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THE CLIQUE NUMBER OF UNIT QUASI-DISK GRAPHS

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RÉSUMÉ :

Pour $e \in [0,1]$, un e-quasi-disque unité est un compact connexe Q du plan tel qu'il existe un point P tel que $D(P, 1-e) \subseteq Q \subseteq D(P, 1)$, où $D(C, r)$ désigne le disque de centre C et de rayon r. Nous montrons que pour tout e>0 fixé, le problème de la clique maximum sur la classe des graphes d'intersection de e-quasi-disques unité est NP-complet

MOTS CLÉS :

Clique maximum, graphe d'intersection, quasi-disque

ABSTRACT:

For $e \in [0,1]$, a unit e-quasi-disk is a connected compact set Q of the plane such that there exists a point P such that $D(P, 1-e) \subseteq Q \subseteq D(P, 1)$, where $D(C, r)$ denotes the disk of centre C and radius r. We prove that for any fixed $e > 0$, the clique number problem on the class of intersection graphs of unit e-quasi-disks is NP-complete

KEY WORDS :

Clique number, intersection graph, quasi-disk

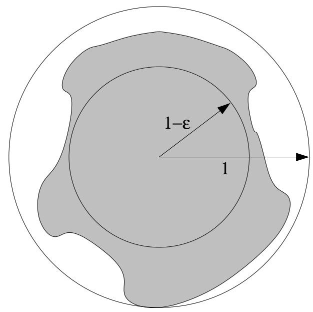

Figure 1: A unit ϵ-quasi-disk.

1 Introduction

Unit disk graphs (also known as *proximity graphs*) have received a lot of attention recently because of their applications to radio telecommunication and mobile phone networks ([5, 2, 10, 9, 11]). The set $\mathcal{V}$ of centres of the disks represents the set of transmitters, and to avoid interferences, two transmitters whose corresponding disks intersect must have distinct channel. Then the minimum number of channels turns out to be the chromatic number $\chi(G)$ of the corresponding unit disk graph G ([12, 4, 8]). Of course, $\chi(G)$ is at least the clique number $\omega(G)$. In [5], it is proved that $\chi(G) \leq 3\omega(G) - 2$, and a $O(n^{4.5})$ algorithm to compute $\omega(G)$ is given (the input being $\mathcal{V}$). Very recently, Raghavan and Spinrad [13] gave an polynomial time algorithm whose input is the abstract graph, and the author [3] extended this result to intersection graph of translates of a fixed convex set.

But the modeling of the covering area of a transmitter as a disk is unrealistic, and a modeling using *unit ϵ-quasi-disks* has been introduced [1]. A unit ϵ-quasi-disk (or *quasi-disk* when the context is clear) is a connected compact set Q of the plane such that there exists a point O such that $D(O, 1-\epsilon) \subseteq Q \subseteq D(O, 1)$, where $D(C, r)$ denotes the disk of centre C and radius r (see Fig. 1).

In this paper, we prove the following theorem :

Theorem 1 *The clique number on the class of intersection graphs of unit ϵ-quasi-disk is NP-complete for any $\epsilon > 0$.*

2 Proof of the theorem

The Euclidean plane is given with an orthonormal set of axis (O, Ox, Oy). The coordinates of a point P are denoted (x_p, y_p). For P a point of the plane and $r \geq 0$, $D(P, r)$ (resp. $C(P, r)$) denotes the closed disk (resp. circle) of centre P and radius r. We denote by ρ_α the rotation of centre O and of angle α. For S a subset of the plane and $\vec{t}$ a vector, $S + \vec{t}$ is the translated of S by vector $\vec{t}$.

To prove Theorem 1, we use a reduction of the following problem :
Problem : Maximum independent set (MIS) on cubic graphs.
Instance : A cubic graph G; an integer K.
Question : Is there an independent set X of G such that $|X| \geq m$?

This problem is known to be NP-complete, even if restricted to planar cubic graphs [7, 6].

