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SIGNED DISTANCE FUNCTIONS AND VISCOSITY SOLUTIONS OF DISCONTINUOUS HAMILTON-JACOBI EQUATIONS

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Signed distance functions and viscosity solutions of discontinuous Hamilton-Jacobi Equations

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&

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Abstract

In this paper, we first review some properties of the signed distance function. In particular, we examine the skeleton of a curve in $\mathbb{R}^2$ and get a complete description of its closure. We also give a sufficient condition for the closure of the skeleton to be of zero Lebesgue's measure. We then make a complete study of the PDE: $\frac{\partial u}{\partial t} + sign(u_0(x))\,(|Du| - 1) = 0$, which is closely related to the signed distance function. The existing literature provides no mathematical results for such PDEs. Indeed, we face the difficulty of considering a discontinuous Hamiltonian operator with respect to the space variable. We state an existence and uniqueness theorem, giving in particular an explicit Hopf-Lax formula for the solution as well as its asymptotic behaviour. This generalizes classical results for continous Hamitonian. We then get interested in a more general class of PDEs: $\frac{\partial u}{\partial t} + sign(u_0(x))H(Du) = 0$, with H convex Under some technical but reasonable assumptions, we obtain the same kind of results. As far as we know, they are new for discontinuous Hamiltonians.

Key-words: Signed distance function, skeleton, PDE, viscosity solutions, Hamilton-Jacobi equations, Hopf-Lax formulae.

AMS subject classifications: 35A, 35C, 35D.

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1. Introduction

In image processing, the signed distance function plays a key-role due to its good properties. For example in image segmentation a main question is to detect the contour of objects in a scene. A very powerful method consists in making the zero level sets of an initial distance function u_0 evolve untill they reach the desired contours. Unfortunately, the partial differential equation (PDE) governing the evolution does not guarantee the evolving function $v(x,t)$ to remain a signed distance function. In particular its gradient can become unbounded , and this is the origin of serious numerical problems. To overcome this difficulty it was proposed by J-M Morel to reinitialize the process by running each five or ten iterations the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + sign(v(x,t_0))\,(|Du|-1) = 0 \\ u(x,0) = v(x,t_0) \end{cases} \tag{1.1}$$

where $v(x,t_0)$ is the evolving function at time t_0 ($t_0 = 5$ or 10). We refer the reader to the abundant literature on this subject ([4, 37, 25, 11, 31, 36, 28, 1, 3, 10]) for a more detailed insight.

Though equation (1.1) is often used in image processing, there exists no real mathematical study of it in the existing literature. PDE (1.1) is of Hamilton-Jacobi type:

$$\frac{\partial u}{\partial t} + H(x, Du) = 0 \tag{1.2}$$

The theory of viscosity solutions proved to be well-adapted to state existence and uniqueness results for such PDEs (see [12, 13, 7, 20, 18, 14, 24, 8, 5, 15, 6, 2]). But all these works are generally conducted under the asumption that the Hamiltonian H is continuous with respect to all its variables, whereas PDE (1.1) has a singularity on $\Gamma = \{x/v(x,t_0) = 0\}$. This is indeed the specifity of our work: our Hamiltonians will not be continuous with respect to the space variable x. We will nevertheless give an existence and uniqueness result, and we will also get an explicit Hopf-Lax formula. The existing literature on viscosity solutions for discontinuous Hamiltonians is not very large. The case of discontinuous solutions is discussed in [7, 2], and the case of a discontinuity in the second member of the PDE in [21, 35, 34, 16]. As far as we know, there is no previous work which deals with the discontinuity we face for PDE (1.1).

The paper is organized as follows. We first review some properties of the signed distance function in Section 2. We also consider the skeleton of a curve in $\mathbb{R}^2$ and get a complete description of its closure. We give a sufficient condition for the closure of the skeleton of a curve in $\mathbb{R}^2$ to be of zero Lebesgue's measure. This specifies a result of Matheron in [32].

We then completely study PDE (1.1). We state an existence and uniqueness result, giving in particular an explicit formula for the solution as well as its asymptotic behaviour, generalizing classical results for continous Hamitonian. Indeed, we show that the function $u(x,t)$ defined as

$$u(x,t) = \begin{cases} \epsilon_x \inf_{|y|\le t} (\epsilon_x u_0(x+y) + t) & \text{if } t \le t_x \\ \epsilon_x d(x,\Gamma) & \text{if } t > t_x \end{cases} \tag{1.3}$$

where

$$\begin{cases} u_0(x) = v(x, t_0) \\ \epsilon_x = sign(u_0(x)) \\ t_x = \inf\{t \in \mathbb{R}_+ / \inf_{|y|\leq t} (\epsilon_x u_0(x+y)) = 0\} = d(x, \Gamma) \end{cases}$$

is the unique uniformly continuous viscosity solution of (1.1) vanishing on $\Gamma = \{x / u_0(x) = 0\}$. In Section 3, we get interested in a more general class of PDEs. Under some technical but reasonable assumptions, we obtain the same kind of results. As far as we know, they are new for such discontinuous Hamiltonians.

2. Euclidean signed distance function and the reinitialization equation

In this first section, we study some elementary but useful properties of the Euclidean signed distance function to closed curves. One of our results is a characterization of the skeleton of a closed curve. Then we analyse in detail the reinitialization equation (1.1) for which we give an explicit solution in the class of uniformly continuous viscosity solutions. We begin by recalling some definitions.

2.1 Properties of the signed distance function

2.1.1 Some definitions

Let us consider $\mathbb{R}^2$ endowed with the usual Euclidean norm and the induced topology. Throughout this section, we will consider Γ a curve in $\mathbb{R}^2$ such that:

(H 2.1) Γ is piecewise C^2.

(H 2.2) Γ has a finite number of connected component Γ_i.

(H 2.3) For all i , Γ_i is parameterized by its curvilinear abscissa $\gamma : [0; L_i] \to \mathbb{R}^2$ (where L_i is the length of Γ_i) such that the curvature $\kappa(\gamma(t))$ changes only a finite number of times on $[0; L_i]$ (κ takes its values in $\overline{\mathbb{R}}$).

(H 2.4) The points where Γ is not C^2 are angular points (i.e. of infinite curvature).

(H 2.5) For all i , Γ_i est geometrically closed, ie $\gamma(0) = \gamma(L_i)$.

(H 2.6) Γ will always be oriented in the indirect way.

In all Section 2, we will only consider curves Γ verifying these hypotheses (except if explicitly mentioned), and we will call such curves Γ closed curves in $\mathbb{R}^2$. We define Ω as the interior of Γ (denoted also by $\partial\Omega$).

Definition 2.1. Let Γ be a closed curve in $\mathbb{R}^2$, and let u be a function:

$$u : \mathbb{R}^2 \mapsto \mathbb{R}$$

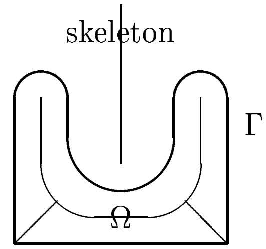


Figure 1: Exemple of skeleton

such that $u(\Gamma)$=0.
We will say that u is a signed distance function to the curve Γ if:

$$u(x) = \begin{cases} \epsilon \mathrm{d}(x,\Gamma) & \text{if } x \text{ lies outside } \Gamma \\ -\epsilon \mathrm{d}(x,\Gamma) & \text{if } x \text{ lies inside } \Gamma \end{cases}$$

where:

1. $d(x, \Gamma) = inf_{y \in \Gamma} \|x - y\|$
2. $\epsilon \in \{\pm 1\}$
3. $\|x\|$ is here the Euclidean norm of x.

In what follows, we will take the convention that $\epsilon = 1$, and we will call u the signed distance function.

Remark: It is possible to generalize definition 2.1 to a any distance d on $\mathbb{R}^2$ (not associated to the Euclidean norm). This point will be examine in Section 3.

Definition 2.2. We will call skeleton of Γ the set
$\{ x \in \mathbb{R}^2$ / there exist at least two distinct points y and z in Γ such that:

$$\|x - y\| = \|x - z\| = d(x, \Gamma) \}$$

We note S the skeleton, and $\bar{S}$ its closure.

Remark: Let $x \in \mathbb{R}^2$. If Γ is a closed curve, it is well-known that there exists at least one element $y \in \Gamma$ such that

$$\|x - y\| = d(x, \Gamma)$$

We will note $y = p(x)$.

Definition 2.3. We will call contact point of degree 2 for Γ each point $x \in \mathbb{R}^2$ such that:

1. x does not belong to Γ.
2. $\kappa(p(x)) = \frac{1}{\|x - p(x)\|} = \frac{1}{d(x,\Gamma)} = \frac{1}{u(x)}$ where u(x) is the signed distance function to Γ, $p(x) \in \Gamma$ as defined in definition 2.2, and $\kappa(y)$ is the curvature of Γ at y.

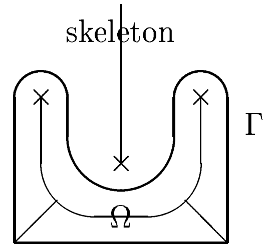


Figure 2: contact points of degree 2 (the crosses stand for the contact points of degree 2)

Remarks:

1. A contact point of degree 2 may belong to the skeleton (for instance in the case when Γ is a circle, then S reduces to the center of the circle, which is also a contact point of degree 2).

2. If x is a contact point of degree 2, then the circle of center x and radius $\|x - p(x)\|$ is in fact the osculator circle to Γ at p(x). That is why we call it contact point of degree 2.

2.1.2 Properties of the signed distance function

We give in this subsection some properties of the signed distance function associated to the Euclidean norm. We state the following well-known result without proof.

Proposition 2.4.

1. *The signed distance function u to a closed curve Γ in $\mathbb{R}^2$ is 1-Lipschitz.*

2. *If u is a signed distance function to a closed curve Γ in $\mathbb{R}^2$, and if $u \in C^1(\mathbb{R}^2 \backslash \bar{S} \to \mathbb{R})$ (where $\bar{S}$ stands for the closure of the skeleton of Γ), then we have for all $x \in \mathbb{R}^2 \backslash \bar{S}$:*

$$\|\nabla u(x)\| = 1$$

2.1.3 A characterization of the signed distance function

We do not know if the following characterization of the signed distance function exists in the literature, but we state it because it was the starting point for discovering an explicit solution of the reinitialization equation (1.1).

Proposition 2.5. *Let Γ be a closed curve in $\mathbb{R}^2$. Then*

1. *u is the distance function to Γ if and only if*

$$\begin{cases} u(x) \geq 0 & \forall x \in \mathbb{R}^2 \\ u(x) = 0 & \textit{if } x \in \Gamma \\ u(x) = \inf_{|y|=h} (u(x+y) + h) & \forall h \in [0, d(x, \Gamma)] \end{cases}$$

2. *u is the opposite of the distance function to Γ if and only if*

$$\begin{cases} u(x) \leq 0 & \forall x \in \mathbb{R}^2 \\ u(x) = 0 & \textit{if } x \in \Gamma \\ u(x) = \sup_{|y|=h} (u(x+y) - h) = -\inf_{|y|=h} (-u(x+y) + h) & \forall h \in [0, d(x, \Gamma)] \end{cases}$$

3. u is the signed distance function to Γ *if and only if*

$$\begin{cases} u(x) = 0 & \text{if } x \in \Gamma \\ u(x) \geq 0 & \text{if } x \text{ lies in the exterior of } \Gamma \\ u(x) \leq 0 & \text{if } x \text{ lies in the interior of } \Gamma \\ u(x) = \epsilon_x \inf_{|y|=h} (\epsilon_x u(x+y) + h) & \forall h \in [0, d(x,\Gamma)] \end{cases} \tag{2.1}$$

$$\text{where } \epsilon_x = \begin{cases} 1 & \text{if } x \text{ lies in the exterior of } \Gamma \\ -1 & \text{if } x \text{ lies in the interior of } \Gamma \end{cases}$$

Proof: We only show the first point of the proposition. The demonstration for the two other points is similar.

Step 1: We assume that u is the signed distance function to Γ. It is therefore obvious that: $u(x) = 0 \Leftrightarrow x \in \Gamma$ (since Γ is closed). Let $x \in \mathbb{R}^2 \backslash \Gamma$. We want to show that:

$$u(x) = \inf_{|y|=h} (u(x+y) + h) \ , \forall h \in [0, u(x)]$$

(here $u(x) = d(x, \Gamma)$). This is equivalent to:

$$\inf_{z \in \Gamma} |x - z| = \inf_{|y|=h} (\inf_{z \in \Gamma} |x + y - z| + h) \ , \forall h \in [0, u(x)]$$

i.e.

$$\inf_{z \in \Gamma} |x - z| = \inf_{z \in \Gamma} (\inf_{|y|=h} |x + y - z| + h) \ , \forall h \in [0, u(x)] \tag{2.2}$$

Let us show this last equality.

- Thanks to the triangular inequality, we have:

$$\underbrace{|x - z| - |y| + h}_{=|x-z| \ if \ |y|=h} \leq |x + y - z| + h$$

So :

$$|x - z| \leq \inf_{|y|=h} (|x + y - z|) + h$$

Hence :

$$\inf_{z \in \Gamma} |x - z| \leq \inf_{z \in \Gamma} (\inf_{|y|=h} |x + y - z| + h) \ , \forall h \in [0, u(x)]$$

And thus we have shown:

$$u(x) \leq \inf_{|y|=h} (u(x+y) + h) \ , \forall h \in [0, u(x)] \tag{2.3}$$

- By contradiction, let us assume there exists $h \in]0, u(x)]$ such that (2.3) is strict, i.e.

$$u(x) = \inf_{z \in \Gamma} |x - z| < \inf_{z \in \Gamma} (\inf_{|y|=h} |x + y - z| + h)$$

Or :

$$u(x) < h + \underbrace{\inf_{z \in \Gamma} (\inf_{|y|=h} |x + y - z|)}_{\inf_{z \in \Gamma} (\inf_{|w-x|=h} |w-z|)}$$

As Γ is closed, we know there exists $\bar{z} \in \Gamma$ such that $u(x) = d(x, \Gamma) = |x - \bar{z}|$.
So

$$u(x) < h + \inf_{|w-x|=h} |w - \bar{z}| \tag{2.4}$$

Let us choose:

$$\bar{w} = \frac{h}{u(x)}\bar{z} + \frac{-h + u(x)}{u(x)}x$$

that is

$$\bar{w} - x = \frac{h}{u(x)}(\bar{z} - x)$$

which implies

$$|\bar{w} - x| = h\frac{|\bar{z} - x|}{u(x)} = h$$

And so $\bar{w}$ is admissible. Moreover we have:

$$|\bar{w} - \bar{z}| = \frac{u(x) - h}{u(x)}|x - \bar{z}| = u(x) - h$$

We therefore get, with (2.4), that $u(x) < u(x)$, which is obviously wrong .
This completes the proof of (2.2).

