \pi\}$ that is the maximum number of vertices in the path so that there exists a virtual graph of diameter less than h that loads G less than π.

Definition 7. Given an interval I, a vertex $x \in I$ is said to be *outward* if it is adjacent in H to at least one vertex of $\overline{I}$. An interval is said to be *outward* if all its vertices are outward.

A.1 Preliminaries : $h = 1$ and $h = 2$

In the case $h = 1$, this is like determining the minimal embedding congestion of the graph of diameter $D \leqslant h$ in G. This problem is a classic one in the case of $h = 1$ because H is isomorph to K_n and $\pi(G, \mathcal{AA}, 1)$ is the edge-forwarding index of G studied in [Gau95,MHS89].

Lemma 8. *The minimal load induced by the complete virtual graph on the physical edge $(i-1, i)$ of the path P_n is $i(n-i)$ for $1 \leqslant i \leqslant n-1$.*

Proof. The routing function P is unique (if indeed there is only one path) then as we can see in the figure 2, the number of virtual edges which share the physical edge $(i-1, i)$ is equal to the number of edges of type $(j, i+k)$ for $0 \leqslant j \leqslant i-1$ and $0 \leqslant k \leqslant n-1-i$ that is $i(n-i)$.

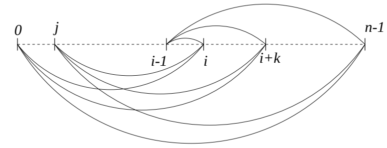


Fig. 2. The load of the edge $(i-1, i)$ is $i(n-i)$.

Corollary 9 (The edge-forwarding index [MHS89]). $\pi(P_n, 1) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$.

Proof. From the Lemma 8, $\pi(P_n, 1) = \max\limits_{1 \leqslant i \leqslant n-1} i(n-i) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$.

Lemma 10. *The minimal load induced by a virtual graph of diameter 2 on the physical edge $(i-1, i)$ of P_n is at least* $\min\{i, n-i\}$.

Proof. Let $I_i = \{0, 1, \ldots, i-1\}$ and $\overline{I_i} = \{i, \ldots, n-1\}$. If there exists a non-outward vertex of I_i, because the diameter is 2, each vertex of $\overline{I_i}$ has a neighbour in I_i, so the load of the edge $(i-1, i)$ is at least $n-i$. Otherwise , each vertex of I_i is an outward vertex and so the load of $(i-1, i)$ is at least i. Then the load is greater than $\max\limits_{0 \leqslant i \leqslant n} (\min\{i, n-i\}) = \left\lfloor \frac{n}{2} \right\rfloor$.

Corollary 11. $\pi(P_n, 2) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. The bound is reached by connecting the vertex $\left\lfloor \frac{n}{2} \right\rfloor$ with all the other vertices.

In order to compute $\pi(P_n, h)$, we will determine $n_{\mathcal{AA}}(\pi, h)$. Therefore we can reverse the result to get $\pi(P_n, h)$.

A.2 Upper bound for $n_{\mathcal{AA}}(\pi, h)$

Remark 12 (interval). Given a virtual graph H of diameter h on P_n, there exists an integer k and k vertices $a_1 < a_2 < \ldots < a_k$ such that

– $\forall 0 \leqslant i \leqslant k$: all the vertices in $[a_i + 1, a_{i+1} - 1]$ are outward vertices.

– $\forall 0 \leqslant i \leqslant k-1$ a vertex in $[a_i+1, a_{i+1}]$ is not an outward vertex.

(with $a_0 = -1$ and $a_{k+1} = n$). We will denote by J_i the interval $[a_i+1, a_{i+1}]$ and I_i the outward interval $[a_i+1, a_{i+1}-1]$.

