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MATHEMATICAL STATEMENT TO ONE DIMENSIONAL PHASE UNWRAPPING: A VARIATIONAL APPROACH

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RÉSUMÉ :

Beaucoup d'algorithmes de déroulement de phase ont été développés et formulés dans le domaine discret durant ces dix dernières années. Nous proposons ici, une formulation variationnelle pour résoudre le problème. Cette étude dans le domaine continu va nous permettre d'imposer quelques contraintes sur la régularité de la solution et de les implémenter efficacement. Cette méthode est présentée dans le cas unidimensionnel, et servira de base pour nos développement futurs pour le cas réel en 2D.

MOTS CLÉS :

modèle variationnel, minimisation, espace de Sobolev H1, BV (espace des fonctions à variations bornées), minimisation semi-quadratique, interférométrie RSO(Radar à Ouverture Synthétique), déroulement de phase, développement de phase.

ABSTRACT:

Over the past ten years, many phase unwrapping algorithms have been developed and formulated in a discrete setting. Here we propose a variational formulation to solve the problem. This continuous framework will allow us to impose some constraints on the smoothness of the solution and to implement them efficiently. This method is presented in the one dimensional case, and will serve as a basis for future developments in the real 2D case.

KEY WORDS :

variational model, minimization, Sobolev space H1, BV (space of functions of bounded variations), half-quadratic minimization, SAR interferometry, phase unwrapping, phase retrieval.

Mathematical statement to one dimensional Phase Unwrapping: a variational approach

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Abstract

Over the past ten years, many phase unwrapping algorithms have been developed and formulated in a discrete setting. Here we propose a variational formulation to solve the problem. This continuous framework will allow us to impose some constraints on the smoothness of the solution and to implement them efficiently. This method is presented in the one dimensional case, and will serve as a basis for future developments in the real 2D case.

Key words: variational model, minimization, Sobolev space H^1, BV (space of functions of bounded variations), half-quadratic minimization, SAR interferometry, phase unwrapping, phase retrieval.

Contents

1 Introduction

Interferometric radar techniques have been widely used to produce high-resolution ground digital elevation models (DEM's). In space-borne SAR (Synthetic Aperture Radar) interferometry, two images of the same scene are acquired using two different geometries. The phase difference between the registered images (the so-called *interferogram*) is related to a desired physical quantity of interest such as the surface topography. The phase difference can be registered only modulo 2π and interferometric techniques consist mainly of recovering the absolute phase (the unwrapped phase) from the registered one (the wrapped phase).
Over the past ten years, many phase unwrapping algorithms have been developed. Commonly they first differentiate the phase field and subsequently reintegrate it, adding the missing integral cycles to obtain a more continuous result. Three basic classes are representative of these algorithms:

• Residue-cut "tree" algorithms: Branch cut methods (Goldstein et al.,1988 [9]) unwrap by integrating the estimated neighboring pixel differences of the unwrapped phase along paths that avoid the regions where these estimated differences are inconsistent.

• Least-square algorithms were adapted to SAR interferometry by Ghiglia and Romero [8]. They applied a mathematical formalism to determine the vector gradient of the phase field and then integrate it subject to regularity constraints.

• Fornaro and al. [7], and recently Lyuboshenko et Maître [11] proposed a phase unwrapping algorithm based on the Green function. This kind of method has been shown to be mathematically equivalent to least-squares solution, but differs in computational efficiency.

It is interesting to note that most of these algorithms are formulated in a discrete setting. Instead, we investigate a continuous formulation of the problem with a variational approach [10]. Traditionally developed in physics and mechanics, this framework has been intensively applied in image analysis since the 1990s. The reasons are that the models can be justified theoretically, and that suitable numerical schemes exist for computing the solution.

Section 2 of this paper is dedicated to the mathematical statement of the unwrapping problem. The two following sections deal with the proposed variational approach depending on the regularity of the wrapped phase. Section 3 deals with the case of regular unwrapped phase corresponding to a smooth ground surface. The unwrapped phase is obtained by including in the energy regularity constraints at phase jumps in the wrapped signal. To achieve these constraints, we propose to minimize a set of functionals. In section 4, we allow the absolute phase to have discontinuities apart from the phase jumps. In order to preserve these ground discontinuities, the minimization is then performed in BV (space of functions of bounded variation). Numerical schemes and results on a synthetic noisy wrapped signal are given.

2 1D phase unwrapping: problem statement

In this section we mathematically state the problem of phase unwrapping. The goal is to formalize the contraints on the resulting phase in a continuous setting.

Let $\varphi_m : I \subset \mathbb{R} \longrightarrow [-\pi, \pi[$ be the given phase difference. The unwrapping problem consists in finding a function $\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$W(\varphi) = \varphi_m \tag{1}$$

where W is the wrapping operator defined by:

$$\begin{aligned} W : \mathbb{R} &\longrightarrow [-\pi,\ \pi[\\ u &\longmapsto W(u) \\ \forall u \in \mathbb{R},\ \exists k_u &\in \mathbb{Z} : W(u) = u - 2k_u\pi. \end{aligned}$$

k_u is an integer such that $u - 2k_u\pi \in [-\pi, \pi[$. Let φ_m be the observation obeying for a.e. $x \in I \subset \mathbb{R}$, $\varphi_m(x) \in [-\pi, \pi[$. The observation φ_m is a function defined modulo 2π on a bounded interval I of $\mathbb{R}$. Let $I =]a, b, [$. By definition φ and φ_m share the same regularity property except at points x where the value of k_u changes. We denote by

$$P_{\varphi_m} = \{d_0, ..., d_D\} \tag{2}$$

these points refered to as phase jumps. It follows that φ_m is a discontinuous function. In this work we assume that phase jumps d_k are known. For real application they will be estimated from the bi-dimensional interferogram.