Step 2: Conversely to show that $u(x)$ is a distance function, by choosing $h = d(x, \Gamma)$, we get :

$$u(x) = \underbrace{\inf_{|y|=d(x,\Gamma)} (u(x + y))}_{=0} + d(x, \Gamma)$$

■

2.1.4 Closure of the skeleton

We close this subsection by stating our main result concerning the closure of the skeleton of a closed curve. We give next an application to the measure of the skeleton.

Theorem 2.6. *The closure $\bar{S}$ of the skeleton S of a closed curve Γ in $\mathbb{R}^2$ satisfying hypotheses (H2.1)-(H2.6) is composed of S, of contact points of degree 2, and of points belonging to Γ where Γ is not C^2.*

Proof: We are going to use the sequential characterization of closed sets.
So let $(x_n) \in S^{\mathbb{N}}$ and $((y_n),(z_n)) \in \Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ (with $y_n \neq z_n$ $\forall n$) such that $\forall$ n $\in \mathbb{N}$:

$$\|x_n - y_n\| = \|x_n - z_n\| = d(x_n, \Gamma) \tag{2.5}$$

Let us suppose that $x_n \to x \in \mathbb{R}^2$. We want to know where x is. Let r>0. As $x_n \to x$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $x_n \in B(x, r)$. Let ρ=d(x,Γ). Then:

$$\|x_n - y_n\| = d(x_n, \Gamma) \leq \|x - x_n\| + d(x, \Gamma) \leq r + \rho$$

which implies:

$$\|y_n - x\| \leq \|y_n - x_n\| + \|x_n - x\| \leq \rho + 2r$$

And for the same reasons we get:

$$\|z_n - x\| \leq \rho + 2r$$

We therefore have: for all $n \geq n_0$, y_n and $z_n \in$ B(x,ρ+2r). The sequences (y_n) et (z_n) being bounded in $\mathbb{R}^2$, there exist $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^2$ such that (up to subsequences):

$$y_n \to y \text{ and } z_n \to z$$

Moreover, since (y_n) et (z_n) are in Γ which is a closed set, we have y and z in Γ.
By passing to the limit in (2.5), we get:

$$\|x - y\| = \|x - z\| = d(x, \Gamma)$$

Two cases may now occur:

Case 1: y et z are two distinct points, in which case $x \in S$.

Case 2: $y = z = p(x)$. In this case:

$$y_n \to y \text{ and } z_n \to y$$

Hence, for n large enough, y_n and z_n belong to a same connected component of Γ (indeed, the connected components of Γ being closed, the distance between two of them is positive). We want to show that x is either a contact point of degree 2, or a point on Γ where Γ is not C^2 (so an angular point thanks to hypotheses (H2.4)).

According to hypothesis (H2.3), Γ is locally concave or convex. We parameterize Γ around y in the indirect way.

Step 1: Let us assume first that Γ is C^2 around y. We are going to show that the curvature is locally positive. By contradiction, let us assume that there exists $r > 0$ such that $\kappa(z) \leq 0$ for all $z \in \mathcal{V}(r) = B(y,r) \bigcap \Gamma$, where $B(y,r)$ is an open disc of center y and radius r. Let $\mathcal{C}$ be the convex envelope of $\mathcal{V}(r)$ (see figure 3). Since the projection on a convex set is unique, we have $y_n = z_n$, which by hypothesis is absurd. The curvature of Γ must therefore be positive around y, i.e. in $\mathcal{V}(r)$.

Let us write $r_n = \|x_n - y_n\|$. The curvature of the circle $\mathcal{C}_n$ whose center is x_n and radius r_n, is equal to $\frac{1}{r_n}$. As $\mathcal{C}_n$ is tangential to Γ (inside) in y_n (since $\|x_n - y_n\| = d(x_n, \Gamma)$), we have:

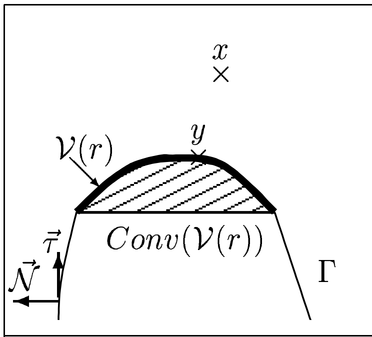
$$\kappa(y_n) \leq \frac{1}{r_n}$$

Indeed, if $\kappa(y_n) > \frac{1}{r_n}$, then locally, around y, the curve Γ lies inside the open disc whose boundary is $\mathcal{C}_n$. And this cannot hold since $\|x_n - y_n\| = d(x_n, \Gamma)$.
And we can prove by the same way that:

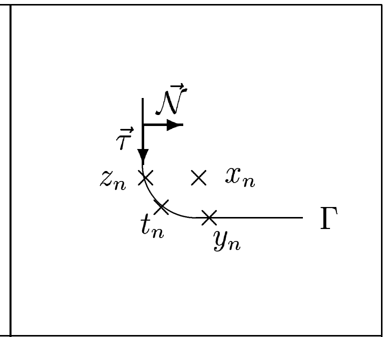
$$\kappa(z_n) \leq \frac{1}{r_n}$$

But as the curvature of Γ is non negative on $\mathcal{V}(r)$, then necessarily there exists t_n in $\mathcal{V}(r)$ between y_n and z_n such that:

$$\kappa(t_n) \geq \frac{1}{r_n}$$



Case when the curvature is non positive



Case when the curvature is positive

Figure 3: Local convexity or concavity of Γ

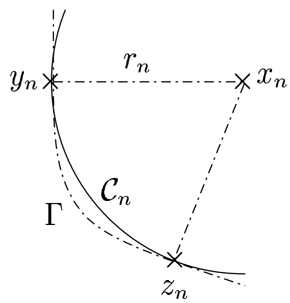


Figure 4: Curvature of Γ

Indeed, if we assume that t_n does not exist, then in each point on Γ between y_n and z_n, we have: $0 < \kappa(t) < \frac{1}{r_n}$. But this cannot hold (see figure 4).

Moreover, as (y_n) and (z_n) tend to y, so does (t_n). Therefore passing to the limit as $n \to \infty$ in the following inequality: $\kappa(y_n) \leq \frac{1}{r_n} \leq \kappa(t_n)$, we get since κ is C^0 in y: $\frac{1}{r} = \kappa(y) = \kappa$ where $r = \|x - y\|$.
The circle $\mathcal{C}$, whose center is x and radius r, is the osculator circle to Γ in y. Thus y is a double contact point between Γ and $\mathcal{C}$. Hence x is a contact point of degree 2.

Step 2: Let us assume now that Γ is not C^2 in y.

Case 1: If we suppose there exists $r > 0$ such that the curvature of Γ is non positive in $\mathcal{V}(r)\backslash\{y\} = \{B(y,r) \bigcap \Gamma\}\backslash\{y\}$, we get the same contradiction as in the case when Γ is not C^2 around y.

Case 2: Let us now examine the other case: for all $r > 0$, there exists $t \in \mathcal{V}(r)\backslash\{y\} = \{B(y,r) \bigcap \Gamma\}\backslash\{y\}$ such that $\kappa(t) > 0$. By eventually choosing r smaller, we may assume that $\mathcal{V}(r)$ has two connected components $\mathcal{V}_1(r)$ and $\mathcal{V}_2(r)$. According to hypotheses (H2.3), the curvature of Γ is thus positive on $\mathcal{V}_1(r)$ and/or $\mathcal{V}_2(r)$. For instance, let us assume that the curvature Γ be positive on $\mathcal{V}_1(r)$.
Then, as in the case where there exists $r > 0$ such that the curvature of Γ is positive on $B(y,r) \bigcap \Gamma$, we get:

$$\kappa(y_n) \leq \frac{1}{r_n}$$

And then, by passing to the limit (and remembering that according to hypothesis (H2.3), $\kappa(y) = \infty$), we get $r = 0$, i.e. $x \in \Gamma$ and Γ is not C^2 in x.
In fact, this case is a degenerate case from the preceeding one.

■

With similar arguments to those given in Theorem 2.6, we can prove the following lemma.

Lemma 2.7. *Let Γ be a closed curve in $\mathbb{R}^2$ satisfying hypotheses (H2.1)-(H2.6), and S its skeleton. If $x \in \mathbb{R}^2 \backslash S$ is a contact point of degree 2, then $|\kappa(p(x))|$ is a local strict maximum for $|\kappa|$, and Γ turns its concavity towards x around $p(x)$. Moreover, each strict local maximum of κ is associated at most to one contact point of degree 2.*

From Theorem 2.6, we can deduce

Corollary 2.8.

1. *The skeleton S of a closed curve Γ in $\mathbb{R}^2$ has a zero Lebesgue's measure.*
2. *The closure of the skeleton $\bar{S}$ of a closed curve Γ in $\mathbb{R}^2$ satisfying hypotheses (H2.1)-(H2.6) has zero Lebesgue's measure.*

Proof:

Part 1: The first part of the corollary is a consequence of the fact that the Euclidean signed distance function is 1-Lipschitz. Indeed, Rademacher's theorem (see [14]) enables us to assert that u is almost everywhere differentiable. As the skeleton of Γ is embedded in $\mathbb{R}^2$, S has zero Lebesgue's measure.

Part 2: Let us now show the second point from the first one. We know that $meas(S) = 0$, and we want to show that $meas(\bar{S}) = 0$. From Theorem 2.6, $\bar{S}$ is embedded in $S \bigcup S_1 \bigcup S_2$, with:

$$S_1 = \{\text{The set containing the contact points of degree 2 which do not belong to } S\}$$
$$S_2 = \{\text{The set containing the points of } \Gamma \text{ where } \Gamma \text{ not } C^2\}$$

According to Lemma 2.7 and hypothesis (H2.3), S_1 has at most a finite number of points, so $meas(S_1) = 0$. And S_2 is embedded in Γ which has zero Lebesgue's measure. So $meas(S_2) = 0$. Hence $meas(\bar{S}) = 0$.

■

Remark: In [32] , G.Matheron shows that the skeleton of any curve in $\mathbb{R}^2$ has an empty interior (in the topological sense). But he also gives an example of a curve in $\mathbb{R}^2$ whose skeleton has a closure of positive Lebesgue's measure. The second point of Corollary 2.8 may thus be considered as optimal. As far as we know, the fact that the closure of the skeleton of any curve in $\mathbb{R}^2$ has zero Lebesgue's measure remains an open question. We give here a sufficient condition.

2.2 The reinitialization equation

The purpose of this subsection is to study the following Hamilton-Jacobi PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + sign(u_0(x))\,(|Du| - 1) = 0 \\ u(.,0) = u_0(x) \end{cases} \tag{2.6}$$

where

$$sign(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases} \tag{2.7}$$

Equation (2.6) is used in image processing for active contours model (see [4, 37, 25, 10, 11, 36, 28, 3]). We will show in particular that the signed distance function is asymptotically solution of the PDE (2.6).

We can remark that in equation (2.6) the Hamiltonian is discontinuous with respect to the x variable. In the case when the Hamitonian is continuous, the viscosity solutions theory gives existence and uniqueness results for such PDEs (see [7, 20, 18, 15, 5, 6]). But works dealing with discontinuous Hamiltonian are just initiated (see [21, 35, 34, 16]). The existing literature gives answer neither to the existence nor to the uniqueness of viscosity solutions for (2.6).

2.2.1 Viscosity solutions: basic facts

The viscosity solutions theory deals with equations defined on an open set Ω embedded in $\mathbb{R}^N$, having the following form:

$$F(x, u, Du, D^2u) = 0 \tag{2.8}$$

It is immediate to extend it to evolutions equations:

$$\frac{\partial u}{\partial t} + F(x, u, Du, D^2u) = 0 \tag{2.9}$$

In equations (2.8) and (2.9), notations Du and D^2u stand respectively for the first and second derivatives in the space variable.
The notion of viscosity solutions is a powerful tool to prove existence and uniqueness of continuous solutions for first and second order PDEs. The definition of viscosity solutions is closely related to the maximum principle. For further information on this theory, we refer the reader to [7, 12, 13].

Let f be a locally bounded function. f_* and f^* will denote respectively the lower semi-continuous envelope (lsc) and the upper semi-continuous envelope (usc) of f.

$$f_*(x) = \liminf_{y \to x} f(y)$$

$$f^*(x) = \limsup_{y \to x} f(y)$$

Remark: We only recall the definition of viscosity solutions for stationary equations (as (2.8)), but it is immediate to generalize it to evolution equations (as (2.9)).

Definition 2.9. Let F: $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ be defined everywhere and locally bounded.

(i) A function u locally bounded, usc on Ω, is a viscosity subsolution of (2.8) if and only if for all $\Phi \in C^2(\Omega)$ and $x_0 \in \Omega$ local maximum point of $u - \Phi$ then:

$$F_*(x_0, u(x_0), D\Phi(x_0), D^2\Phi(x_0)) \leq 0 \tag{2.10}$$

(ii) A function u locally bounded, lsc on Ω, is a viscosity supersolution of (2.8) if and only if for all $\Phi \in C^2(\Omega)$ $x_0 \in \Omega$ local minimum point of $u - \Phi$ then:

$$F^*(x_0, u(x_0), D\Phi(x_0), D^2\Phi(x_0)) \geq 0 \tag{2.11}$$

(iii) We will call a viscosity solution of (2.8) every continuous function satisfying (2.10) and (2.11).

2.2.2 Existence and uniqueness of a viscosity solution for (2.6)

Throughout this subsection, we will consider a function u_0 uniformly continuous on $\mathbb{R}^2$. We will note $\Gamma = \{x / u_0(x) = 0\}$. For $x \in \mathbb{R}^2$, we will write $\epsilon_x = sign(u_0(x))$ and $d(x, \Gamma)$ the distance from x to Γ. The goal of this subsection is to prove that the function $u : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}^2$ defined by:

$$u(x,t) = \begin{cases} \epsilon_x \inf_{|y| \leq t} (\epsilon_x u_0(x+y) + t) & \text{if } t \leq t_x \\ \epsilon_x d(x, \Gamma) & \text{if } t > t_x \end{cases} \tag{2.12}$$

where

$$t_x = \inf\{t \in \mathbb{R}_+ / \inf_{|y| \leq t} (\epsilon_x u_0(x+y)) = 0\} = d(x, \Gamma) \tag{2.13}$$

is a viscosity solution of (2.6).