Now we call the intervals J_i *blocks of level 1* of the topology H. We note that the graph of the interval $\{J_i\}_{1 \leqslant i \leqslant k-1}$ (or blocks) of level 1 of H is also a virtual topology on the path but with k vertices; we note it H_1. The diameter of this topology is at most $h-2$ because each of the interval has at least one non-outward vertex. In an iterative way, we define some blocks of blocks. The blocks from H_1 are called *blocks of level 2*. Formally :

– The *blocks of level* l from a virtual graph H of diameter h are the blocks of level 1 from the topology H_{l-1}.
– The topology H_l is the graph formed by the blocks from H_{l-1}.
– The vertices from P_n are the blocks of level 0.
– $H_0 = H$.

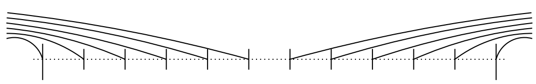
Proposition 13. *For* $i \leqslant \frac{h}{2} - 1$ *the diameter of the topology* H_i *is at most* $h - 2i$.

Proof. Directly by induction.

Definition 14. The *available load* of an interval $I \in \{I_i\}_i$ $(\forall i, I_i$ is an interval and $\cup_i I_i = V)$ is $\pi - \pi_I$ where π_I is the load induced by the edges between I and all the other intervals.

Proposition 15. *The cardinal of an outward interval with an available load* l *is at most* $2l$.

Proof. If the interval has k vertices, k edges load the 2 extreme edges of the interval. Then $k \leqslant 2l$.



Definition 16. Let denote by $B(l, c)$ the *maximum number of vertices in a block of level* l *with an available load* c.

Proposition 17. *for* $h = 2p+1$ *odd,* $n_{\mathcal{AA}}(\pi, h) \leqslant \overline{n_{\mathcal{AA}}}(\pi, h)$ *with*
$\overline{n_{\mathcal{AA}}}(\pi, h) = \max\limits_{k \leqslant 2\sqrt{\pi}+1} \left\{ \sum_{i=0}^{k-1} B(p, \pi - i(k-i-1)) + B(p, \pi) \right\}$

Proof. Let H be a virtual graph of diameter $h = 2p+1$ and load π on the path P_n. The graph H_p of the blocks of level p is the complete graph according to the Proposition 13. Let us fix $k = |V(H_p)|$. Now we consider the blocks of level p $\{J_i\}_{0 \leqslant i \leqslant k-1}$; they induce a complete graph with a load $i(k-i-1)$[1] on all the physical edges of the block J_i because of the lemma 8. Then the available load of the block J_i is at most $\pi - i(k-i-1)$. So for $0 \leqslant i \leqslant k-1$, $|J_i| \leqslant B(p, \pi - i(k-i-1))$. To finish we have to consider the interval J_k; this interval is an outward one and so its cardinal is lower than the one of the block of level p then $|J_k| \leqslant B(p, \pi)$.Then, $n_{\mathcal{AA}}(\pi, 2) \leqslant \left(\sum_{i=0}^{k-1} B(p, \pi - i(k-i-1)) \right) + B(p, \pi)$.

[1] $k-i-1$ because we don't count the load induced by J_i into the complete graph.

Remark 18. We can consider that $k \leqslant 2\sqrt{\pi}+1$, because if $k > 2\sqrt{\pi}+1$ then it exists i so that $\pi - i(k-i-1) < 0$, that's a nonsense.

Proposition 19. *for $h = 2p$ even, $n_{\mathcal{AA}}(\pi, h) \leqslant \overline{n_{\mathcal{AA}}}(\pi, h)$ with*
$\overline{n_{\mathcal{AA}}}(\pi, h) = \max\limits_{k \leqslant 2\pi} \left\{ \left(2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} B(p-1, \pi - i) \right) + B(p-1, \pi) \right\}$