In order to retrieve the unwrapped phase φ from the observation φ_m, it is necessary to add the following conditions, for every discontinuity $d_k \in P_{\varphi_m}$:

$$\varphi_{|]d_k, d_{k+1}[} \text{ and } \varphi_{m|]d_k, d_{k+1}[} \text{ share the same regularity} \tag{3}$$

$$\begin{aligned} &\varphi \text{ is continuous at } d_k \\ &\varphi'^{+}(d_k) = \varphi_m'^{+}(d_k) \text{ and } \varphi'^{-}(d_k) = \varphi_m'^{-}(d_k) \end{aligned} \tag{4}$$

Let us comment on these conditions:

- We assume in condition (4) that the limits $\lim_{x \to d_k^+} \varphi_m'(x)$ and $\lim_{x \to d_k^-} \varphi_m'(x)$ exist, and we denote by $\varphi_m'^{+}(d_k)$ and $\varphi_m'^{-}(d_k)$ these limits.
- In (3), the regularity of φ_m depends on the observed areas. Two cases may arise. The first one occurs when the only discontinuities of φ_m are due to phase jumps and so we suppose that φ_m is piecewise differentiable on $I \backslash \{d_0, ..., d_D\}$. The second one arises when the wrapped function φ_m admits a finite number of ground discontinuities $\{t_j\}_j$. We assume they are located between two phase jumps. The $\{t_j\}_j$ model ground discontinuities with a jump small enough not to generate points d_k or unrecoverable areas. Note that the phase discontinuities $\{d_k\}_k$ are assumed to be known, while the ground discontinuities $\{t_j\}_j$ are unknown. We suppose that φ_m is regular except at the points $\{d_0, ..., d_D\}$ and $\{t_j\}_j$.

Now the goal is to describe a functional to be minimized with respect to φ such that φ satisfies conditions (1), (3) and (4).

3 Variational approach for the phase unwrapping problem in $H^1(I)$

3.1 Description of the functional

In this section we assume that:

- $\varphi_m \in \mathcal{C}^p(\overline{I}\backslash P_{\varphi_m})$ for some $p > 1$ and has no ground discontinuities.

In this case the distributional derivative $D\varphi_m$ can be decomposed as the sum of a regular measure (absolutely continuous with respect to the Lebesgue measure) and a sum of Dirac masses (a singular part) (see [2], [6]).

$$D\varphi_m = \underbrace{\varphi_m' dx}_{\text{regular}} + \underbrace{D_s\varphi_m}_{\text{singular}} .$$

We suppose that $\varphi_m' \in L^2(I)$ and we consider the quadratic functional $E(\varphi)$ defined by:

$$E(\varphi) = E_1(\varphi) + \lambda E_2(\varphi) \tag{5}$$

where:

$$E_1(\varphi) = \int_I |\varphi' - \varphi_m'|^2 dx \tag{6}$$

$$E_2(\varphi) = \int_I (\varphi - \varphi_m)^2 \chi_{I_0} dx \tag{7}$$

and $I_0 =]a_0, b_0[\subset I$ is an open interval of reference that does not contain any phase jumps of φ_m. χ_{I_0} is the standard characteristic function of I_0.
The problem is to minimize $E(\varphi)$ on the Sobolev space (see [6] and [2] for the definition and properties of Sobolev spaces)

$$H^1(I) = \{\varphi \in L^2(I) | \;\; D\varphi \in L^2(I)\}$$

Let us comment on the energy terms:

- $E_1(\varphi)$ is the data term which contains the requirements (1) and (3). Indeed $W(\varphi) = \varphi_m$ implies that $\varphi' = \varphi_m'$ a.e. $I\backslash S_{\varphi_m}$.
- $E_2(\varphi)$ is a reference term. It will play an important role in the uniqueness of the phase φ we want to retrieve. We can easily prove the existence and uniqueness of a minimizer $\varphi \in H^1(I)$ for (5).
  Moreover in one dimension we know that $H^1(I) \subset \mathcal{C}(\overline{I})$, the space of continuous functions on $\overline{I}$ (see [3], [5]). More precisely in the class of $\varphi \in H^1(I)$ there exists a continuous representation of φ. Or equivalently, φ is almost everywhere equal to a continuous function.