Let $\epsilon > 0$. Throughout this proof, quasi-disk stands for unit ϵ-quasi-disk. Clearly, the MIS problem is also NP-complete on the class of *connected* cubic graphs distinct from K_4. So let $G = (V, E)$ be a connected cubic graph, different from K_4. Then by Brook's theorem, G is 3-colorable. Let C_1, C_2, C_3 a partition of V in 3 independent sets. To prove Theorem 1, we are going to construct a set of quasi-disks $\{Q_1, Q_2, \dots, Q_n\}$ whose intersection graph is the complement $\overline{G}$ of G. Thus the clique number of $\overline{G}$ is exactly the size of an MIS of G, and this proves that the clique number is NP-complete on the class of unit ϵ-quasi-disk graphs

For S a subset of $\{1, 2, \dots, n\}$, we construct a quasi-disk D^n_S the following way : let $D(O, 1)$ be the unit disk centered in the origin. Remove from it the open half plane $\Pi : x > 1 - \epsilon$, let D' be the resulting set. Let A and B be the two endpoints of the segment $D(O, 1) \cap \Pi$, A being the one with positive ordinate. Let O', A', B' denote the middle of $[A, B]$, $[A, O']$, $[B, O']$ respectively. Let A'' (resp. B'') denotes the point of $C(O, 1)$ having the same ordinate as A' (resp. B') and positive abscissae. We denote by l this positive abscissae (an easy computation gives $l = \frac{1}{2}\sqrt{4 - 2\epsilon + \epsilon^2}$ and $y_{A'} = \frac{1}{2}\sqrt{2\epsilon - \epsilon^2}$). Note that the rectangle $A'A''B''B'$ is between the disks $D(O, 1)$ and $D(O, 1 - \epsilon)$. We consider the $n + 2$ points $P_0 = B', P_1, P_2, \dots, P_n, P_{n+1} = A'$ that subdivide regularly the segment $[A'B']$, and we denote by μ half the distance between two consecutive points, i.e. $\mu = y_{A'}/(n + 1)$. We denote by R_i the rectangle with vertices $(1 - \epsilon, y_{P_i} - \mu/2)$, $(1 - \epsilon, y_{P_i} + \mu/2)$, $(l, y_{P_i} + \mu/2)$ and $(l, y_{P_i} - \mu/2)$.

For $S \subseteq \{1, 2, \dots, n\}$, the set D^n_S is defined as $D' \cup \bigcup_{i \in S} R_i$ (see Fig 2). For $i \in \{1, 2, \dots, n\}$, we denote by $E^n_{S,i}$ the set $D^n_S \cap \rho_{\pi/3}(D^n_{\{n-i+1\}})$ (see Fig. 3). It is easy to

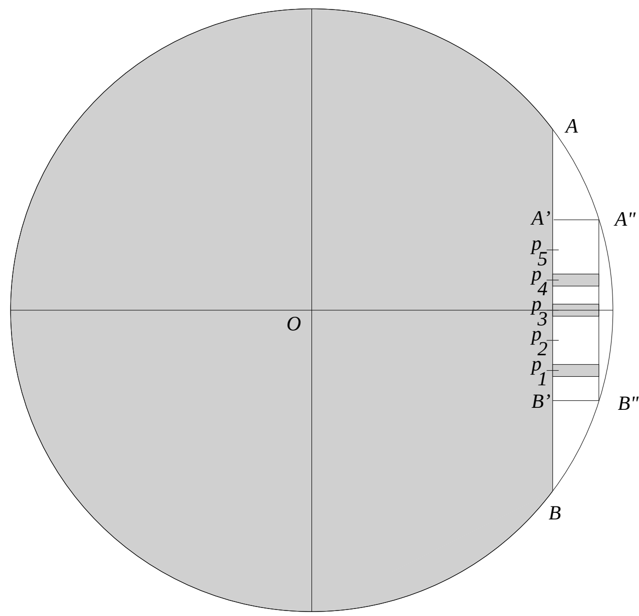

Figure 2: The set $D^5_{\{1,3,4\}}$, in grey.

see that $E^n_{S,i}$ is a quasi-disk. The points of $E^n_{S,i} \cap \Pi$ is a set of $|S|$ rectangles called *the connector* of $E^n_{S,i}$. Similarly, the rectangle $E^n_{S,i} \cap \rho_{\pi/3}(\Pi)$ is called *the identifier* of $E^n_{S,i}$.