Let us write $\Omega_+ = \{x / u_0(x) > 0\}$ the outside (in the strict sense) of Γ, and $\Omega_- = \{x / u_0(x) < 0\}$ the inside (in the strict sense) of Γ ($\mathbb{R}^2 = \Omega_+ \bigcup \Omega_- \bigcup \Gamma$).

Proposition 2.10. *Let u_0 be a uniformly continuous function on $\mathbb{R}^2$ and set $\Gamma = \{x / u_0(x) = 0\}$. Then u defined by (2.12) is a uniformly continuous function on $\mathbb{R}^2 \times \mathbb{R}_+$.*

Remark: A consequence of the definition of t_x is that $x \mapsto t_x$ is continuous on $\mathbb{R}^2$.

Proof: Let us prove that u given by (2.12) is uniformly continuous on $\Omega \times [0, T]$, $\forall\, T > 0$. To do this, we are going to show that u is uniformly continous on $\Omega_+ \times [0, T]$, $\forall\, T > 0$, on $\Omega_- \times [0, T]$, $\forall\, T > 0$, and on $\mathcal{V} \times [0, T]$, $\forall\, T > 0$ (where $\mathcal{V}$ is a neighbourhood in $\mathbb{R}^2$ of $\{x / u_0(x) = 0\}$).

(i) First case:
Let us therefore show that u is uniformly continuous on $\Omega_+ \times [0, T]$,$\forall\, T > 0$. We first set $T > 0$. Let (x, t) and $(\hat{x}, \hat{t})$ in $\Omega_+ \times [0, T]$.

Step 1: Let us first assume that $\hat{t} \leq t_{\hat{x}}$ and $t \leq t_x$. We note ρ the modulus of continuity of u_0 (defined by $\rho(r) = \sup(|u_0(x) - u_0(y)|; |x-y| \leq r)$. We have:

$$u(x,t) - u(\hat{x},\hat{t}) = \inf_{\{|y|\leq t\}} (u_0(x+y)+t) - \inf_{\{|y|\leq \hat{t}\}} \left(u_0(\hat{x}+y)+\hat{t}\right) \tag{2.14}$$

As u_0 is continuous, there exists $\hat{b} \in B(\hat{x},\hat{t})$ (where $B(z,r)$ denotes the open ball in $\mathbb{R}^2$ centered in z and of radius r) such that

$$u_0(\hat{b}) = \inf_{\{|y|\leq \hat{t}\}} (u_0(\hat{x}+y))$$

Since $\hat{b} \in B(\hat{x},\hat{t})$, there exists $\lambda \in [0,1]$ such that $|\hat{x} - \hat{b}| = \lambda \hat{t}$. Let a be on $C(x, \lambda t)$ (where $C(z,r)$ denotes the circle in $\mathbb{R}^2$ centered in z and of radius r) such that $(\hat{x}-\hat{b}, x-a) = -|\hat{x}-\hat{b}||x-a|$ (where $(,)$ stands for the Euclidean scalar product). In particular, we have $a \in B(x,t)$ (since $|\lambda| \leq 1$). Then (2.14) becomes:

$$\begin{aligned} u(x,t) - u(\hat{x},\hat{t}) &= \inf_{\{|y|\leq t\}} (u_0(x+y)+t) - u_0(\hat{b}) - \hat{t} \\ &\leq u_0(a) - u_0(\hat{b}) + t - \hat{t} \\ &\leq \rho(|a-\hat{b}|) + t - \hat{t} \end{aligned}$$

But

$$\begin{aligned} |a-\hat{b}| &= |a - x + x - \hat{x} + \hat{x} - \hat{b}| \\ &\leq \underbrace{|a - x + \hat{x} - \hat{b}|}_{=\lambda|t-\hat{t}|\leq|t-\hat{t}|} + |x-\hat{x}| \end{aligned}$$

Since $\rho(r+s) \leq \rho(r) + \rho(s)$, we deduce that:

$$u(x,t) - u(\hat{x},\hat{t}) \leq \rho(|x-\hat{x}|) + |t-\hat{t}| + \rho(|t-\hat{t}|)$$

By the same way we can show that:

$$u(x,t) - u(\hat{x},\hat{t}) \geq -\left(\rho(|x-\hat{x}|) + |t-\hat{t}| + \rho(|t-\hat{t}|)\right)$$

Thus:

$$|u(x,t) - u(\hat{x},\hat{t})| \leq \rho(|x-\hat{x}|) + |t-\hat{t}| + \rho(|t-\hat{t}|) \tag{2.15}$$

Step 2: Let us now assume that $t \geq \hat{t} \geq \max(t_x, t_{\hat{x}})$. We then have:

$$\begin{aligned} |u(x,t) - u(\hat{x},\hat{t})| &= |u(x,t_x) - u(\hat{x},t_{\hat{x}})| \\ &= |d(x,\Gamma) - d(\hat{x},\Gamma)| \end{aligned}$$

Hence, as the Euclidean distance is 1-Lipschitz:

$$|u(x,t) - u(\hat{x},\hat{t})| \leq |x-\hat{x}| \tag{2.16}$$

Step 3: Let us now assume that $\hat{t} \geq t_{\hat{x}}$ and $t \leq t_x$. We then have:

$$|u(x,t) - u(\hat{x},\hat{t})| = |u(x,t) - u(\hat{x},t_{\hat{x}})|$$

The difference $|u(x,t)-u(\hat{x},t_{\hat{x}})|$ can now be estimated as in Step 1. So, using (2.15), we have:

$$|u(x,t)-u(\hat{x},t_{\hat{x}})| \leq \rho(|x-\hat{x}|)+|t-t_{\hat{x}}|+\rho(|t-t_{\hat{x}}|)$$

Hence:

$$|u(x,t)-u(\hat{x},\hat{t})| \leq \rho(|x-\hat{x}|)+|t_x-t_{\hat{x}}|+\rho(|t_x-t_{\hat{x}}|)$$

But

$$|t_x-t_{\hat{x}}| = |d(x,\Gamma)-d((\hat{x},\Gamma)| \leq |x-\hat{x}|$$

and so:

$$|u(x,t)-u(\hat{x},\hat{t})| \leq 2\rho(|x-\hat{x}|)+|x-\hat{x}|) \tag{2.17}$$

From Steps 1 to 3, we deduce the uniform continuity of u on $\Omega_+\times[0,T]$ since as u_0 uniformly continuous, we have $\rho(|t-\hat{t}|)\to 0$ as $t\to\hat{t}$, and $\rho(|x-\hat{x}|)\to 0$ as $x\to\hat{x}$.

(ii) Second case:
We can show with an identical proof that u is also uniformly continuous on $\Omega_-\times[0,T]$, $\forall T>0$.

(iii) Third case:
Let us show now that u is uniformly continuous in any neighbourhood $\mathcal{V}$ in $\mathbb{R}^2$ of $\{x/u_0(x)=0\})$. In fact, thanks to the first two cases, we only have to show that u is continuous on $\{x/u_0(x)=0\}$.
Let us choose $T>0$. Let $\epsilon>0$. Since u_0 is uniformly continuous, for all $\epsilon>0$, there exists $\delta>0$ such that: $(0\leq r\leq\delta)\Rightarrow(\rho(r)\leq\frac{\epsilon}{3})$. Moreover, there exists $\eta>0$ such that: $(d(x,\Gamma)\leq\eta)\Rightarrow(|u_0(x)|\leq\frac{\epsilon}{3})$
Let $x\in\mathbb{R}^2$ such that $d(x,\Gamma)\leq\min\left(\delta,\eta,\frac{\epsilon}{3}\right)$. Recall that $t_x=d(x,\Gamma)$. Two cases can occur:

Step 1: If $t\leq t_x$, then:

$$|u(x,t)| = |\inf_{\{|y|\leq t\}}(u_0(x+y)+t)| \leq \inf_{\{|y|\leq t\}}|u_0(x+y)|+t$$

and since $t_x=d(x,\Gamma)$ and $t\leq t_x$, we have:

$$\begin{aligned}|u(x,t)| &\leq |u_0(x)|+\rho(t)+t\\ &\leq |u_0(x)|+\rho(t_x)+t_x\\ &\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ &\leq \epsilon\end{aligned}$$

Step 2: If $t>t_x$, then:

$$|u(x,t)| = d(x,\Gamma)\leq\frac{\epsilon}{3}\leq\epsilon$$

Thus, $u(x,t)$ is continuous on $\{x/u_0(x)=0\}\times[0,T]$.

From the three cases, we conclude that u is uniformly continuous on $\mathbb{R}^2\times\mathbb{R}_+$.

■

We can now give the main result of this section.

Theorem 2.11. *Let u be defined by (2.12). Then u is the unique viscosity solution of (2.6) uniformly continuous on $\mathbb{R}^2\times[0,T]$, $\forall$ $T>0$, and vanishing on Γ, $\forall t\in[0,T]$.*

Proof: We split the proof in two parts

Part1: Existence of a solution.
Let us first prove that u defined by (2.12) is a viscosity subsolution of (2.6). From Proposition 2.10, we already know that u is uniformly continuous on $\mathbb{R}^2 \times \mathbb{R}_+$. Let us set $(x_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}_+$, and let Φ of class C^2 such that $u - \Phi$ has a local maximum in (x_0, t_0).

(i) Step 1: Let us first assume $u_0(x_0) \neq 0$. For instance, we will assume $u_0(x_0) > 0$.

If $t_0 > t_{x_0}$: We have $u(x_0, t_0) = d(x_0, \Gamma)$ and $\frac{\partial u}{\partial t}(x_0, t_0) = 0$. Let us show that $F(x_0, D\Phi(x_0, t_0)) \leq 0$, i.e. $-1 + |D\Phi(x_0)| \leq 0$.
For the sake of clarity, we will not write the t-dependency which plays no role here. For x small enough, we have (if $h \in [0, d(x, \Gamma]$):

$$h + u(x_0 + x) - \Phi(x_0 + x) \leq u(x_0) - \Phi(x_0) + h$$

$$\Leftrightarrow u(x_0 + x) + h \leq u(x_0) - \Phi(x_0) + \Phi(x_0 + x) + h$$

$$\Rightarrow \underbrace{\inf_{|x|=h} (u(x_0 + x) + h)}_{= u(x_0) \text{ thanks to Proposition 2.5 } (u(x_0) = d(x_0, \Gamma))} \leq u(x_0) - \Phi(x_0) + \inf_{|x|=h} (\Phi(x_0 + x) + h)$$

Hence

$$\inf_{|x|=h} (\Phi(x_0 + x) - \Phi(x_0)) \geq -h$$

But we have

$$\Phi(x_0 + x) - \Phi(x_0) = (D\Phi(x_0), x) + 0(h^2)$$

Thus

$$\begin{aligned} \inf_{|x|=h} (\Phi(x_0 + x) - \Phi(x_0)) &= \inf_{|x|=h} (-|x||D\phi(x_0)|) + 0(h^2) \\ &= -h|D\phi(x_0)| + 0(h^2) \end{aligned}$$

And

$$|D\phi(x_0)| \leq 1 + 0(h)$$

We deduce from this that u is a viscosity subsolution of (2.6) in (x_0, t_0).

Remarks:

1. If $u_0(x_0) < 0$, it sufffices to repeat the same proof as above by starting with the inequality:
$$-h + u(x_0 + y) - \Phi(x_0 + y) \leq u(x_0) - \Phi(x_0) - h$$
2. We can also notice that in fact we have shown the signed distance function is a viscosity solution of the stationnary equation: $sign(u_0)\,(|Du| - 1) = 0$.

If $t_0 \leq t_{x_0}$: For h and y small enough ($h > 0$), we have :

$$h + u(x_0 + y, t_0 - h) - \Phi(x_0 + y, t_0 - h) \leq u(x_0, t_0) - \Phi(x_0, t_0) + h$$

which implies

$$\inf_{|y|\leq h} (u(x_0 + y, t_0 - h) + h) \leq \inf_{|y|\leq h} (\Phi(x_0 + y, t_0 - h)) - \Phi(x_0, t_0) + h + u(x_0, t_0)$$

But

$$\begin{aligned} \inf_{|y|\leq h} (u(x_0 + y, t_0 - h) + h) &= \inf_{|y|\leq h} \left(\inf_{|z|\leq t_0 - h} (u_0(x_0 + y + z) + t_0) \right) \\ &= \underbrace{\inf_{|w|\leq t_0} (u_0(x_0 + w) + t_0)}_{= u(x_0, t_0) \text{ thanks to (2.12)}} \end{aligned}$$

Hence

$$\Phi(x_0, t_0) \leq h + \inf_{|y|\leq h} (\Phi(x_0 + y, t_0 - h))$$

And thus

$$\Phi(x_0, t_0) - \Phi(x_0, t_0 - h) \leq \inf_{|y|\leq h} (\Phi(x_0 + y, t_0 - h) - \Phi(x_0, t_0 - h)) + h$$

But

$$\Phi(x_0, t_0) - \Phi(x_0, t_0 - h) = h \frac{\partial \Phi}{\partial t}(x_0, t_0) + 0(h^2)$$

and

$$\Phi(x_0 + y, t_0 - h) - \Phi(x_0, t_0 - h) = (D\Phi(x_0, t_0 - h), y) + 0(h^2)$$

thus

$$\begin{aligned} \inf_{|y|\leq h} (\Phi(x_0 + y, t_0 - h) - \Phi(x_0, t_0 - h)) &= \inf_{|y|\leq h} (-|y||D\Phi(x_0, t_0 - h)|) + 0(h^2) \\ &= -h|D\Phi(x_0, t_0 - h)| + 0(h^2) \end{aligned}$$

from which we deduce

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) \leq -|D\Phi(x_0, t_0 - h)| + 0(h) + 1$$

But

$$D\Phi(x_0, t_0 - h) = D\Phi(x_0, t_0) + 0(h)$$

Thus we obtain

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) + \underbrace{(|D\Phi(x_0, t_0 - h)| - 1)}_{= F(x_0, D\Phi(x_0, t_0))} \leq 0$$

i.e. u is a vicosity subsolution of (2.6) in (x_0, t_0).