Proof. Let H be a virtual graph of diameter $h = 2p$ and load π on the path P_n. The graph H_p of the blocks of level p is a graph of diameter 2 according to the Proposition 13. We fix $k = |V(H_{p-1})|$ and now we consider the blocks of level $p-1$ $\{J_i\}_{0 \leqslant i \leqslant k-1}$; they induce a graph of diameter 2 with a load $\min\{i, n-i-1\}^2$ on all the physical edges of the block J_i because of the lemma 10. Then the available load of the block J_i is at most $\pi - \min\{i, k-i-1\}$. So for $0 \leqslant i \leqslant k-1$, $|J_i| \leqslant B(p, \pi - \min\{i, k-i-1\})$. To finish we have to consider the interval J_k; this interval is an outward one and so its cardinal is lower than the one of the block of level p ; therefore $|J_k| \leqslant B(p, \pi)$. Then, we obtain the result because : $\min\{i, k-i-1\} = i$ if $i \leqslant \frac{k-1}{2}$ and $\min\{i, k-i-1\} = k-i-1$ otherwise.

A.3 Lower bound for $n_{\mathcal{AA}}(\pi, h)$

The techniques employed in the proof of the lower bound tell us that the good virtual topologies with diameter $2p+1$ may be constructed with blocks of level p linked together in a complete graph. We can note that the blocks of level p are sets of vertices which we can get out of in p jumps exactly. Then we can choose as blocks of level p some maximal trees with eccentricity p, the same as the one described by Gerstel, Wool and Zaks in [GWZ95b]. An equivalent proof can be given for the case h is even.

Proposition 20. *for all $h = 2p+1$ odd, for all k, $\left(\frac{k}{2}\right)^2 \leqslant \pi$ we have : $n_{\mathcal{AA}}(h, \pi) \geqslant \underline{n_{\mathcal{AA}}}(\pi, h)$ with $\underline{n_{\mathcal{AA}}}(\pi, h) = \max\limits_{k \leqslant 2\sqrt{\pi}} \left\{ \sum_{i=0}^{k-1} ball(p, \pi - i(k-i)) \right\}$ with $ball(p, c) = \sum_{i=0}^{\min\{p,c\}} 2^i \binom{l}{i} \binom{r}{i}$ the number of internal points in the p-dimensional Sphere of radius c.*

Proof. We obtain the construction as follows : we choose k vertices $a_0, a_1, \ldots, a_{k-1}$ on the path and we link them with a complete graph, then we root 2 trees in each vertex (one on the right and one on the left) which use all of the available load and with eccentricity p as shown by the figure 3. We note $n(p, c)$ the maximal number of vertices that a tree of available load c and eccentricity p can contain. Then we have the following recurrence relation : $n(p, c) = n(p, c-1) + n(p-1, c-1) + n(p-1, c) = \frac{1}{2} ball(p, c) + \frac{1}{2}$, with $n(1, c) = c$ and $n(p, 0) = 1$ (There are the trees used in [GWZ95b] for the *One-To-All*-case).

The available load : on the left of a_0 is π, between a_{i-1} and a_i is $\pi - i(k-i)$ for $0 \leqslant i \leqslant k$, on the right of a_k is π. Then we can root : on the left of a_0 a tree of eccentricity p with a load π which contains $n(p, \pi)$ vertices, for

[2] as in the Proposition 17 we don't count the load induced by J_i.

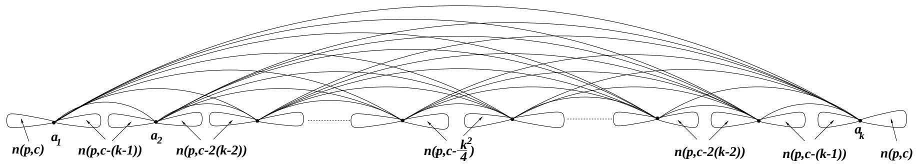


Fig. 3. Optimal construction on the path for h odd.