The goal of what follows is to construct an algorithm that gives this continuous representation at points d_k. In fact we want more: we want that the solution satisfied (4). The idea is to add a term of the form:

$$E_3(\varphi) = \sum_{d_k \in P_{\varphi_m}} e_{3,k}(\varphi) \quad \text{with:}$$

$$e_{3,k}(\varphi) = \left(\varphi'^+(d_k) - \varphi_m'^+(d_k)\right)^2 + \left(\varphi'^-(d_k) - \varphi_m'^-(d_k)\right)^2 + (\varphi^+(d_k) - \varphi^-(d_k))^2.$$

However since we are looking for a function φ in $H^1(I)$, we cannot give a sense to the derivatives φ'^{+} and φ'^{-}. So we introduce a parameter $\alpha \in \mathbb{R}^+$ for approximating the backward and forward derivatives of φ at points d_k. We define:

$$
\begin{aligned}
E_{3,\alpha}(\varphi) &= \sum_{d_k \in P_{\varphi_m}} e_{3,k}^{\alpha}(\varphi) \quad \text{with:} && (8)\\
e_{3,k}^{\alpha}(\varphi) &= \left(\frac{\varphi(d_k+\alpha)-\varphi(d_k)}{\alpha} - \varphi'_m(d_k+\alpha)\right)^2 \\
&\quad + \left(\frac{\varphi(d_k)-\varphi(d_k-\alpha)}{\alpha} - \varphi'_m(d_k-\alpha)\right)^2 \\
&\quad + \Big(\varphi(d_k+\alpha)-\varphi(d_k-\alpha)-\alpha\left(\varphi'_m(d_k+\alpha)+\varphi'_m(d_k-\alpha)\right)\Big)^2.
\end{aligned}
$$

The term $\alpha\left(\varphi'_m(d_k+\alpha)+\varphi'_m(d_k-\alpha)\right)$ is introduced to enforce the continuity of φ. Hence the functional (5) is approximated by the functional E_α defined by:

$$E_\alpha(\varphi) = E_1(\varphi) + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi). \quad (9)$$

Therefore to find the unwrapped phase we minimize in $H^1(I)$ the functional E_α, without any constraint.

In the next section we study the existence and uniqueness of φ_α, of a minimizer in $H^1(I)$ for the functional E_α.

3.2 Existence and uniqueness of φ_α

In the previous part we have introduced a sequence of functionals $(E_\alpha)_{\alpha\in\mathbb{R}^+}$, that takes into account regularity constraints at phase jump points of the wrapped signal. The phase unwrapping is seen as a minimization problem:

$$\inf_{\varphi \in H^1(I)} E_\alpha(\varphi) \quad (10)$$

This section is dedicated to the proof of existence and uniqueness in $H^1(I)$ of a minimizer of problem (10).

Theorem 1 *Assume that:*

- $\varphi_m \in \mathcal{C}^p(I \backslash P_{\varphi_m})$ *(case of no ground discontinuities)*
- φ'_m *(the regular part of $D\varphi_m$) belongs to $L^2(I)$.*

Then the minimization problem (10) admits a unique solution φ_α in $H^1(I)$.

Proof: For α fixed, the functional $E_\alpha(\varphi)$ is strictly convex and lower semi-continuous for the weak topology of $H^1(I)$. So if a solution φ_α exists the uniqueness is deduced from the strict-convexity of the functional. The existence is classical. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a minimizing sequence of $E_\alpha(\varphi)$ i.e.

$$E_\alpha(\varphi_n) \underset{n\to\infty}{\longrightarrow} \inf_{\varphi\in H^1(I)} E_\alpha(\varphi)$$

Then we easily obtain

$$\int_I |\varphi_n' - \varphi_m'|^2 dx \le C_\alpha \text{ and } \int_{I_0} |\varphi_n - \varphi_m|^2 dx \le C_\alpha$$

Therefore

$$|\varphi_n'|_{L^2(I)} \le C_\alpha \text{ and } |\varphi_n|_{H^1(I_0)} \le C_\alpha$$

where C_α is a universal constant which may change one line to another. According to the compact Sobolev injection, of $H^1(I_0)$ in $C(\overline{I}_0)$, we deduce:

$$|\varphi_n|_{L^\infty(I_0)} \le C_\alpha.$$

Now we are looking for an estimation of $|\varphi_n|_{H^1(I)}$. So let x be in I, and let p be a fixed point in I_0, we have (if φ_n is a continuous representation) :

$$\varphi_n(x) = \int_p^x \varphi_n'(t)dt + \varphi_n(p)$$

Thus

$$|\varphi_n(x)| \le \int_p^x |\varphi_n'(t)|dt + |\varphi_n(p)| \le |\varphi_n'|_{L^1(I)} + |\varphi_n|_{L^\infty(I_0)}$$

and so φ_n is uniformly bounded in $L^\infty(I)$ and in $H^1(I)$ with respect to n. Therefore we can extract a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$, (always denoted by $(\varphi_n)_{n\in\mathbb{N}}$) which weakly converges in $H^1(I)$ to a function φ_α. The lower semi-continuity of E_α implies

$$E_\alpha(\varphi_\alpha) \le \lim\inf_{n\to\infty} E_\alpha(\varphi_n) \le E_\alpha(\varphi), \forall \varphi \in H^1(I).$$

and so φ_α is a minimizer of E_α.

♢

3.3 Optimality conditions for φ_α

Previously, we have shown that the minimization problem

$$E_\alpha(\varphi_\alpha) = \inf_{\varphi\in H^1(I)} E_\alpha(\varphi) \tag{11}$$

admits only one solution, denoted as φ_α. In this section we establish the optimality conditions satisfied by φ_α.