Remark : If $u_0(x_0) < 0$, it suffices to start from:

$$-h + u(x_0 + y, t_0 - h) - \Phi(x_0 + y, t_0 - h) \leq u(x_0, t_0) - \Phi(x_0, t_0) - h$$

It can be shown by a similar proof that u is a viscosity supersolution of (2.6) in (x_0, t_0). So we have shown that u is a viscosity solution of (2.6) in the case $u_0(x_0) \neq 0$ (since thanks to Proposition 2.10, u is also continuous).

(ii) Step 2: Let us now assume $u_0(x_0) = 0$

We have

$$F_*(x_0, D\Phi(x_0, t_0)) = \min(0, |D\Phi(x_0, t_0)| - 1, 1 - |D\Phi(x_0, t_0)|)$$

and

$$F^*(x_0, D\Phi(x_0, t_0)) = \max(0, |D\Phi(x_0, t_0)| - 1, 1 - |D\Phi(x_0, t_0)|)$$

Thus

$$F_*(x_0, D\Phi(x_0, t_0)) \leq 0 \text{ and } F^*(x_0, D\Phi(x_0, t_0)) \geq 0$$

and so u is still a viscosity solution of (2.6) in this case.

Part 2: Uniqueness of the solution.

Let us recall that:

$$\begin{cases} u_0(x) = 0 & \text{if } x \in \Gamma \\ u_0(x) \geq 0 & \text{if } x \in \Omega_+ \\ u_0(x) \leq 0 & \text{if } x \in \Omega_- \end{cases} \tag{2.18}$$

Let us consider the two following problems:

$$\begin{cases} \frac{\partial u}{\partial t} + |Du| - 1 = 0 \\ u(.,0) = u_0\lfloor_{\Omega_+} \\ u(x,t) = 0 \text{ si } x \in \Gamma \end{cases} \tag{2.19}$$

$$\begin{cases} \frac{\partial u}{\partial t} - (|Du| - 1) = 0 \\ u(.,0) = u_0\lfloor_{\Omega_-} \\ u(x,t) = 0 \text{ si } x \in \Gamma \end{cases} \tag{2.20}$$

Let us recall that u_0 is assumed to be uniformly continuous.

From classical results on viscosity solutions (see Theorem 1 in [20]), (2.19) (resp. (2.20)) has at most one viscosity solution uniformly continuous on $\Omega_+ \times [0, T]$, $\forall$ T>0 (resp. on $\Omega_- \times [0, T]$, $\forall$ T>0). By reconsidering the arguments given for the existence, it can be shown that the solution of (2.19) (resp. of (2.20)) is the restriction of the function (2.12) to $\Omega_+ \times \mathbb{R}_+$ (resp. to $\Omega_- \times \mathbb{R}_+$). And since a viscosity solution of (2.6) has to be continuous, the only possible solution is the function (2.12).
The proof of Theorem 2.11 is thus complete.

■

Remark on Theorem 2.11: Equation (2.6) is used to reinitialize a function as the signed distance function to a closed curve in $\mathbb{R}^2$. This reinitialization is only performed in a narrow band in which Γ is embedded (see [25, 4]). But according to the form of the solution of (2.6), the first reinitialized values of u are the closest ones to Γ. This explain why this reinitialization method is so fast. Moreover, an immediate consequence of the definition of t_x is that $u(x, t_x) = \epsilon_x t_x$. Thus, if we choose the bandwidth around Γ on which we want a signed distance function, we know how many iterations equation (2.6) are necessary.

Commentary: We comment here the introduction of t_x in the definition of (2.12). It could appear arbitrary. Let us consider the function $v : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ defined by

$$v(x,t) = \epsilon_x \inf_{|y| \leq t} (\epsilon_x u_0(x+y) + t) \tag{2.21}$$

We recall that $\epsilon_x = sign(u_0(x))$. By reconsidering the arguments of the existence part of Theorem 2.11, we see that v is a viscosity solution of (2.6) on $\Omega_+ \times \mathbb{R}_+$ and on $\Omega_- \times \mathbb{R}_+$. But v is not necessarily a continuous function on $\mathbb{R} \times \mathbb{R}_+$, and thus is not a viscosity solution of (2.6) on $\mathbb{R} \times \mathbb{R}_+$. The following 1-D example illustrates this remark.

Let $\Omega = (-1, 1)$, and let us take as initial condition:

$$u_0(x) = 2\,(|x| - 1)$$

In this case, it is easy to compute both functions u and v given by (2.12) and (2.21). We get:

$$u(x,t) = \begin{cases} 2\,(|x|-1)+t & \text{if } |x| \leq 1 \text{ and } t \leq 1 - |x| = t_x \\ |x| - 1 & \text{if } |x| \leq 1 \text{ and } t \geq 1 - |x| = t_x \\ 2\,(|x|-1)-t & \text{if } |x| \geq 1 \text{ and } t \leq |x| - 1 = t_x \\ |x| - 1 & \text{if } |x| \geq 1 \text{ and } t \geq |x| - 1 = t_x \end{cases} \tag{2.22}$$

$$v(x,t) = \begin{cases} 2\,(|x|-1)+t & \text{if } |x| \leq 1 \\ 2\,(|x|-1)-t & \text{if } |x| \geq 1 \text{ and } t \leq |x| \\ t - 2 & \text{if } |x| \geq 1 \text{ and } t \geq |x| \end{cases} \tag{2.23}$$

u is continuous on $\mathbb{R} \times \mathbb{R}^+$, but this is not the case for v: we just have to consider $v(1+\epsilon, t)$ and $v(1-\epsilon, t)$ for t large enough to convince ourselves.

■

Remark: Let us consider the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + sign(u)\,(|Du| - 1) = 0 \\ u(.,0) = u_0 \end{cases} \tag{2.24}$$

This PDE is very close to (2.6), but the discontinuity of the Hamiltonian $H(r,p) = sign(r)\,(|p| - 1)$ is with respect to r.
u defined by (2.12) is still a uniformly continuous viscosity solution of (2.24) (it suffices to remark that for all $x \in \mathbb{R}^2$, we have for all $t \in \mathbb{R}_+$: $sign(u(x,t)) = sign(u_0(x))$; and the arguments are then the same as the one in the proof of Proposition 2.11 (Part 1)).
But as far as we know, the uniqueness of a solution of (2.24) remains an open problem.

3. A more general equation

In this section, we want to extend some results of the previous section to more general discontinuous Hamiltonian. Let us consider the first order Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + sign(u_0(x))H(Du) = 0 \\ u(.,0) = u_0(x) \end{cases} \tag{3.1}$$

In the sequel we will always make the following hypotheses (for the reader's convenience, we recall all the hypotheses which we use in this section at the end of the paper):

(H 3.1) u_0 is uniformly continuous on $\mathbb{R}^2$.

(H 3.2) H is a convex function on $\mathbb{R}^2$.

(H 3.3) $\lim_{|p|\to+\infty} \dfrac{H(p)}{|p|} = +\infty$.

(H 3.4) $H(0) < 0$ and $\{p/H(p) = 0\}$ is non-void and symmetric with respect to 0 (i.e. $\{H(p) = 0\} \Rightarrow \{H(-p) = 0\}$).

We will note Ω_+ (resp. Ω_-) the open set $\{x/u_0(x) > 0\}$ (resp. $\{x/u_0(x) < 0\}$), and Γ the common boundary to Ω_+ and Ω_-, i.e. $\Gamma = \{x/u_0(x) = 0\}$.

We consider the function:

$$u(x,t) = \begin{cases} u_0(x) & \text{if } x \in \Gamma \\ \inf_{z\in\mathbb{R}^2} [u_0(x - tz) + tH^*(z)] & \text{if } x \in \Omega_+ \text{ and } t \leq t_x \\ d_L(x,\Gamma) & \text{if } x \in \Omega_+ \text{ and } t > t_x \\ \sup_{z\in\mathbb{R}^2} [u_0(x + tz) - tH^*(z)] & \text{if } x \in \Omega_- \text{ and } t \leq t_x \\ -d_L(x,\Gamma) & \text{if } x \in \Omega_- \text{ and } t > t_x \end{cases} \tag{3.2}$$

where t_x is defined by:

$$t_x = \begin{cases} \inf\{t \in \mathbb{R}_+ / \inf_{z\in\mathbb{R}^2} [u_0(x - tz) + tH^*(z)] = d_L(x,\Gamma)\} & \text{if } x \in \Omega_+ \\ \inf\{t \in \mathbb{R}_+ / \sup_{z\in\mathbb{R}^2} [u_0(x + tz) - tH^*(z)] = -d_L(x,\Gamma)\} & \text{if } x \in \Omega_- \end{cases} \tag{3.3}$$

Our goal is to prove that with some additional (and technical) hypotheses the function $u(x,t)$ defined by (3.2) is the unique viscosity solution of (3.1) vanishing on Γ.

d_L is the distance associated to the Hamiltonian H and will be defined in subsection 3.1.2. We recall that H^* stands for the Legendre-Fenchel transform of H:

$$H^*(z) = \sup_{p\in R^N} ((p,z) - H(p))$$

Remarks:

1. Most of the results of Section 3 are still valid when (H 3.3) is replaced by $\lim_{|p|\to+\infty} H(p) = +\infty$.

2. We draw reader's attention to the asymptotic behaviour of the solution $u(x,t)$ as $t \to +\infty$. This type of result, to the best of our knowledge, seems new for discontinuous Hamiltonian.

3. t_x is such that $u(x,t_x) = d_L(x,\Gamma)$ if $x \in \Omega_+$ and $u(x,t_x) = -d_L(x,\Gamma)$ if $x \in \Omega_-$, and $x \mapsto d_L(x,\Gamma)$ is the viscosity solution of the stationary equation $H(Du) = 0$ (see corollary 3.5).

3.1 Technical preliminaries

3.1.1 Convex analysis tools

We will use some classical tools from convex analysis.

1. H^* is always convex, and $H^*(0) = -\inf H(p)$

2. As we assume that $\lim_{|p|\to+\infty} \dfrac{H(p)}{|p|} = +\infty$, we also have (see [14]):

$$\lim_{|p|\to+\infty} \frac{H^*(p)}{|p|} = +\infty \tag{3.4}$$

3. If F_i is a family of functions from $\mathbb{R}^N$ into $(-\infty,\infty)$, we have (see [29]):

$$\left(\inf_i F_i\right)^* = \sup_i F^*{}_i \tag{3.5}$$

$$\forall\lambda > 0\ ,\ (\lambda F)^* = \lambda F^*\left(\frac{u}{\lambda}\right) \tag{3.6}$$

4. If H is convex, we have $\forall x \in R^2$, $\forall t \geq \inf_{\mathbb{R}^2} H$:

$$\sup_{\{p/H(p)=t\}} (x,p) = \inf_{\{\lambda>0\}} \left(t\lambda + \lambda H^*\left(\frac{x}{\lambda}\right)\right) \tag{3.7}$$

This last formula comes from [24].

3.1.2 Distance associated to the Hamiltonian H

Definition 3.1. For H satisfying hypotheses (H 3.2) and (H 3.4), we define:

$$L(x) = \max_{\{p/H(p)=0\}}(x,p)$$

where (.,.) denotes the Euclidean product in $\mathbb{R}^2$. L is a norm on $\mathbb{R}^2$. The associated distance to this norm is given by:

$$d_L(x,y) = L(x-y)$$

We will note

$$d_L(x,\Gamma) = \inf_{y\in\Gamma} L(x-y)$$

L is called norm associated to the Hamiltonian H and d_L a L-distance (to distinguish it from the Euclidean distance).

L verifies the following proposition:

Proposition 3.2. *If Γ is a closed curve in $\mathbb{R}^2$, then:*

(i) u is the L-distance function to Γ if and only if

$$\begin{cases} u(x) \geq 0 & \forall x \in \mathbb{R}^2 \\ u(x) = 0 & \textit{if } x \in \Gamma \\ u(x) = \inf_{\{L(y)=h\}} \left(u(x+y)+h\right) & \forall h \in [0, d_L(x,\Gamma)] \end{cases}$$

(ii) u is the opposite of the L-distance function to Γ if and only if

$$\begin{cases} u(x) \leq 0 & \forall x \in \mathbb{R}^2 \\ u(x) = 0 & \textit{if } x \in \Gamma \\ u(x) = \sup_{\{L(y)=h\}} \left(u(x+y)-h\right) = -\inf_{\{L(y)=h\}} \left(-u(x+y)+h\right) & \forall h \in [0, d_L(x,\Gamma)] \end{cases}$$

(iii) u is the signed L-distance function to Γ if and only if

$$\begin{cases} u(x) = 0 & \textit{if } x \in \Gamma \\ u(x) \geq 0 & \textit{if } x \textit{ lies in the exterior of } \Gamma \\ u(x) \leq 0 & \textit{if } x \textit{ lies in the interior of } \Gamma \\ u(x) = \epsilon_x \inf_{\{L(y)=h\}} \left(\epsilon_x u(x+y)+h\right) & \forall h \in [0, d_L(x,\Gamma)] \end{cases} \tag{3.8}$$

where $\epsilon_x = \begin{cases} 1 & \textit{if } x \textit{ lies outside } \Gamma \\ -1 & \textit{if } x \textit{ lies inside } \Gamma \end{cases}$

Proof: As L is a norm, the proof is the same as the one given for Proposition 2.5.