$0 \leqslant i \leqslant k$, between a_i and a_{i+1} 2 trees of eccentricity p with a load $\pi - i(k-i)$ which contain $2n(p, \pi - i(k-i))$ on the right of a_k : a tree of eccentricity p with a load π which contains $n(p, \pi)$ vertices. So the total number of vertices is $2\sum_{i=0}^{k-1} n(p, \pi - i(k-i)) - k$.

Remark 21. A similar construction can be realized in the case h is even $(h = 2p)$, then the graph of blocks of level $p-1$ is isomorph to the graph of diameter 2 used in the case $h = 2$. Each vertex of this graph is taken over from an optimal tree of eccentricity $p-1$ and a maximal available load.

A.4 Determination of $n_{\mathcal{AA}}(\pi, h)$

$\overline{n_{\mathcal{AA}}}(\pi, h)$: calculation of the upper bound of $n_{\mathcal{AA}}(\pi, h)$. We proceed as follows : the available load of a block of level l is c, then we determine the maximal number of vertices $B(l, c)$ that it can contain according to $B(l-1, i)$. This relation is given by the following recursion equation :

Lemma 22. $B(l,c) = 2(B(l-1,1) + B(l-1,2) + \cdots + B(l-1,c)) + B(l-1,c)$

Proof. A block of level l with an available c is formed with some blocks of level $l-1$ and all (except one) have an outward vertex. We can suppose without lost of generality that all the edges don't cross each other (Gerstel, Wool and Zaks [GWZ95b]). Then, a block of level l with an available load c is formed with at most : 2 blocks of level $l-1$ with an available load equal to 1, 2 blocks of level $l-1$ with an available load equal to $2, \ldots,$ 2 blocks of level $l-1$ with an available load equal to c. Now we have to add a block of level $l-1$ with an available load equal to at most c (this is the block which is not outward), as we can see in the figure 4.

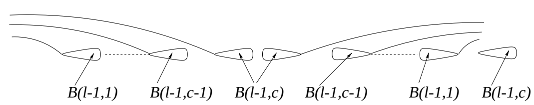


Fig. 4. Composition of a block of level l with an available load c.

In order to determine $B(l,c)$, we introduce the generating function : $\mathcal{B}(l,z) = \sum_{i=0}^{\infty} B(l,i) z^i$. The blocks of level 0 are reduced to vertices, $B(0,c) = 1$ and then $\mathcal{B}(0,z) = \frac{1}{1-z}$. Then we have, $B(l,c) = \frac{2^l c^l}{l!} + o(c^l)$.

Proposition 23. *for all $h = 2p+1$ odd,*
$$\overline{n_{\mathcal{AA}}}(\pi, h) = 2^p \pi^{h/2} \sup_{0 \leqslant \alpha \leqslant 2} \left\{ \sum_{j=0}^{p} \frac{\alpha j! (-\alpha^2)^j}{(p-j)!(2j+1)!} \right\} + o(\pi^{h/2})$$

Proof. according to Lemma 22 : $\overline{n_{\mathcal{AA}}}(\pi,h) = \frac{2^p}{p!} \max_{k \leqslant 2\sqrt{\pi}+1} \left\{ \sum_{i=0}^{k} (\pi - i(k-i))^p \right\} + o(\pi^{h/2})$ because of $\sum_{i=0}^{k} (\pi - i(k-i))^p = \sum_{j=0}^{p} \binom{p}{j} \pi^{p-j} (-1)^j \sum_{i=0}^{k} i^j (k-i)^j$ and we can approach $\sum_{i=0}^{k} i^j (k-i)^j$ using the *Euler-Mac Laurin*'s summation formula, in $\frac{(j!)^2}{(2j+1)!} k^{2j+1} + o(k^{2j+1})$. So we have :
$\overline{n_{\mathcal{AA}}}(\pi,h) = 2^p \pi^{p+1/2} \max_{0 \leqslant \alpha \leqslant 2+O(n^{1/h})} \alpha \sum_{j=0}^{p} \frac{j!}{(p-j)!(2j+1)!} (-\alpha^2)^j + o(\pi^{h/2})$