Theorem 2 *Let us assume the same hypotheses as in Theorem 1. Let φ_α be the unique solution of the minimization problem (11). Then necessarily φ_α verifies:*

$$\begin{cases} -(\varphi_\alpha' - \varphi_m')' + \lambda(\varphi_\alpha - \varphi_m)\chi_{I_0} = 0, \ \textit{in } L^2(I) & (12) \\ \varphi_\alpha'(a) = \varphi_m'(a), \ \textit{and } \varphi_\alpha'(b) = \varphi_m'(b) & (13) \\ \textit{and for all } d_k \in P_{\varphi_m} : & (14) \\ \dfrac{\varphi_\alpha(d_k+\alpha) - \varphi_\alpha(d_k)}{\alpha} = \varphi_m'(d_k+\alpha) & \\ \dfrac{\varphi_\alpha(d_k) - \varphi_\alpha(d_k-\alpha)}{\alpha} = \varphi_m'(d_k-\alpha) & \\ \varphi_\alpha(d_k+\alpha) - \varphi_\alpha(d_k-\alpha) = \alpha\left(\varphi_m'(d_k+\alpha) + \varphi_m'(d_k-\alpha)\right) & \end{cases}$$

Proof: The Euler-Lagrange equations satisfied by φ_α are obtained by examining

$$\lim_{\theta\to 0}\frac{E_\alpha(\varphi_\alpha+\theta\psi)-E_\alpha(\varphi_\alpha)}{\theta},\ \ \forall\psi\in H^1(I).$$

Since φ_α is a minimizer of E_α, we have

$$E_\alpha(\varphi_\alpha)\leq E_\alpha(\varphi_\alpha+\theta\psi),\ \ \forall\psi\in H^1(I)$$

from which we deduce

$$\int_I(\varphi'_\alpha-\varphi'_m)\psi' dx+\lambda\int_I \chi_{I_0}(\varphi_\alpha-\varphi_m)\psi dx+\mu\sum_{d_k\in P_{\varphi_m}}\frac{\psi(d_k+\alpha)-\psi(d_k)}{\alpha}a_{k,\alpha}$$
$$+\mu\sum_{d_k\in P_{\varphi_m}}\frac{\psi(d_k)-\psi(d_k-\alpha)}{\alpha}b_{k,\alpha}+\mu\sum_{d_k\in P_{\varphi_m}}(\psi(d_k+\alpha)-\psi(d_k-\alpha))\,c_{k,\alpha}=0. \quad (15)$$

where

$$\begin{cases} a_{k,\alpha}=\dfrac{\varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k)}{\alpha}-\varphi'_m(d_k+\alpha)\\ b_{k,\alpha}=\dfrac{\varphi_\alpha(d_k)-\varphi_\alpha(d_k-\alpha)}{\alpha}-\varphi'_m(d_k-\alpha)\\ c_{k,\alpha}=\varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k-\alpha)-\alpha\left(\varphi'_m(d_k+\alpha)+\varphi'_m(d_k-\alpha)\right)\end{cases} \quad (16)$$

- **Step 1:** Euler-Lagrange equation

Let us consider the closed vector space V defined by:

$$V=\{\psi\in\mathcal{C}^\infty(I)|\ \psi(d_k-\alpha)=\psi(d_k)=\psi(d_k+\alpha),\ \ \forall d_k\in P_{\varphi_m}\}.$$

Let $\psi\in V$, then (15) becomes after an integration by parts:

$$-\int_I(\varphi'_\alpha-\varphi'_m)'\psi dx+\int_{\partial I}(\varphi'_\alpha-\varphi'_m)\psi ds+\lambda\int_{I_0}(\varphi_\alpha-\varphi_m)\psi dx=0. \quad (17)$$

Therefore if $\psi\in V\cap\mathcal{C}_0^\infty(I)$, where $\mathcal{C}_0^\infty(I)$ is the set of functions $\mathcal{C}^\infty(I)$ with compact support in I, we get

$$\int_I\left(-(\varphi'_\alpha-\varphi'_m)'+\lambda(\varphi_\alpha-\varphi_m)\chi_{I_0}\right)\psi dx=0.$$

Hence

$$<-(\varphi'_\alpha-\varphi'_m)'+\lambda(\varphi_\alpha-\varphi_m)\chi_{I_0},\psi>_{\mathcal{D}',\mathcal{D}}=0,\ \ \forall\psi\in V\cap\mathcal{C}_0^\infty(I).$$

By density we have the same result for all $\psi\in V\cap H_0^1(I)$, where $H_0^1(I)$ is the subspace of $H^1(I)$ such that $\psi_{|\partial I}=0$.
In order to conclude that in the distributionnal sense we have:

$$-(\varphi'_\alpha-\varphi'_m)'+\lambda(\varphi_\alpha-\varphi_m)\chi_{I_0}=0$$

we use the following lemma.

Lemma 1 *Let consider the linear set V, defined by:*

$$V = \{g \in \mathcal{C}^\infty(I) |\; g(d_k - \alpha) = g(d_k) = g(d_k + \alpha),\;\; \forall d_k \in P_{\varphi_m}\}.$$

Then for each function $\psi \in \mathcal{C}_0^\infty(I)$, there exists $\psi_\varepsilon \in V \cap H_0^1(I)$ such that:

$$\lim_{\varepsilon \to 0} \int_I f\psi_\varepsilon dx = \int_I f\psi dx,\;\; \forall f \in L^1(I).$$

The proof of Lemma 1 will be given at the end of Theorem 2.

Thanks to the lemma 1 we deduce that:

$$\int_I \Big(-(\varphi'_\alpha - \varphi'_m)' + \lambda(\varphi_\alpha - \varphi_m)\chi_{I_0} \Big) \psi = 0,\;\; \forall \psi \in H_0^1(I).$$

thus it follows in the distributional sense:

$$-(\varphi'_\alpha - \varphi'_m)' + \lambda(\varphi_\alpha - \varphi_m)\chi_{I_0} = 0. \tag{18}$$

Moreover since the function $(\varphi_\alpha - \varphi_m)\chi_{I_0}$ is in the space $L^2(I)$, the equation is true in $L^2(I)$. Hence equation (12) is obtained.