■

3.1.3 Some more tools

We will also need the following lemma:

Lemma 3.3. *Let $x \in \mathbb{R}^2$, $h \geq 0$ and $t \geq 0$. The following formula holds:*

$$\inf_z \left[u_0\left(x-(t+h)z\right)+(t+h)H^*(z)\right] =$$
$$\inf_\xi \inf_y \left[u_0\left(x-h\xi-ty\right)+tH^*(y)+hH^*(\xi)\right]$$

Proof: Let us write:

$$\begin{aligned} A &= \inf_z \left[u_0\left(x-(t+h)z\right)+(t+h)H^*(z)\right] \\ B &= \inf_\xi \inf_y \left[u_0\left(x-h\xi-ty\right)+tH^*(y)+hH^*(\xi)\right] \end{aligned}$$

It is clear that $B \leq A$. Let us show the reverse inequality: Let ξ and y in $\mathbb{R}^2$, and h and $t > 0$ (the case when h or $t = 0$ is obvious). Let $z = \frac{h\xi+ty}{h+t}$. We have:

$$u_0\left(x-(t+h)z\right)+(t+h)H^*(z) =$$
$$u_0\left(x-h\xi-ty\right)+(t+h)H^*\left(\frac{h\xi}{h+t}+\frac{ty}{h+t}\right)$$

We then use the fact that H^* is convex:

$$(t+h)H^*\left(\frac{h\xi+ty}{h+t}\right) \leq hH^*(\xi)+tH^*(y)$$

We deduce:

$$\begin{array}{c} u_0\left(x-(t+h)z\right)+(t+h)H^*(z) \leq \\ u_0\left(x-h\xi-ty\right)+hH^*(\xi)+tH^*(y) \end{array}$$

which implies (by taking the infimum in z in the left-hand-side and the maximum in y and ξ in the right-hand-side)

$$A \leq B$$

■

3.2 Viscosity inequalities and uniqueness for equation (3.1)

In order to show that $u(x,t)$ given by (3.2) is a right solution of (3.1), we have examine four points:

(i) $u(x,t)$ verifies the viscosity inequalities (2.10) and (2.11).

(ii) $u(x,t)$ is the unique solution of (3.1) vanishing on Γ.

(iii) The existence and continuity of t_x.

(iv) $u(x,t)$ is uniformly continuous.

We will begin by proving the two first points in this subsection. We will need to make further assumptions on u_0 to get the two last points.

3.2.1 Viscosity inequalities

Proposition 3.4. *Let us assume hypotheses (H 3.1)-(H 3.4) hold. If $x \mapsto t_x$ and $(x,t) \mapsto u(x,t)$ given by (3.2) are continuous respectively on $\mathbb{R}^2\backslash\Gamma$ and $\mathbb{R}^2 \times \mathbb{R}_+$, then u is a viscosity solution of (3.1) vanishing on Γ.*

Proof: The proof is not very different from the case when the Hamiltonian H is continuous (see [5, 15, 6]). But our proof appears more direct and natural to us than those existing in the literature which are often based on optimal control theory.
To show that u is a viscosity solution of (3.1), it suffices to show that u is both a super and subsolution (in the viscosity sense). We only show that u is a subsolution (proving that u is also a supersolution is similar).
So let $(x_0,t_0) \in \mathbb{R}^2 \times \mathbb{R}^+$, and Φ smooth such that (x_0,t_0) is a local maximum of $(u-\Phi)(x,t)$.

(i) First case: We first assume that $x_0 \in \Omega_+$ (the case ($x_0 \in \Omega_-$) is identical).

Step 1: if $t_0 \leq t_{x_0}$, then:

For $\epsilon < 1$ small enough, if $h \leq \epsilon^2$ and $|y| \leq \frac{1}{\epsilon}$ (with $h > 0$), we have :

$$u(x_0 - hy, t_0 - h) - \Phi(x_0 - hy, t_0 - h) \leq u(x_0, t_0) - \Phi(x_0, t_0) \tag{3.9}$$

We add $hH^*(y)$ to the two members of the inequality (3.9), then:

$$\begin{aligned} u(x_0 - hy, t_0 - h) + hH^*(y) \leq \\ u(x_0, t_0) + [\Phi(x_0 - hy, t_0 - h) - \Phi(x_0, t_0) + hH^*(y)] \end{aligned} \tag{3.10}$$

This implies

$$\inf_{|y| \leq \frac{1}{\epsilon}} (u(x_0 - hy, t_0 - h) + hH^*(y)) \leq \inf_{|y| \leq \frac{1}{\epsilon}} (\Phi(x_0 - hy, t_0 - h) - \Phi(x_0, t_0) + hH^*(y)) + u(x_0, t_0) \tag{3.11}$$

which implies:

$$\inf_{y \in \mathbb{R}^2} (u(x_0 - hy, t_0 - h) + hH^*(y)) \leq \inf_{|y| \leq \frac{1}{\epsilon}} (\Phi(x_0 - hy, t_0 - h) - \Phi(x_0, t_0) + hH^*(y)) + u(x_0, t_0) \tag{3.12}$$

But from (3.2), we have (since $t_{x_0} - h < t_{x_0 - hy}$ for h small enough as $x \mapsto t_x$ is continuous on Ω_+):

$$\begin{aligned} & \inf_{y \in \mathbb{R}^2} (u(x_0 - hy, t_0 - h) + hH^*(y)) \\ = & \inf_{y \in \mathbb{R}^2} \left(\inf_{z \in \mathbb{R}^2} (u_0(x_0 - hy - (t_0 - h)z) + hH^*(y) + (t_0 - h)H^*(z)) \right) \end{aligned}$$

So, thanks to Lemma 3.3:

$$\inf_{y \in \mathbb{R}^2} (u(x_0 - hy, t_0 - h) + hH^*(y)) = \underbrace{\inf_{w \in \mathbb{R}^2} (u_0(x_0 - t_0 w) + t_0 H^*(w))}_{= u(x_0, t_0)} \tag{3.13}$$

Hence, using (3.12):

$$\Phi(x_0, t_0) \leq \inf_{|y| \leq \frac{1}{\epsilon}} (\Phi(x_0 - hy, t_0 - h) + hH^*(y))$$

And thus

$$\Phi(x_0, t_0) - \Phi(x_0, t_0 - h) \leq \inf_{|y| \leq \frac{1}{\epsilon}} (\Phi(x_0 - hy, t_0 - h) - \Phi(x_0, t_0 - h) + hH^*(y)) \tag{3.14}$$

But

$$\Phi(x_0, t_0) - \Phi(x_0, t_0 - h) = h \frac{\partial \Phi}{\partial t}(x_0, \tau(h)) \tag{3.15}$$

where $\tau(h) \in [t_0 - h, t_0]$. And (remember that $D\Phi$ stands for the derivative in the space variable):

$$\Phi(x_0 - hy, t_0 - h) - \Phi(x_0, t_0 - h) = -h(D\Phi(\xi(h, y), t_0 - h), y) \tag{3.16}$$

where $\xi(h,y) = \lambda(x_0 - hy) + (1-\lambda)x_0$ (with $\lambda \in [0,1]$).
thus
$$h\frac{\partial \Phi}{\partial t}(x_0, \tau(h)) \leq \inf_{|y|\leq \frac{1}{\epsilon}} (-h(D\Phi(\xi(h,y), t_0 - h), y) + hH^*(y)) \tag{3.17}$$
hence
$$\frac{\partial \Phi}{\partial t}(x_0, \tau(h)) \leq \inf_{|y|\leq \frac{1}{\epsilon}} (-(D\Phi(\xi(h,y), t_0 - h), y) + H^*(y)) \tag{3.18}$$

If $\epsilon \to 0$, then (since $h \leq \epsilon^2$ and $|y| \leq \frac{1}{\epsilon}$):
$$\tau(h) \to t_0$$
and
$$\xi(h,y) \to x_0$$
Thus
$$\frac{\partial \Phi}{\partial t}(x_0, t_0) \leq \inf_{y\in\mathbb{R}^2} (-(D\Phi(x_0, t_0), y) + H^*(y)) \tag{3.19}$$
But
$$\begin{aligned}\inf_y (-(D\Phi(x_0,t_0),y) + H^*(y)) &= -\sup_y ((D\Phi(x_0,t_0),y) - H^*(y)) \\ &= -H^{**}(D\Phi(x_0,t_0)) \\ &= -H(D\Phi(x_0,t_0))\end{aligned}$$
Hence
$$\frac{\partial \Phi}{\partial t}(x_0, t_0) \leq -H(D\Phi(x_0, t_0)) \tag{3.20}$$
and u is a viscosity subsolution of (3.1).

Step 2: if $t_0 > t_{x_0}$, then:
We have $\frac{\partial u}{\partial t}(x_0,t_0) = 0$ (u is then differentiable with respect to the variable t), and thus $\frac{\partial \Phi}{\partial t}(x_0,t_0) = 0$. We therefore want to show that
$$H\left(D\Phi(x_0,t_0)\right) \leq 0$$

To make the proof clearer, we will not write the t-dependency which here is unsignificant. For x small enough, we have:
$$h + u(x_0 + x) - \Phi(x_0+x) \leq u(x_0) - \Phi(x_0) + h$$
$$\Leftrightarrow u(x_0+x) + h \leq u(x_0) - \Phi(x_0) + \Phi(x_0+x) + h$$
$$\Rightarrow \quad \underbrace{\inf_{L(x)=h} (u(x_0+x)+h)}_{\substack{= u(x_0) \text{ from Proposition 3.2} \\ (u(x_0) = d_L(x_0,\Gamma))}} \leq u(x_0) - \Phi(x_0) + \inf_{L(x)=h} (\Phi(x_0+x)+h)$$
Hence
$$\inf_{L(x)=h} (\Phi(x_0+x) - \Phi(x_0)) \geq -h$$

But

$$\Phi(x_0 + x) - \Phi(x_0) = (D\Phi(x_0), x) + o(h)$$

So

$$\inf_{L(x)=h} (D\Phi(x_0), x) \geq -h + o(h) \tag{3.21}$$

Remember that:

$$L(x) = L(x,0) = \sup_{\{H(p)=0\}} (x,p)$$

(3.21) can then be written as:

$$\sup_{L(x)=h} (D\Phi(x_0), x) \leq h + o(h) \tag{3.22}$$

We then use (3.7) with $t = 0$ to compute L:

$$L(x) = \sup_{\{H(p)=0\}} (x,p) = \inf_{\{\lambda>0\}} \left(\lambda H^* \left(\frac{x}{\lambda}\right)\right)$$

And by using (3.6), we get:

$$L(x) = \inf_{\{\lambda>0\}} ((\lambda H)^*(x))$$

But, from (3.7) with $t = h$

$$\sup_{\{L(x)=h\}} (x,v) = \inf_{\{\lambda>0\}} \left(h\lambda + \lambda L^* \left(\frac{v}{\lambda}\right)\right)$$

and the expression of L^* is:

$$L^* \left(\frac{v}{\lambda}\right) = \left(\inf_{\{\mu>0\}} \left((\mu H)^* \left(\frac{v}{\lambda}\right)\right)\right)^*$$

By using (3.5), we have:

$$L^* \left(\frac{v}{\lambda}\right) = \sup_{\{\mu>0\}} \left((\mu H)^* \left(\frac{v}{\lambda}\right)\right)^* = \sup_{\{\mu>0\}} \left((\mu H)^{**} \left(\frac{v}{\lambda}\right)\right)$$

But $H^{**} = H$, thus:

$$L^* \left(\frac{v}{\lambda}\right) = \sup_{\{\mu>0\}} \left(\mu H \left(\frac{v}{\lambda}\right)\right)$$

Hence:

$$\sup_{\{L(x)=h\}} (x,v) = \inf_{\{\lambda>0\}} \left(h\lambda + \lambda \left(\sup_{\{\mu>0\}} \mu H \left(\frac{v}{\lambda}\right)\right)\right) \tag{3.23}$$

Let us set:

$$G_h(v) = \sup_{\{L(x)=h\}} (x,v) \tag{3.24}$$

From (3.23), we have:

$$G_h\left(D\Phi(x_0)\right) \leq h + o(h) \tag{3.25}$$

We remark that if $H\left(\frac{v}{\lambda}\right) > 0$, then $\sup_{\{\mu>0\}} \mu H\left(\frac{v}{\lambda}\right) = +\infty$
So (3.23) can be written as:

$$G_h(v) = \inf_{\{\lambda>0/H\left(\frac{v}{\lambda}\right)\leq 0\}} \left(h\lambda + \lambda\left(\sup_{\{\mu>0\}} \mu H\left(\frac{v}{\lambda}\right)\right)\right) \tag{3.26}$$

But, if $H\left(\frac{v}{\lambda}\right) \leq 0$, we have $\sup_{\{\mu>0\}} \mu H\left(\frac{v}{\lambda}\right) = 0$ (it suffices to take $\mu > 0$ arbitrarily close to 0). So if $\lambda > 0$, we have $\lambda h + \lambda \sup_{\{\mu>0\}} \mu H\left(\frac{v}{\lambda}\right) = \lambda h$
Thus (3.26) implies that:

$$G_h(v) = \inf_{\{\lambda>0/H\left(\frac{v}{\lambda}\right)\leq 0\}} (\lambda h) \tag{3.27}$$

And as $h > 0$, we therefore have:

$$G_h(v) = h \inf_{\{\lambda>0/H\left(\frac{v}{\lambda}\right)\leq 0\}} (\lambda) \tag{3.28}$$

And putting it back into (3.25), we then get (as $h > 0$):

$$\inf\{\lambda \in \mathbb{R}_+^* / H\left(\frac{D\Phi(x_0)}{\lambda}\right) \leq 0\} \leq 1 + o(1) \tag{3.29}$$

Since H is assumed to be convex and as $H(0) < 0$, we deduce that:

$$H\left(D\Phi(x_0)\right) \leq 0 \tag{3.30}$$

which means that u is a viscosity subsolution in the case $\alpha(x_0) > 0$

(ii) Second case: If $x_0 \in \Omega_-$, we have the same result (the computations are analogous).

(iii) Third case: If $x_0 \in \Gamma$. We then use the definition of viscosity subsolution for a discontinuous Hamiltonian. We want to check that:

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) + F_*\left(x_0, D\Phi(x_0, t_0)\right) \leq 0$$

where

$$F\left(x, Du\right) = sign(u_0(x))H(Du)$$

But in this case, we have $\forall t \geq 0$:

$$u(x_0, t) = 0$$

and thus

$$\frac{\partial u}{\partial t}(x_0, t) = 0$$

It therefore suffices to check that:

$$F_*\left(x_0, D\Phi(x_0, t_0)\right) \leq 0 \tag{3.31}$$

But

$$F_*\left(x_0, D\Phi(x_0, t_0)\right) = \min\left(0, H(D\Phi(x_0, t_0)), -H(D\Phi(x_0, t_0))\right)$$

Hence (3.31) holds.

■

Remark: In fact, while proving Proposition 3.4, we showed in addition the following result:

Corollary 3.5. *Let us assume (H 3.3)-(H 3.4) hold. Then* $u : x \mapsto d_L(x, \Gamma)$ *is a viscosity solution in* $\Omega_+ = \{x/u_0(x) > 0\}$ *of the stationnary equation* $H(Du) = 0$.