Recall that l is the abscissae A', and let $O_1 = O$, $O_2 = (2l, 0)$ and $O_3 = (l, \sqrt{3}l)$ be the vertices of an equilateral triangle of size $2l$. The quasi-disks $E^n_{S,i}$ are defined to satisfy the following property :

Lemma 1 *Let $Q_1 = E^n_{S,i}$ and $Q_2 = \rho_{2\pi/3}(E^n_{S',i'}) + \overrightarrow{O_1O_2}$ be two quasi-disks. They intersect if and only if $i' \in S$.*

Proof. Due to the value of l and the relative positions of Q_1 and Q_2, these two quasi-disks intersect if and only if the identifier of Q_2 intersects a rectangle of the connector of Q_1.

By construction, the rectangles of the connector of Q_1 are vertically centered in the y_{P_j}'s, $j \in S$. It is easy to see that the identifier of Q_2 is vertically centered in $y_{P_{i'}}$. Thus this rectangle intersects the connector of Q_1 if and only if $i' \in S$. ∎

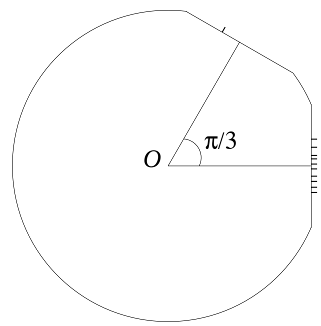

Figure 3: A set $E^n_{S,i}$

For any vertex v_i of G, we note S_i the set of indices of the non-neighbours of v_i : $S_i = \{j \neq i \ : \ \{v_i, v_j\} \notin E\}$, and we associate to v_i the quasi-disk Q_i defined as

- $E^n_{S_i,i}$ if $v_i \in C_1$;
- $\rho_{2\pi/3}(E^n_{S_i,i}) + \overrightarrow{O_1O_2}$ if $v_i \in C_2$;
- $\rho_{-2\pi/3}(E^n_{S_i,i}) + \overrightarrow{O_1O_3}$ if $v_i \in C_3$;

Thus the quasi-disks associated to the elements of C_i are centered in O_i, see Fig 4.

Now we shall prove that the intersection graph of this set of quasi-disks is the complement of G.

Lemma 2 *Two vertices v_i and v_j are adjacent in G if and only if Q_i and Q_j are disjoint.*

First suppose that v_i and v_j are adjacent, thus they do not belong to the same color class. Without loss of generality, we may suppose that $v_i \in C_1$ and $v_j \in C_2$. By definition of S_i, $j \notin S_i$, and thus by Lemma 1, Q_i and Q_j do not intersect.

Now suppose conversely that v_i and v_j are not adjacent. If they belong to the same color class C_k then Q_i and Q_j intersect obviously. Otherwise, as previously we may suppose without loss of generality that $v_i \in C_1$ and $v_j \in C_2$, and then again by Lemma 1 we prove that Q_i and Q_j intersect. ■

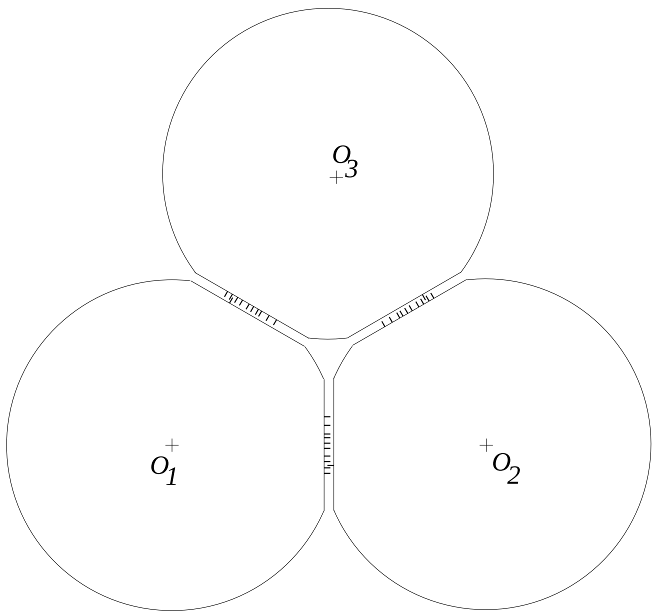

Figure 4: Configuration of the Q_i's.

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