- **Step 2:** Boundary conditions

By taking the test function ψ in the linear set V (and not in $V \cap \mathcal{C}_0^\infty(I)$), and multiplying (18) by ψ we get

$$\int_I (\varphi'_\alpha - \varphi'_m)\psi' dx - \int_{\partial I} (\varphi'_\alpha - \varphi'_m)\psi ds + \lambda \int_I (\varphi_\alpha - \varphi_m)\psi\chi_{I_0} dx = 0.$$

Comparing the previous formulation with the functional (15), we obtain for all $\psi \in V$

$$\psi(a)\left(\varphi'_\alpha(a) - \varphi'_m(a)\right) - \psi(b)\left(\varphi'_\alpha(b) - \varphi'_m(b)\right) = 0$$

from which we deduce by taking a good choice of functions ψ

$$\begin{aligned} \varphi'_\alpha(a) &= \varphi'_m(a) \\ \varphi'_\alpha(b) &= \varphi'_m(b) \end{aligned} \tag{19}$$

- **Step 3:** Continuity conditions

Now we multiply (18) by $\psi \in \mathcal{C}^\infty(I)$ (and not in V) and we make an integration by parts:

$$\int_I (\varphi'_\alpha - \varphi'_m)\psi' dx - [\psi(b)(\varphi'_\alpha(b) - \varphi'_m(b)) - \psi(a)(\varphi'_\alpha(a) - \varphi'_m(a))]$$
$$+ \lambda \int_I (\varphi_\alpha - \varphi_m)\psi\chi_{I_0} dx = 0.$$

Remember that the solution φ_α satisfies (19), thus the previous equation reduces to:

$$\int_I (\varphi'_\alpha - \varphi'_m)\psi' dx + \lambda \int_I (\varphi_\alpha - \varphi_m)\psi\chi_{I_0} dx = 0.$$

Comparing this equality with the variational formulation (15) we have for all test-functions $\psi \in \mathcal{C}^\infty(I)$:

$$\sum_{d_k \in P_{\varphi_m}} \frac{\psi(d_k+\alpha)-\psi(d_k)}{\alpha} a_{k,\alpha} + \sum_{d_k \in P_{\varphi_m}} \frac{\psi(d_k)-\psi(d_k-\alpha)}{\alpha} b_{k,\alpha} + \sum_{d_k \in P_{\varphi_m}} \left(\psi(d_k+\alpha)-\psi(d_k-\alpha)\right) c_{k,\alpha} = 0. \quad (20)$$

with:

$$\begin{cases} a_{k,\alpha} = \dfrac{\varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k)}{\alpha} - \varphi'_m(d_k+\alpha) \\ b_{k,\alpha} = \dfrac{\varphi_\alpha(d_k)-\varphi_\alpha(d_k-\alpha)}{\alpha} - \varphi'_m(d_k-\alpha) \\ c_{k,\alpha} = \varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k-\alpha) - \alpha\left(\varphi'_m(d_k+\alpha)+\varphi'_m(d_k-\alpha)\right) \end{cases}$$

Let us consider ψ in $H^1(I)$ such that $\forall d_k \in P_{\varphi_m}$, $\psi(d_k+\alpha)=\psi(d_k)$. That leads to

$$b_{k,\alpha} = -\alpha c_{k,\alpha}, \quad \forall d_k \in P_{\varphi_m}. \quad (21)$$

Taking now $\psi \in H^1(I)$ such that $\psi(d_k)=\psi(d_k-\alpha)$ $\forall d_k \in P_{\varphi_m}$, yields

$$a_{k,\alpha} = -\alpha c_{k,\alpha}, \quad \forall d_k \in P_{\varphi_m}. \quad (22)$$

Moreover the examination of the expression of (16) shows that $a_{k,\alpha}$, $b_{k,\alpha}$, $c_{k,\alpha}$ are linked through:

$$a_{k,\alpha} + b_{k,\alpha} = \frac{c_{k,\alpha}}{\alpha}, \quad \forall d_k \in P_{\varphi_m}. \quad (23)$$

Thus the only solution of linear system composed with (21), (22), (23) is $a_{k,\alpha} = b_{k,\alpha} = c_{k,\alpha} = 0$. That implies the expression of the continuity conditions for all $d_k \in P_{\varphi_m}$:

$$\begin{cases} \dfrac{\varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k)}{\alpha} - \varphi'_m(d_k+\alpha) = 0 \\ \dfrac{\varphi_\alpha(d_k)-\varphi_\alpha(d_k-\alpha)}{\alpha} - \varphi'_m(d_k-\alpha) = 0 \\ \varphi_\alpha(d_k+\alpha)-\varphi_\alpha(d_k-\alpha) - \alpha\left(\varphi'_m(d_k+\alpha)+\varphi'_m(d_k-\alpha)\right) = 0 \end{cases}$$

Hence the optimality conditions satisfied by φ_α are those described in Theorem 2.