3.2.2 Uniqueness

We deal here with the uniqueness of a solution of (3.1).

Proposition 3.6. *Let us assume hypotheses (H 3.1)-(H 3.4) hold. Then (3.1) has at most one viscosity solution which is uniformly continuous on* $\mathbb{R}^2 \times [0, T]$, $\forall$ $T>0$ *and which vanishes on* Γ.

Proof: It is the same as the one for the uniqueness in Theorem 2.11. It relies on a theorem by Ishii (see [18]).

■

3.3 Existence of t_x and continuity of $u(x,t)$

In order to completely show that $u(x,t)$ given by (3.2) is a viscosity solution of (3.1), it remains to prove that $u(x,t)$ is continuous. It is the most difficult point (due to the discontinuity of the Hamiltonian of (3.1)).

3.3.1 Further hypotheses

As we mentioned earlier, we need to make further assumptions on u_0 to go on. In the sequel, we will always make one of the two following hypotheses.

(H 3.5)

$$\begin{cases} |u_0(x)| \leq d_L(x, \Gamma) \ \forall x \in \mathbb{R}^2 \\ u_0 \text{ is bounded on } \mathbb{R}^2. \end{cases}$$

(H 3.6)

$$|u_0(x)| \geq d_L(x, \Gamma) \ \forall x \in \mathbb{R}^2$$

We refer the reader to subsection 3.3.4 to see what can happen when (H 3.5) or (H 3.6) is not verified.

Remark: Note that these two hypotheses cannot hold simultaneously. And we will have to distinguish these two cases in the following proofs.

We define $v : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ by:

$$v(x,t) = \begin{cases} u_0(x) & \text{if } x \in \Gamma \\ \inf_{z \in \mathbb{R}^2} [u_0(x - tz) + tH^*(z)] & \text{if } x \in \Omega_+ \\ \sup_{z \in \mathbb{R}^2} [u_0(x - tz) - tH^*(z)] & \text{if } x \in \Omega_- \end{cases} \tag{3.32}$$

and $f : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ by:

$$f(x,t) = v(x,t) - \epsilon_x d_L(x, \Gamma) \tag{3.33}$$

where $\epsilon_x = \begin{cases} 1 \text{ if } x \in \Omega_+ \\ -1 \text{ if } x \in \Omega_- \end{cases}$

$v(x,t)$ coincides with $u(x,t)$ when $t \leq t_x$. We study t_x through the zeros of f.

For the moment the time t_x defined by (3.3) is such that $v(x,t_x) = \epsilon_x d_L(x,\Gamma)$, i.e. $f(x,t_x) = 0$. We can remark that hypothesis (H 3.5) (resp. (H 3.6)) implies $f(x,t) \leq 0$ for all $t \in [0,t_x]$ (resp. $f(x,t) \geq 0$) since $f(x,0) = u_0(x) - d_L(x,\Gamma) \leq 0$ (resp. $f(x,0) \geq 0$). We will have to assume more about the nature of this zero. Depending on the fact that (H 3.5) or (H 3.6) hold, we will use one of the two following hypotheses:

(H 3.7) For all $x \in \mathbb{R}^2 \backslash \Gamma$, there exists $\beta_x > 0$ such that:

$$\begin{cases} f(x,t) \leq 0 & \forall t \in [0,t_x] \\ f(x,t) > 0 & \forall t \in]t_x, t_x + \beta_x] \end{cases}$$

(H 3.8) For all $x \in \mathbb{R}^2 \backslash \Gamma$, there exists $\beta_x > 0$ such that:

$$\begin{cases} f(x,t) \geq 0 & \forall t \in [0,t_x] \\ f(x,t) < 0 & \forall t \in]t_x, t_x + \beta_x] \end{cases}$$

We will also use the following assumptions:

(H 3.9) u_0 is C^1 on $\mathbb{R}^2$.

(H 3.10)

$$\inf_{x \in \mathbb{R}^2} (H^*(x) - |x||\nabla u_0|_\infty) > 0$$

where $|\nabla u_0|_\infty = \sup_{x\in\mathbb{R}^2} |\nabla u_0(x)|$.
Remark that when H is radial, (H 3.10) is equivalent to:

$$H(|\nabla u_0|_\infty) < \infty$$

(H 3.11) H^* is radial and $H^{*''}(x) > 0$, $\forall x \in \mathbb{R}$.

These two last hypotheses are not empty. For instance, let us consider the Hamiltonian $H(t) = t^2 - 1$. It verifies (H 3.11). And if u_0 is such that $|\nabla u_0|_\infty < 1$, H and u_0 verify (H 3.10).
Beside hypotheses (H 3.1)-(H 3.4), we will now assume (H 3.5) or (H 3.6).

3.3.2 Continuity of $u(x,t)$ under hypothesis (H 3.5)

We first verify that t_x given by (3.3) exists.

Lemma 3.7. *Let us assume hypotheses (H 3.1)-(H 3.5) hold. Then for all x in $\mathbb{R}^2$, t_x exists and is finite. Moreover,*

$$t_x \leq \frac{d_L(x,\Gamma) + \sup_{x\in\mathbb{R}^2} |u_0(x)|}{-H(0)} \tag{3.34}$$

Proof: As u_0 is assumed bounded, there exists $M > 0$ such that $\forall x \in \mathbb{R}^2$, $|u_0(x)| \leq M$. Moreover, since v (defined by (3.32)) is uniformly continuous on Ω_+ and Ω_- (see [6]), f (defined by (3.33)) is also uniformly continuous on Ω_+ and Ω_-.
Let us set $x \in \Omega_+$ (the case $x \in \Omega_-$ is similar). From (H 3.5), we have $u(x,0) = u_0(x) \leq d_L(x,\Gamma)$, i.e. $f(x,0) \leq 0$. On the other hand, we have:

$$\begin{aligned} v(x,t) &= \inf_z \left(u_0(x-tz) + tH^*(z)\right) \\ &\geq -M + t\inf_z H^*(z) \end{aligned} \tag{3.35}$$

Since $H(0) < 0$, we have for all $z \in \mathbb{R}^2$:

$$H^*(z) \geq -H(0) > 0 \tag{3.36}$$

Thus

$$\lim_{t\to\infty} v(x,t) = +\infty$$

And then:

$$\lim_{t\to\infty} f(x,t) = +\infty$$

And we conclude thanks to the mean value theorem that t_x exists and is finite (remember that by definition $f(x,t_x) = 0$). (3.34) is an immediate consequence of (3.35) and (3.36).

■

The following lemma gives the behaviour of $u(x,t)$ close to Γ.

Lemma 3.8. *Let us assume that hypotheses (H 3.1)-(H 3.5) hold.*

- *If $x \in \Omega_+$, then for all $t \geq 0$ we have:*

$$0 \leq u(x,t) \leq d_L(x,\Gamma) \tag{3.37}$$

- *If $x \in \Omega_-$, then for all $t \geq 0$ we have:*

$$-d_L(x,\Gamma) \leq u(x,t) \leq 0 \tag{3.38}$$

In particular, we have for all $t \geq 0$:

$$\lim_{x\to\Gamma} u(x,t) = 0 \tag{3.39}$$

which means that $u(x,t)$ is continuous on $\Gamma \times \mathbb{R}_+$.

Proof: We only show the first point of the lemma. Let $x \in \Omega_+$, since $u(x,0) = u_0(x) \leq d_L(x,\Gamma)$, we have thanks to (3.3) and (H 3.5), $f(x,t) \leq 0$ if $t \leq t_x$, and thanks to (3.2), $u(x,t) = d_L(x,\Gamma)$ if $t \geq t_x$. Therefore, for all $t \geq 0$:

$$u(x,t) \leq d_L(x,\Gamma) \tag{3.40}$$

On the other hand (we recall that thanks to (3.7) we have $\forall t>0$ and $\forall y\in\mathbb{R}$ $tH^*\left(\frac{y}{t}\right)\geq L(y)$), thus for all $t\leq t_x$

$$\begin{aligned}
u(x,t) &= \inf_y \left(u_0(x-y)+tH^*\left(\frac{y}{t}\right)\right)\\
&\geq \inf\left(\inf_{y/u_0(x-y)\geq 0}\left(\underbrace{u_0(x-y)}_{\geq 0}+\underbrace{tH^*\left(\frac{y}{t}\right)}_{\geq L(y)}\right), \inf_{y/u_0(x-y)<0}\left(\underbrace{u_0(x-y)}_{\geq -d_L(x-y,\Gamma)}+\underbrace{tH^*\left(\frac{y}{t}\right)}_{\geq L(y)}\right)\right)\\
&\geq \inf\left(\inf_{y/u_0(x-y)\geq 0} L(y), \inf_{y/u_0(x-y)<0}(-d_L(x-y,\Gamma)+L(y))\right)\\
&\geq \inf\left(0, \inf_{y/u_0(x-y)<0}(-d_L(x-y,\Gamma)+L(y))\right)
\end{aligned}$$

But if $u_0(x-y)<0$, then there exists z on the segment $[x,x-y]$ such that $u_0(z)=0$. z is given by: $z=(1-\lambda)x+\lambda(x-y)=x-\lambda y$ for some $\lambda\in[0,1]$.
Since $d_L(x-y,\Gamma)$ stands for the L distance from $x-y$ to Γ, and since $z\in\Gamma$, we have:

$$d_L(x-y,\Gamma)\leq L(x-y-z)=(1-\lambda)L(y) \tag{3.41}$$

Thus, we get:

$$u(x,t)\geq 0 \tag{3.42}$$

Hence the lemma.

■

The main difficulty to show that $u(x,t)$ defined by (3.2) is continuous is to prove that $x\mapsto t_x$ is continuous on $\mathbb{R}^2\backslash\Gamma$. We will need hypothesis (H 3.7) to go further.

Lemma 3.9. *Let us assume that hypotheses (H 3.1)-(H 3.5) hold. If hypothesis (H 3.7) holds too, then the function $x\mapsto t_x$ is continuous on $\mathbb{R}^2\backslash\Gamma$.*

Proof: Let $x\in\mathbb{R}^2\backslash\Gamma$. Let (x_n) be a sequence of points in $\mathbb{R}^2$ such that $x_n\to x$ when $n\to\infty$. We want to show that $t_{x_n}\to t_x$ when $n\to\infty$.
As (x_n) is bounded in $\mathbb{R}^2$ (since it converges to x), so is t_{x_n} (thanks to (3.34)). Therefore, there exists $\xi_x\in\mathbb{R}^2$ such that (up to a subsequence): $t_{x_n}\to\xi_x$.
From hypothesis (H 3.7), there exists $\beta_x>0$ such that:

$$f(x,t)\leq 0 \qquad \forall t\in[0,t_x] \tag{3.43}$$

$$f(x,t)>0 \quad \forall t\in]t_x,t_x+\beta_x] \tag{3.44}$$

We then remark that (3.43) can be rewritten as

$$f(x,t)\,(t_x-t)_+\leq 0 \text{ for all } t\in\mathbb{R}_+ \tag{3.45}$$

where $(y)_+=\max(y,0)$.
We also have ($\forall t\in\mathbb{R}_+$):

$$f(x_n,t)\,(t_{x_n}-t)_+\leq 0 \tag{3.46}$$

So passing to the limit, we get (for all t in $\mathbb{R}_+$):

$$f(x,t)\,(\xi_x-t)_+\leq 0 \tag{3.47}$$

By contradiction, let us suppose that $\xi_x > t_x$. Reducing, if necessary, β_x, we may assume $\xi_x > t_x + \beta_x$. Then, for any t in $]t_x, t_x + \beta_x[$, we obtain from (3.47) $f(x,t) \leq 0$. This contradicts (3.44). We therefore deduce that:

$$\xi_x \leq t_x \tag{3.48}$$

But by definition of t_{x_n}, we have $f(x_n, t_{x_n}) = 0$. So passing to the limit, we get

$$f(x, \xi_x) = 0 \tag{3.49}$$

Hence, as t_x is the first zero of $f(x,t)$ (by definition), we obtain:

$$t_x \leq \xi_x \tag{3.50}$$

Hence

$$t_x = \xi_x \tag{3.51}$$

We therefore deduce that t_{x_n} has only one value of adherence t_x, and all the sequence (t_{x_n}) converges to t_x.
We conclude that $x \mapsto t_x$ is continuous on $\mathbb{R}^2 \backslash \Gamma$.

■

We are now in position to prove the continuity of $u(x,t)$ defined by (3.2).

Proposition 3.10. *Let us assume that (H 3.1)-(H 3.5) hold. If (H 3.7) holds, then the function $u(x,t)$ defined by (3.2) is continuous on $\mathbb{R}^2 \times \mathbb{R}_+$.*

Proof: Let us first recall that from [6], the function v defined by (3.32) is uniformly continuous on Ω_+ and Ω_- (since (H 3.3) holds). Let $(x, \hat{x}, t, \hat{t})$ be in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}_+$. We can assume that $t_x \leq t_{\hat{x}}$.

First case: We begin by showing that u is continuous on $\Omega_+ \times \mathbb{R}_+$ (the case $\Omega_- \times \mathbb{R}_+$ is similar). Assume that x and $\hat{x}$ are both in Ω_+.

Step 1: If $t \leq t_x$ and $\hat{t} \leq t_{\hat{x}}$, then:

$$u(x,t) - u(\hat{x}, \hat{t}) = v(x,t) - v(\hat{x}, \hat{t}) \tag{3.52}$$

And since v is uniformly continuous, u is continuous.

Step 2: If $t \geq t_x$ and $\hat{t} \geq t_{\hat{x}}$, then:

$$u(x,t) - u(\hat{x}, \hat{t}) = d_L(x, \Gamma) - d_L(\hat{x}, \Gamma) \tag{3.53}$$

And since d_L is continuous, so is u.