Proposition 24. *for all $h = 2p$ even, $\overline{n_{\mathcal{AA}}}(\pi,h) = \frac{2^p \pi^p}{p!} + o(\pi^{h/2})$*

Proof. $\overline{n_{\mathcal{AA}}}(\pi,h) = \max_{k \leqslant 2\pi} \left\{ \left(2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor -1} B(p-1, \pi - i) \right) + B(p-1,\pi) \right\}$
$\overline{n_{\mathcal{AA}}}(\pi,h) = \frac{2^p}{(p-1)!} \max_{k \leqslant 2\pi} \left\{ \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor -1} (\pi - i)^{p-1} \right\} + o(\pi^p)$

Clearly, this function of k reaches its maximum when k is maximum, i.e. $k = 2\pi$. Using the integral calculation, we obtain : $\overline{n_{\mathcal{AA}}}(\pi,h) = \frac{2^p \pi^p}{p!} + o(\pi^p)$.

$\underline{n_{\mathcal{AA}}}(\pi,h)$: calculation of the lower bound of $n_{\mathcal{AA}}(\pi,h)$.

Proposition 25. *for all h, $\underline{n_{\mathcal{AA}}}(\pi,h) = \overline{n_{\mathcal{AA}}}(\pi,h) + o(\pi^{h/2})$.*

Proof. Note that $ball(p,c) = \sum_{i=0}^{\min(p,c)} 2^i \binom{l}{i} \binom{c}{i} = \frac{2^p c^p}{p!} + o(c^p)$ if $p \ll c$.
In the odd case : $h = 2p+1$, with the Proposition 20, we can say that : $\underline{n_{\mathcal{AA}}}(\pi,h) = \frac{2^p}{p!} \max_{k \leqslant 2\sqrt{\pi}} \left\{ \sum_{i=0}^{k} (\pi - i(k-i))^p \right\} + o(\pi^{h/2})$. Then $\underline{n_{\mathcal{AA}}}(\pi,h) = \overline{n_{\mathcal{AA}}}(\pi,h) + o(\pi^{h/2})$. The same is true in the even case.

Corollary 26. $n_{\mathcal{AA}}(\pi,h) = \frac{2^p \pi^p}{p!} + o(\pi^{h/2})$ *for $h = 2p$*
$n_{\mathcal{AA}}(\pi,h) = 2^p \pi^{(2p+1)/2} \sup_{0 \leqslant \alpha \leqslant 2} \left\{ \alpha \sum_{j=0}^{p} \frac{j!(-\alpha^2)^j}{(p-j)!(2j+1)!} \right\} + o(\pi^{h/2})$ *for $h = 2p+1$*

Proof. $\underline{n_{\mathcal{AA}}}(\pi,h) \leqslant n_{\mathcal{AA}}(\pi,h) \leqslant \overline{n_{\mathcal{AA}}}(\pi,h)$ and $\underline{n_{\mathcal{AA}}}(\pi,h) = \overline{n_{\mathcal{AA}}}(\pi,h) + o(\pi^{h/2})$ then $n_{\mathcal{AA}}(\pi,h) = \underline{n_{\mathcal{AA}}}(\pi,h) + o(\pi^{h/2}) = \overline{n_{\mathcal{AA}}}(\pi,h) + o(\pi^{h/2})$.

Theorem 27. $\pi(P_n,h) = \frac{(p!)^{1/p}}{2} n^{2/h} + o(n^{2/h})$ *for $h = 2p$*
$\pi(P_n,h) = \frac{2^{1/h}}{2(M_p)^{2/h}} n^{2/h} + o(n^{2/h})$ *for $h = 2p+1$ with $M_p = \sup_{0 \leqslant \alpha \leqslant 2} \left\{ \alpha \sum_{j=0}^{p} \frac{j!(-\alpha^2)^j}{(p-j)!(2j+1)!} \right\}$*

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