◊

Proof of Lemma: Let $\psi \in \mathcal{C}_0^\infty(I)$, we are going to construct a sequence $(\psi_\varepsilon)_\varepsilon$ of $V \cap H_0^1(I)$, such that:

$$\lim_{\varepsilon \to 0} \int_I f \psi_\varepsilon dx = \int_I f \psi dx, \quad \forall f \in L^1(I).$$

For $\varepsilon < \frac{\alpha}{2}$ let us denote by $J_{k,\varepsilon}$ the set defined by:

$$J_{k,\varepsilon} = L_{k-\alpha,\varepsilon} \cup L_{k+\alpha,\varepsilon}$$

with:

$$\begin{aligned} L_{k-\alpha,\varepsilon} &=]d_k - \alpha - \varepsilon, d_k - \alpha + \varepsilon[\\ L_{k+\alpha,\varepsilon} &=]d_k + \alpha - \varepsilon, d_k + \alpha + \varepsilon[\end{aligned}$$

The choice of ψ_ε will be made in such a way ψ_ε in $V \cap H_0^1(I)$ (see Figure 1):

$$\begin{cases} \psi_\varepsilon = \psi & \text{on } I \setminus \left(\bigcup_{k=0,\ldots D} J_{k,\varepsilon} \right) \\ \psi_\varepsilon \text{ is piecewise linear} & \text{on } \bigcup_{k=0,\ldots D} J_{k,\varepsilon} \\ \quad \text{and} & \psi_\varepsilon(d_k - \alpha) = \psi_\varepsilon(d_k) = \psi_\varepsilon(d_k + \alpha) = M_k, \ \forall k \in \{0, \ldots D\}, \end{cases}$$

where $M_k = \psi(d_k)$.

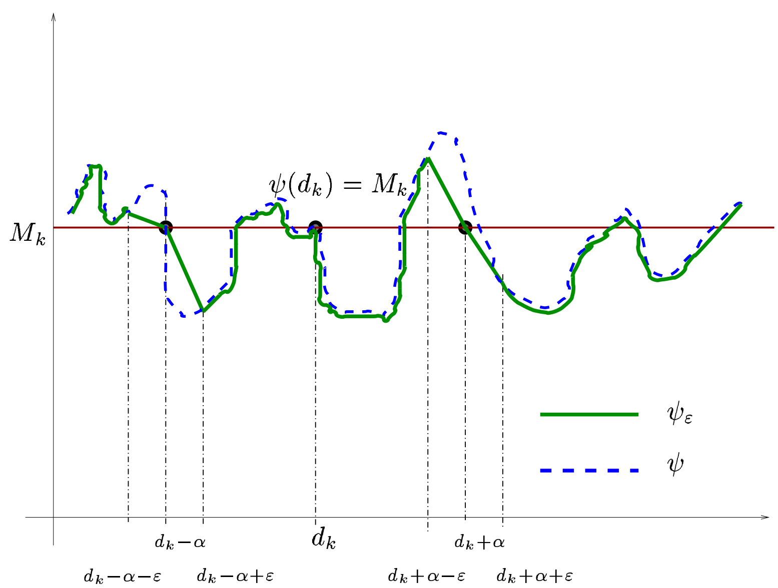

Figure 1: Construction of $\psi_\varepsilon \in H_0^1(I) \cap V$ from $\psi \in \mathcal{C}_0^\infty(I)$.

According to the definition of the intervals $J_{k,\varepsilon}$, the previous integral yields:

$$\int_I f \psi_\varepsilon dx = \int_a^{d_0 - \alpha - \varepsilon} f \psi_\varepsilon dx + \sum_{k=0}^{D} T_{k,\varepsilon} + \sum_{k=0}^{D-1} \int_{d_k + \alpha + \varepsilon}^{d_{k+1} - \alpha - \varepsilon} f \psi_\varepsilon dx + \int_{d_D + \alpha + \varepsilon}^{b} f \psi_\varepsilon dx$$

where:

$$T_{k,\varepsilon} = \int_{J_{k,\varepsilon}} f \psi_\varepsilon dx + \int_{d_k - \alpha + \varepsilon}^{d_k + \alpha - \varepsilon} f \psi_\varepsilon dx$$

The goal is to find $\lim_{\varepsilon \to 0} \int_I f\psi_\varepsilon dx$. By construction of ψ_ε, we have for each $k \in \{0, ...D\}$:

$$\int_I f\psi_\varepsilon dx = \int_a^{d_0-\alpha-\varepsilon} f\psi dx + \sum_{k=0}^{D} \left(\int_{J_{k,\varepsilon}} f\psi_\varepsilon dx + \int_{d_k-\alpha+\varepsilon}^{d_k+\alpha+\varepsilon} f\psi dx \right) + \sum_{k=0}^{D-1} \int_{d_k+\alpha+\varepsilon}^{d_{k+1}-\alpha-\varepsilon} f\psi dx + \int_{d_D+\alpha+\varepsilon}^{b} f\psi dx.$$

Taking the limit when ε tends toward 0 and we easily get:

$$\begin{aligned} &\int_a^{d_0-\alpha-\varepsilon} f\psi dx \underset{\varepsilon\to 0}{\longrightarrow} \int_a^{d_o-\alpha} f\psi dx \quad \text{and} \quad \int_{d_D+\alpha+\varepsilon}^{b} f\psi dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{d_D+\alpha}^{b} f\psi dx \\ &\int_{d_k-\alpha+\varepsilon}^{d_k+\alpha-\varepsilon} f\psi dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{d_k-\alpha}^{d_k+\alpha} f\psi dx \quad \text{and} \quad \int_{d_k+\alpha+\varepsilon}^{d_{k+1}-\alpha-\varepsilon} f\psi dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{d_k+\alpha}^{d_{k+1}-\alpha} f\psi dx \end{aligned} \tag{24}$$