Step 3: If $t_x < t < t_{\hat{x}}$, then we can write (as $u(x, t_x) = v(x, t_x)$ and $v(\hat{x}, t_{\hat{x}}) = u(\hat{x}, t_{\hat{x}})$):

$$u(x,t) - u(\hat{x}, \hat{t}) = u(x,t) - u(x, t_x) + v(x, t_x) - v(\hat{x}, t_{\hat{x}}) + u(\hat{x}, t_{\hat{x}}) - u(\hat{x}, \hat{t}) \tag{3.54}$$

From Lemma 3.9, we know that $y \mapsto t_y$ is continuous on Ω_+, so it is continous in x: we have $\hat{t_x} \to t_x$ when $\hat{x} \to x$. As we consider the case when $t_x < t < t_{\hat{x}}$, we have $t \to t_x$ in (3.54) when $\hat{x} \to x$. Let $\epsilon > 0$. For $|x - \hat{x}|$ small enough, we have $u(x,t) - u(x, t_x) \leq \frac{\epsilon}{3}$ (since $u(x,.)$ is continuous) and $v(x, t_x) - v(\hat{x}, t_{\hat{x}}) \leq \frac{\epsilon}{3}$, and for $|t - \hat{t}|$ small enough, we have $u(\hat{x}, t_{\hat{x}}) - u(\hat{x}, \hat{t}) \leq \frac{\epsilon}{3}$. And consequently $u(x,t)$ is continuous on $\Omega_+ \times \mathbb{R}_+$.

Second case: It remains to show that u is continuous on $\mathcal{V} \times \mathbb{R}_+$ (where $\mathcal{V}$ is an open neighbourhood of Γ in $\mathbb{R}^2$). But this follows from Lemma 3.8 where we have studied the behaviour of $u(x,t)$ for x close to Γ.

■

We then show that $u(x,t)$ is in fact uniformly continuous.

Proposition 3.11. *Let us assume hypotheses (H 3.1)-(H 3.5) and (H 3.7) hold. Then the function $u(x,t)$ defined by (3.2) is uniformly continuous on $\mathbb{R}^2 \times \mathbb{R}_+$.*

Proof: From Proposition 3.10, we know that u is continuous. Furthermore

$$|u(x,t)| \leq \max\left(|v(x,t)|, d_L(x,\Gamma)\right)$$

Let us recall that a uniformly continuous function has at most a linear growth at infinity (see [12, 13]). Since v and d_L are uniformly continuous, we deduce u has at most a linear growth at infinity, i.e. there exists $C > 0$ such that:

$$u(x,t) \leq C(|x| + |t| + 1)$$

As u is a continuous viscosity solution, we deduce from [12, 13] that u is in fact uniformly continuous.

■

The main result of the paper is a consequence of Propositions 3.4, 3.6 and 3.11:

Theorem 3.12. *Let us assume hypotheses (H 3.1)-(H 3.5) and (H 3.7) hold. Then the function $u(x,t)$ defined by (3.2) is the unique viscosity solution of (3.1) which is uniformly continuous on $\mathbb{R}^2 \times [0,T]$, $\forall\, T > 0$ and which vanishes on Γ.*

A natural question is how we can check hypothesis (H 3.7) which is rather technical. We give below a way to verify it.

Proposition 3.13. *Let us assume that hypotheses (H 3.1)-(H 3.5) hold. If u_0 and H satisfy (H 3.9) and (H 3.10), then v defined by (3.32) is such that $\frac{\partial v}{\partial t}(x,t) > 0$ for all $x \in \mathbb{R}^2 \backslash \Gamma$ and for all $t \geq 0$. Hence (H 3.7) is verified.*

Proof: Let $x \in \mathbb{R}^2 \backslash \Gamma$. We will assume $x \in \Omega_+$ (the case $x \in \Omega_-$ is similar). According to (3.32), we therefore have:

$$v(x,t) = \inf_y \left(u_0(x - ty) + tH^*(y)\right) \tag{3.55}$$

Let us note $S(x,t) = \{y(x,t)/v(x,t) = u_0(x - ty(x,t)) + tH^*(y(x,t))\}$. $S(x,t)$ is a compact set of $\mathbb{R}^2$. We thus obtain (cf [23]):

$$\frac{\partial v}{\partial t}(x,t) = \inf_{y \in S(x,t)} \left(y \cdot \nabla u_0(x - ty) + H^*(y)\right) \tag{3.56}$$

Hence

$$\frac{\partial v}{\partial t}(x,t) \geq \inf_{y \in \mathbb{R}^2} \left(y \cdot \nabla u_0(x - ty) + H^*(y)\right)$$

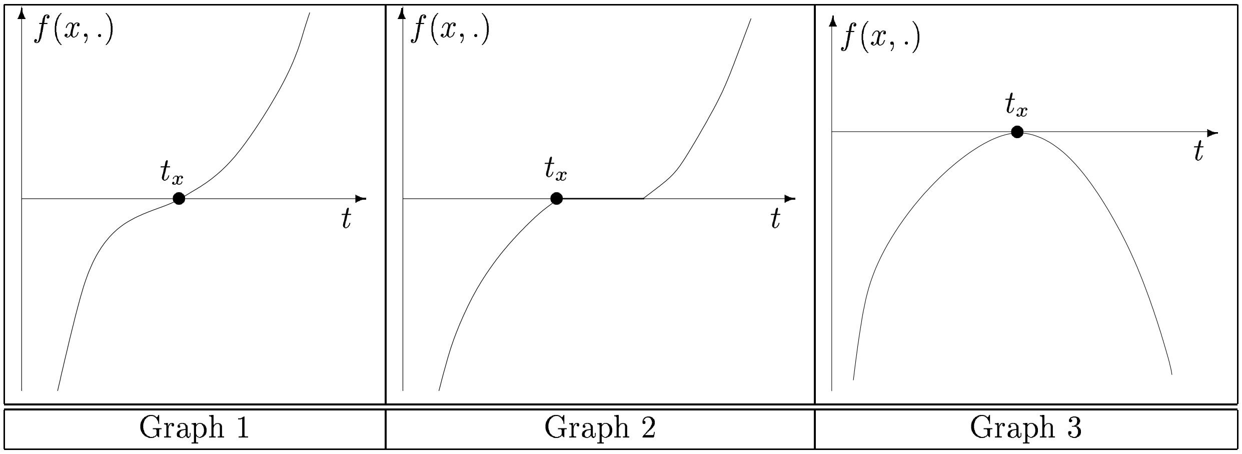


Figure 5: Nature of the zero of $t \mapsto f(x,t)$

So, thanks to (H 3.10)

$$\frac{\partial v}{\partial t}(x,t) > 0 \tag{3.57}$$

■

As an immediate consequence of Theorem 3.12 and Proposition 3.13, we have:

Corollary 3.14. *Assume hypotheses (H 3.1)-(H 3.5), (H 3.9) and (H 3.10) hold. Then the function $u(x,t)$ defined by (3.2) is the unique viscosity solution of (3.1) which is uniformly continuous on $\mathbb{R}^2 \times [0,T]$, $\forall\, T > 0$, and which vanishes on Γ.*

By definition, t_x is a zero of the function $t \mapsto f(x,t)$. In the proof of Lemma 3.9, the graph of $f(x,.)$ in t_x cannot look like Graph 2 or 3 of figure 5. Hypothesis (H 3.10) ensures that in t_x the graph of $f(x,.)$ looks like Graph 1 of figure 5. The following proposition gives a sufficient condition on H under which the graph of $f(x,.)$ cannot look like Graph 2 of figure 5 (but it can nevertheless look like Graph 3).

Proposition 3.15. *Let us assume that hypotheses (H 3.1)-(H 3.5) and (H 3.9) hold. If H satisfies (H 3.11) and if $x \in \mathbb{R}^2 \backslash \Gamma$, then the zeros of $t \mapsto f(x,t)$ are isolated (f is defined by (3.33)).*

Proof: Let $x \in \mathbb{R}^2 \backslash \Gamma$. We will assume $x \in \Omega_+$ (the case $x \in \Omega_-$ is similar). We recall the definition of f (given by (3.33) when $x \in \Omega_+$:

$$f(x,t) = v(x,t) - d(x,\Gamma)$$

So

$$\frac{\partial f}{\partial t}(x,t) = \frac{\partial v}{\partial t}(x,t) \tag{3.58}$$

By contradiction, let us assume that there exists $\alpha_x > 0$ such that $\frac{\partial f}{\partial t}(x,t) = 0$ on $[t_x, t_x + \alpha_x]$, i.e. on $[t_x, t_x + \alpha_x]$:

$$\frac{\partial v}{\partial t}(x,t) \equiv 0 \tag{3.59}$$

Since H^* is radial, we have

$$v(x,t) = \inf_y \left(u_0(x+ty) + tH^*(|y|)\right) \tag{3.60}$$

Let us note $S(x,t) = \{y(x,t)/v(x,t) = u_0(x+ty(x,t)) + tH^*(|y(x,t)|)\}$. $S(x,t)$ is a compact set of $\mathbb{R}^2$ We thus obtain:

$$\frac{\partial v}{\partial t}(x,t) = \inf_{y \in S(x,t)} \left(y.\nabla u_0(x+ty) + H^*(|y|)\right) \tag{3.61}$$

But for all $y \in S(x,t)$, we have:

$$t\nabla u_0(x+ty) + t\frac{y}{|y|}H^{*'}(|y|) = 0 \tag{3.62}$$

Since x does not belong to Γ, we have (thanks to hypothesis (H 3.11)) $t_x > 0$. If $t \neq 0$, we deduce from (3.62) that:

$$y.\nabla u_0(x+ty) = -|y|H^{*'}(|y|) \tag{3.63}$$

Hence

$$\frac{\partial v}{\partial t}(x,t) = \inf_{y \in S(x,t)} \left(-|y|H^{*'}(|y|) + H^*(|y|)\right) \tag{3.64}$$

Let (t_1, t_2) be in $[t_x, t_x + \alpha_x]^2$ (with $t_1 \neq t_2$). We have:

$$\frac{\partial v}{\partial t}(x,t_1) = \frac{\partial v}{\partial t}(x,t_2) = 0$$

Let us set $y_1 \in S(x,t_1)$ and $y_2 \in S(x,t_2)$ (y_1 and $y_2 \neq 0$). We thus have:

$$-|y_1|H^{*'}(|y_1|) + H^*(|y_1|) = 0 \tag{3.65}$$

$$-|y_2|H^{*'}(|y_2|) + H^*(|y_2|) = 0 \tag{3.66}$$

Substracting (3.65) to (3.66), we get:

$$H^*(|y_1| - H^*(|y_2|) - |y_1|H^{*'}(|y_1|) + |y_2|H^{*'}(|y_2|) = 0 \tag{3.67}$$

Of course, we also have:

$$H^*(|y_2| - H^*(|y_1|) - |y_2|H^{*'}(|y_2|) + |y_1|H^{*'}(|y_1|) = 0 \tag{3.68}$$

Since H^* is strictly convex on $\mathbb{R}$ (from hypothesis (H 3.11)), and since H is of class C^1 on $\mathbb{R}$, we have (if $y_1 \neq y_2$):

$$H^*(|y_1|) - H^*(|y_2|) > (|y_1| - |y_2|))H^{*'}(|y_2|) \tag{3.69}$$

$$H^*(|y_2|) - H^*(|y_1|) > (|y_2| - |y_1|))H^{*'}(|y_1|) \tag{3.70}$$

From (3.67) and (3.69), we get:

$$(|y_1| - |y_2|))H^{*'}(|y_2|) - |y_1|H^{*'}(|y_1|) + |y_2|H^{*'}(|y_2|) < 0$$

i.e.

$$|y_1|\left(H^{*'}(|y_2|) - H^{*'}(|y_1|)\right) < 0$$

Hence:

$$H^{*'}(|y_2|) - H^{*'}(|y_1|) < 0 \tag{3.71}$$

But from (3.68) and (3.70), we get:

$$(|y_2| - |y_1|)H^{*'}(|y_1|) - |y_2|H^{*'}(|y_2|) + |y_1|H^{*'}(|y_1|) < 0$$

i.e.

$$|y_2|\left(H^{*'}(|y_1|) - H^{*'}(|y_2|)\right) < 0$$

Hence:

$$H^{*'}(|y_1|) - H^{*'}(|y_2|) < 0 \tag{3.72}$$

But (3.71) and (3.72) are contradictory. We thus conclude that we cannot have simultaneously $\frac{\partial f}{\partial t}(x,t) = 0$ on $[t_x, t_x + \alpha_x]$ and the existence of two distinct points t_1 and t_2 such that

$$S(x,t_1) \neq S(x,t_2) \tag{3.73}$$

As we have assumed (by contradiction) that $\frac{\partial f}{\partial t}(x,t) = 0)$ on $[t_x, t_x + \alpha_x]$, this means that $S(x,t)$ is the same set for all t $[t_x, t_x + \alpha_x]$. There exists thus $y_0 = y_0(x) \in \mathbb{R}^2$ such that $\forall t \in [t_x, t_x + \alpha_x]$:

$$-|y_0|H^{*'}(|y_0|) + H^*(|y_0|) = \inf_{y \in S(x,t)} \left(-|y|H^{*'}(|y|) + H^*(|y|)\right)$$

For such a y_0, we have

$$\begin{aligned} v(x,t) &= \inf_{|y|=|y_0|} (u_0(x+ty) + tH^*(|y|)) \\ &= \inf_{|y|=|y_0|} (u_0(x+ty) + tH^*(|y_0|)) \end{aligned}$$

As this holds $\forall t \in [t_x, t_x + \alpha_x]$, we must have

$$0 = \frac{\partial v}{\partial t}(x,t) = H^*\left(\frac{|y_0|}{t}\right) - \frac{|y_0|}{t}H^{*'}\left(\frac{|y_0|}{t}\right) \tag{3.74}$$

Taking the derivative with respect to t again, we get:

$$-\frac{|y_0|}{t^2}H^{*'}\left(\frac{|y_0|}{t}\right) + \frac{|y_0|}{t^2}H^{*'}\left(\frac{|y_0|}{t}\right) + \frac{|y_0|}{t^3}H^{*''}\left(\frac{|y_0|}{t}\right) = 0$$

i.e.

$$\frac{|y_0|}{t^3}H^{*''}\left(\frac{|y_0|}{t}\right) = 0 \tag{3.75}$$

First case: if $|y_0| \neq 0$, we then deduce that:

$$H^{*''}\left(\frac{|y_0|}{t}\right) = 0$$

and we get a contradiction with (H 3.11).

Second case: if $|y_0| = 0$, i.e. $y_0 = 0$, then:

$$v(x,t) = u_0(x) + tH^*(0)$$

and

$$0 = \frac{\partial v}{\partial t} = H^*(0)$$

But we get a contradiction with (H 3.4) (we have assumed $H(0) < 0$ and thus $H^*(0) > 0$).

■

We can slightly weaken (H 3.10) by considering a non-strict inequality, but doing so we will have to assume more on H.