It remains to study

$$\lim_{\varepsilon\to 0} \int_{J_{k,\varepsilon}} f\psi_\varepsilon dx$$

Let us examine $\int_{d_k+\alpha-\varepsilon}^{d_k+\alpha+\varepsilon} f\psi_\varepsilon dx$, one part of $\int_{J_{k,\varepsilon}} f\psi_\varepsilon dx$. The integral $\int_{d_k+\alpha-\varepsilon}^{d_k+\alpha+\varepsilon} f\psi_\varepsilon dx$ can be bounded above, by using the Cauchy-Schwarz inequality:

$$\int_{d_k+\alpha-\varepsilon}^{d_k+\alpha+\varepsilon} f\psi_\varepsilon dx \leq ||f||_{L^1(]d_k+\alpha-\varepsilon, d_k+\alpha+\varepsilon[)} ||\psi_\varepsilon||_{L^\infty(]d_k+\alpha-\varepsilon, d_k+\alpha+\varepsilon[)} \tag{25}$$

therefore it is sufficient to calculate the integral of ψ_ε over $[d_k+\alpha-\varepsilon, d_k+\alpha+\varepsilon]$. Let us remind that ψ_ε is choosen so that this function is piecewise linear on $[d_k+\alpha-\varepsilon, d_k+\alpha+\varepsilon]$. Since ψ is regular $||\psi_\varepsilon||_{L^\infty(]d_k+\alpha-\varepsilon, d_k+\alpha+\varepsilon[)}$ is bounded, and we have:

$$\int_{d_k+\alpha-\varepsilon}^{d_k+\alpha+\varepsilon} |f| dx \underset{\varepsilon\to 0}{\longrightarrow} 0$$

Hence, we obtain:

$$\lim_{\varepsilon\to 0} \int_{d_k+\alpha-\varepsilon}^{d_k+\alpha+\varepsilon} f\psi_\varepsilon dx = 0. \tag{26}$$

We can show same results for the other terms of $\int_{J_{k,\varepsilon}} f\psi_\varepsilon dx$.

In conclusion, since $\lim_{\varepsilon\to 0} \int_{J_{k,\varepsilon}} f\psi_\varepsilon dx = 0$, and by using (24) we get:

$$\lim_{\varepsilon\to 0} \int_I f\psi_\varepsilon dx = \int_I f\psi dx.$$

$\diamondsuit$

To summarize we have demonstrated the existence and uniqueness of the unwrapped phase φ_α in $H^1(I)$ by minimizing the functional E_α. However φ_α is the solution of the approximated variational formulation. The next section is devoted to the study of the behaviour of φ_α as α tends to 0.

3.4 Behaviour of φ_α when α tends to 0

The aim of this section is to show that the sequence of functions $(\varphi_\alpha)_\alpha$ admits a limit for the weak topology of $H^1(I)$ and to seek the PDE's satisfied by this limit.

Theorem 3 *Let us assume same hypotheses as in Theorem 1. Let φ_α be the solution of the minimization problem*

$$E_\alpha(\varphi_\alpha) = \inf_{\varphi \in H^1(I)} E_\alpha(\varphi).$$

Then there exists a unique function $\tilde{\varphi}$, satisfying:

- $\varphi_\alpha \underset{\alpha \to 0}{\rightharpoonup} \tilde{\varphi}$ *weakly in* $H^1(I)$
- $\tilde{\varphi} \in \mathcal{C}(I)$
- $\tilde{\varphi}'^{+}(d_k) = \varphi_m'^{+}(d_k)$ *and* $\tilde{\varphi}'^{-}(d_k) = \varphi_m'^{-}(d_k)$ *for all* $d_k \in P_{\varphi_m}$

Moreover $\tilde{\varphi}$ satisfies the following Euler-Lagrange equation:

$$\begin{cases} -(\tilde{\varphi}' - \varphi_m')' + \lambda \chi_{I_0}(\tilde{\varphi} - \varphi_m) = 0, & I =]a,b[\\ \\ \tilde{\varphi}'(a) = \varphi_m'(a), \quad \tilde{\varphi}'(b) = \varphi_m'(b). \end{cases} \tag{27}$$

Proof: We first bound $|\varphi_\alpha|_{H^1(I)}$, independently of α. Taking into account the coercivity of the functional E_α, it reduces to get an upper bound for $E_\alpha(\varphi_\alpha)$. Since φ_α is defined as the minimum of E_α, we have:

$$E_\alpha(\varphi_\alpha) \leq E_\alpha(\varphi) \quad \forall \varphi \in H^1(I).$$

In order to simplify the proof we assume that the wrapped phase φ_m has just one discontinuity d. So we denote $I =]a,b[$ the interval of definition, and $I_0 =]d,b[$ the interval of reference. At the end of the proof we will describe the general case.