(H 3.12)

$$\begin{cases} H(|\nabla u_0|_\infty) \leq 0 \\ H \text{ radial} \\ H^{*''}(x) > 0, \ \forall x \in \mathbb{R} \end{cases}$$

Remark: If H verifies (H 3.11) and if u_0 is the viscosity solution of $H(\nabla u) = 0$, then (H 3.12) holds.

As an immediate consequence of Theorem 3.12 and Propositions 3.13 and 3.15, we have:

Corollary 3.16. *Assume hypotheses (H 3.1)-(H 3.5) and (H 3.12) hold. Then the function $u(x,t)$ defined by (3.2) is the unique viscosity solution of (3.1) which is uniformly continuous on $\mathbb{R}^2 \times [0,T]$, $\forall\ T > 0$, and which vanishes on Γ.*

3.3.3 Existence of t_x and continuity of $u(x,t)$ under hypothesis (H 3.6)

We now consider the case when hypothesis (H 3.6) holds.

For $x \in \mathbb{R}^2$, let $\hat{y}_x$ be the projection of x on Γ with respect to the L-distance (in the case where the projection is not unique, we arbitrarily choose one of the projected of x as $\hat{y}_x$), and $y_x = x - \hat{y}_x$ (therefore $L(y_x) = d_L(x,\Gamma)$). Then let us introduce an auxiliary time $t_x^{'}$ defined as:

$$t_x^{'} = \inf\left\{t > 0 / \inf_{t>0}\left(tH^*\left(\frac{y_x}{t}\right)\right) = d_L(x,\Gamma)\right\} \tag{3.76}$$

The following inequalities hold:

Lemma 3.17. *Assume (H 3.1)-(H 3.4) hold. Then for all x in $\mathbb{R}^2$, we have*

$$t_x^{'} \leq \frac{2d_L(x,\Gamma)}{H^*(0)} \tag{3.77}$$

and

$$|u(x,t_x^{'})| \leq d_L(x,\Gamma) \tag{3.78}$$

Proof: We split the proof in two parts.

Part 1: We first prove (3.77). Let us recall that since H is convex, we have thanks to (3.7)

$$L(y) = \inf_{t>0} tH^*\left(\frac{y}{t}\right) \tag{3.79}$$

Now let $A > 0$. Since $\lim_{|p|\to+\infty} \dfrac{H^*(p)}{|p|} = +\infty$, there exists $T > 0$ such that $\forall t \leq T$ one has:

$$\frac{H^*\left(\frac{y_x}{t}\right)}{\frac{|y_x|}{t}} \geq A$$

i.e.

$$tH^*\left(\frac{y_x}{t}\right) \geq A|y_x|$$

which implies

$$\lim_{t\to 0^+} tH^*\left(\frac{y_x}{t}\right) = +\infty \tag{3.80}$$

and thus $t_x^{'} > 0$.
Moreover, when $t \to +\infty$:

$$tH^*\left(\frac{y_x}{t}\right) \sim tH^*(0) \tag{3.81}$$

Therefore

$$\lim_{t\to+\infty} tH^*\left(\frac{y_x}{t}\right) = +\infty$$

and $t_x^{'} < +\infty$. Let us give a more precise bound for $t_x^{'}$. Since $H^*(0) > 0$, there exists $\delta > 0$ such that $|y| \leq \delta$ implies $H^*(y) \geq \frac{H^*(0)}{2}$. Moreover, there exists $T > 0$ such that $t \geq T \Rightarrow \frac{y_x}{t} \leq \delta$. So, if $t \geq T$, then

$$tH^*\left(\frac{y_x}{t}\right) \geq t\frac{H^*(0)}{2}$$

Hence

$$t_x^{'} \leq \frac{2d_L(x,\Gamma)}{H^*(0)} \tag{3.82}$$

Part 2: We now prove (3.78). The result is obvious when x lies on Γ. We now consider the case when x lies outside Γ. Remember that

$$u(x,t) \leq \max(v(x,t), d_L(x,\Gamma)) \tag{3.83}$$

(where v is defined by (3.32)).
We want to compute $v(x,t_x^{'})$. Assume for instance that x lies in Ω_+ (the case when x lies in Ω_- is similar). We have:

$$\begin{aligned} v(x,t_x^{'}) &= \inf_{y\in\mathbb{R}^2}\left[u_0(x-y) + t_x^{'}H^*\left(\frac{y}{t_x^{'}}\right)\right] \\ &\leq u_0(x-y_x) + t_x^{'}H^*\left(\frac{y_x}{t_x^{'}}\right) \end{aligned}$$

But by definition

$$u_0(x-y_x) = 0$$

and

$$L(y_x) = t_x^{'} H^* \left(\frac{y_x}{t_x^{'}} \right) = d_L(x, \Gamma)$$

Thus

$$v(x, t_x^{'}) \leq d_L(x, \Gamma) \tag{3.84}$$

Hence we conclude from (3.83) that (3.78) holds.

■

Lemma 3.18. *Let us assume that hypotheses (H 3.1)-(H 3.4) and (H 3.6) hold. Then:*

$$t_x \leq \frac{2d_L(x, \Gamma)}{H^*(0)} \tag{3.85}$$

Proof: Let us assume $x \in \Omega_+$ (the case $x \in \Omega_-$ is similar). Then according to hypothesis (H 3.6) we have $u(x, 0) = u_0(x) \geq d_L(x, \Gamma)$. Lemma 3.17 tells us that $u(x, t_x^{'}) \leq d_L(x, \Gamma)$. So from the mean value theorem (remember that $u(x, t_x) = d_L(x, \Gamma)$, we deduce that necessarily $t_x \leq t_x^{'}$. And we conclude thanks to Lemma 3.17.

■

Remark: From Lemma 3.18, we get that for all x in $\mathbb{R}^2$, t_x exists and is finite.

The following lemma gives the behaviour of u close to Γ.

Lemma 3.19. *Let us assume hypotheses (H 3.1)-(H 3.4) and (H 3.6) hold. Then for all $t \geq 0$, we have*

$$\lim_{x \to \Gamma} u(x, t) = 0 \tag{3.86}$$

which means that $u(x, t)$ is continuous on $\Gamma \times \mathbb{R}_+$.

Proof: Let us set $\epsilon > 0$. We know that v defined by (3.32) is uniformly continuous on $\mathbb{R}^2 \backslash \Gamma$ (see [6]). Moreover, $v(x, 0) = u_0(x) \to 0$ when $x \to \Gamma$. Hence there exists $A > 0$ and $T > 0$ such that: $d_L(x, \Gamma) \leq A$ and $t \leq T$ imply $|v(x, t)| \leq \epsilon$. From Lemma 3.18, we deduce that there exists $\eta > 0$ such that $t_x \leq T$ when $d_L(x, \Gamma) \leq \eta$. Since $|u(x, t)| \leq \max(|v(x, t)|, d_L(x, \Gamma))$, we thus conclude that $\forall t \geq 0$, $|u(x, t)| \leq \epsilon$ when $d_L(x, \Gamma) \leq \min(\eta, \epsilon)$.

■

We then prove the continuity of $x \mapsto t_x$.

Lemma 3.20. *Let us assume that hypotheses (H 3.1)-(H 3.4) and (H 3.6) hold. If hypothesis (H 3.8) holds too, then the function $x \mapsto t_x$ is continuous on Ω_+ and Ω_-.*

Proof: The proof is the same as the one of Lemma 3.9.

■

We are now in position to prove the continuity of $u(x, t)$ defined by (3.2).

Proposition 3.21. *Let us assume hypotheses (H 3.1)-(H 3.4), (H 3.6) and (H 3.8) hold. Then the function $u(x, t)$ defined by (3.2) is continuous on $\mathbb{R}^2 \times \mathbb{R}_+$.*

Proof: The proof is the same as the one of Proposition 3.10, except the fact that we use Lemma 3.19 instead of Lemma 3.8.

■

As in the preceeding subsection, we deduce from Proposition 3.21 that:

Proposition 3.22.
Let us assume hypotheses (H 3.1)-(H 3.4), (H 3.6) and (H 3.8) hold. Then the function $u(x,t)$ defined by (3.2) is uniformly continuous on $\mathbb{R}^2 \times \mathbb{R}_+$.

The main result of this subsection is a consequence of Propositions 3.4, 3.6 and 3.22:

Theorem 3.23. *Let us assume hypotheses (H 3.1)-(H 3.4), (H 3.6) and (H 3.8) hold. Then the function $u(x,t)$ defined by (3.2) is the unique viscosity solution of (3.1) which is uniformly continuous on $\Omega \times [0,T]$, $\forall\, T > 0$ and which vanishes on Γ.*

Remark: Contrary to the case of the previous subsection (when we assumed that (H 3.5) held), there is no easy sufficient condition on H and u_0 such that (H 3.8) hold. We just have (the proof is the same as the one of Proposition 3.15):

Proposition 3.24. *Let us assume that hypotheses (H 3.1)-(H 3.4),(H 3.6) and (H 3.9) hold. If H satisfies (H 3.11) and if $x \in \mathbb{R}^2 \backslash \Gamma$, then the zeros of $t \mapsto f(x,t)$ are isolated (f is defined by (3.33)).*

3.3.4 Discussion about hypotheses

We explain here why we introduced hypotheses (H 3.5) or (H 3.6). Let us consider an example when they are not verified. Assume that u_0 is given by:

$$u_0(x) = \begin{cases} 2(|x|-1) & \text{if } x \leq 2 \\ -2|x|+6 & \text{if } 2 \leq x \leq \frac{5}{2} \\ 2|x|-4 & \text{if } x \geq \frac{5}{2} \end{cases} \tag{3.87}$$

and, let us choose:

$$H(t) = t^2 - 1 \tag{3.88}$$

It is easy to show that:

$$H^*(t) = \frac{t^2}{4} + 1 \tag{3.89}$$

Let x be in $(2, \frac{5}{2})$. It can be checked that if $x + 4t \leq \frac{5}{2}$ then (v is defined by (3.32)):

$$v(x,t) = -3t - 2x + 6 \tag{3.90}$$

So $v(x,t) = 0$ if and only if $t = \frac{7}{3} - x$.
If we assume that x also satisfies $x \leq \frac{7}{3}$, then (remember (3.3)):

$$t_x = \frac{7}{3} - x$$

When both conditions $t = \frac{7}{3} - x$ and $x + 4t \leq \frac{5}{2}$ hold, we get: $x \geq \frac{41}{18}$. And we have that $\frac{41}{18} < \frac{7}{3} = \frac{42}{18}$.
Thus we have when $\frac{41}{18} \leq x \leq \frac{7}{3}$:

$$t_x = \frac{7}{3} - x \tag{3.91}$$

Let us now choose x in $\left[\frac{7}{3}, \frac{43}{18}\right]$. If $t \leq \frac{1}{36}$, then $x + 4t \leq \frac{5}{2}$. Set $A = \frac{1}{36}$. If $t \leq A$, then

$$u(x,t) = -3t - 2x + 6 \tag{3.92}$$

There exists $\epsilon > 0$ such that: if $\frac{7}{3} - \epsilon \leq x \leq \frac{7}{3}$, then $d(x,\Gamma) \geq d(\frac{7}{3},\Gamma) - A$. Up to take ϵ smaller, we can assume that $\frac{41}{18} < \frac{7}{3} - \epsilon$ and that $t_x \leq A$ if $\frac{7}{3} - \epsilon \leq x \leq \frac{7}{3}$. Thus, if $\frac{7}{3} - \epsilon \leq x \leq \frac{7}{3}$, then

$$u(x,A) = d(x,\Gamma) \geq d\left(\frac{7}{3},\Gamma\right) - A \tag{3.93}$$

And if $\frac{7}{3} \leq x \leq \frac{43}{18}$

$$u(x,A) = -3A - 2x + 6 \leq -3A + d\left(\frac{7}{3},\Gamma\right) \tag{3.94}$$

We therefore check that u is not continuous in $(\frac{7}{3}, A) = (\frac{7}{3}, \frac{1}{36})$.

Comments: This example shows us that with the definition we choose for t_x, we cannot expect to have the continuity of u if we do not make further assumptions on u_0 and H. That is why we introduced hypotheses (H 3.5) and (H 3.6).

A Hypotheses used in section 3

For the reader's convenience, we recall here the hypotheses we use in section 3.

(H 3.1) u_0 is uniformly continuous on $\mathbb{R}^2$.

(H 3.2) H is a convex function on $\mathbb{R}^2$.

(H 3.3) $\lim_{|p|\to+\infty} \dfrac{H(p)}{|p|} = +\infty$.

(H 3.4) $H(0) < 0$ and $\{p/H(p) = 0\}$ is non-void and symmetric with respect to 0 (i.e. $\{H(p) = 0\} \Rightarrow \{H(-p) = 0\}$).

(H 3.5)

$$\begin{cases} |u_0(x)| \leq d_L(x,\Gamma) \ \forall x \in \mathbb{R}^2 \\ u_0 \text{ is bounded on } \mathbb{R}^2. \end{cases}$$

(H 3.6)

$$|u_0(x)| \geq d_L(x,\Gamma) \ \forall x \in \mathbb{R}^2$$

(H 3.7) For all $x \in \mathbb{R}^2 \backslash \Gamma$, there exists $\beta_x > 0$ such that:

$$\begin{cases} f(x,t) \leq 0 & \forall t \in [0, t_x] \\ f(x,t) > 0 & \forall t \in]t_x, t_x + \beta_x] \end{cases}$$

(H 3.8) For all $x \in \mathbb{R}^2 \backslash \Gamma$, there exists $\beta_x > 0$ such that:

$$\begin{cases} f(x,t) \geq 0 & \forall t \in [0, t_x] \\ f(x,t) < 0 & \forall t \in]t_x, t_x + \beta_x] \end{cases}$$

(H 3.9) u_0 is C^1 on $\mathbb{R}^2$.

(H 3.10)

$$\inf_{x \in \mathbb{R}^2} (H^*(x) - |x||\nabla u_0|_\infty) > 0$$

(H 3.11) H^* is radial and $H^{*''}(x) > 0$, $\forall x \in \mathbb{R}$.

(H 3.12)

$$\begin{cases} H(|\nabla u_0|_\infty) \leq 0 \\ H \text{ radial} \\ H^{*''}(x) > 0, \ \forall x \in \mathbb{R} \end{cases}$$

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