We choose a "good function" φ as follows:

$$\varphi(x) = \begin{cases} \varphi_m'^{+}(d)x + q^+ & \text{if } x \in (d, d+\alpha) \\ \varphi_m'^{-}(d)x + q^- & \text{if } x \in (d-\alpha, d) \\ \varphi_m(x) + (d+\alpha)\varphi_m'^{+}(d) + q^+ - \varphi_m(d+\alpha) & \text{if } x \in (d+\alpha, b) \\ \varphi_m(x) + (d-\alpha)\varphi_m'^{-}(d) + q^- - \varphi_m(d-\alpha) & \text{if } x \in (a, d-\alpha) \end{cases} \tag{28}$$

where q^+ and q^- are to be choosen so that φ is continuous on $]a,b[$ and such that $E_\alpha(\varphi)$ is bounded independently of α. Notice that φ is in $H^1(I)$.

- Continuity of φ:

By construction φ is continuous at points $d-\alpha$ and $d+\alpha$. At the discontinuity point d, the continuity of φ implies that:

$$\varphi_m'^{+}(d)d + q^+ = \varphi_m'^{-}(d)d + q^-.$$

which leads to the necessary conditions linking q^+ and q^-:

$$\left(\varphi_m'^{+}(d) - \varphi_m'^{-}(d)\right) d = q^- - q^+ \tag{29}$$

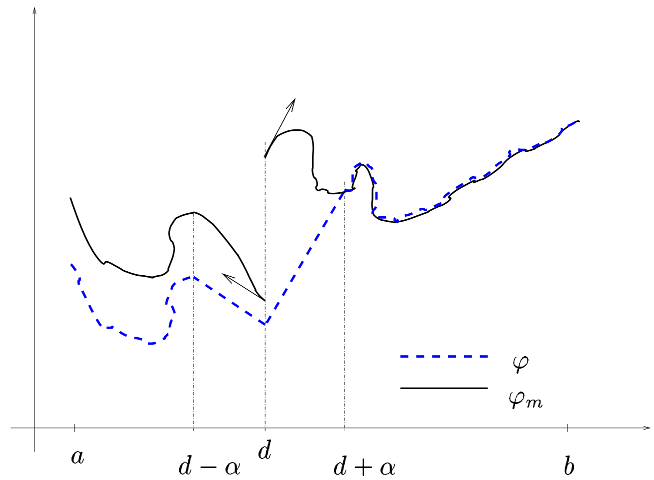

Figure 2: Construction of a "good function" φ from φ_m.

- Calculus of $E_\alpha(\varphi)$:

$$\begin{aligned} E_\alpha(\varphi) = & \int_I (\varphi' - \varphi'_m)^2 dx + \int_I (\varphi - \varphi_m)^2 \chi_{I_0} dx \\ & + \left(\frac{\varphi(d+\alpha) - \varphi(d)}{\alpha} - \varphi'_m(d+\alpha) \right)^2 + \left(\frac{\varphi(d) - \varphi(d+\alpha)}{\alpha} - \varphi'_m(d-\alpha) \right)^2 \\ & + \big(\varphi(d+\alpha) - \varphi(d-\alpha) - \alpha\left(\varphi'_m(d+\alpha) + \varphi'_m(d-\alpha)\right)\big)^2. \end{aligned}$$

On one hand it is easily checked that:

$$\begin{cases} \dfrac{\varphi(d+\alpha) - \varphi(d)}{\alpha} = \varphi'^{+}_m(d) & (30) \\ \dfrac{\varphi(d) - \varphi(d+\alpha)}{\alpha} = \varphi'^{-}_m(d) & (31) \\ \varphi(d+\alpha) - \varphi(d-\alpha) = \alpha\left(\varphi'^{+}_m(d) + \varphi'^{-}_m(d)\right) & (32) \end{cases}$$

On the other hand the derivative $\varphi'(x)$ is given explicitely

$$\varphi'(x) = \begin{cases} \varphi'^{+}_m(d) & \text{for } x \in (d, d+\alpha) \\ \varphi'^{-}_m(d) & \text{for } x \in (d-\alpha, d) \\ \varphi'_m(x) & \text{for } x \in I - (d-\alpha, d+\alpha). \end{cases} \qquad (33)$$

Thus we obtain for the first term of $E_\alpha(\varphi)$:

$$\int_I (\varphi' - \varphi'_m)^2 dx = \int_{d-\alpha}^{d} \left(\varphi'^{-}_m(d) - \varphi'_m(x)\right)^2 dx + \int_d^{d+\alpha} \left(\varphi'^{+}_m(d) - \varphi'_m(x)\right)^2 dx.$$

and for the second term defined on the interval of reference I_0:

$$\int_I (\varphi - \varphi_m)^2 \chi_{I_0} dx = \int_{d+\alpha}^{b} \left(\varphi'^{+}_m(d)(d+\alpha) + q^+ - \varphi_m(d+\alpha)\right)^2 dx + \int_d^{d+\alpha} \left(\varphi'^{+}_m(d)x + q^+ - \varphi_m(x)\right)^2 dx.$$

If we choose

$$q^+ = \varphi_m(d+\alpha) - (d+\alpha)\varphi_m'^+(d).$$

After a substitution in the previous expression, we deduce

$$\int_{d+\alpha}^{b} \left(\varphi_m'^+(d)(d+\alpha) + q^+ - \varphi_m(d+\alpha)\right)^2 dx = 0.$$

It follows that:

$$\begin{aligned} E_\alpha(\varphi) \quad &= \int_{d-\alpha}^{d} \left(\varphi_m'^-(d) - \varphi_m'(x)\right)^2 dx + \int_{d}^{d+\alpha} \left(\varphi_m'^+(d) - \varphi_m'(x)\right)^2 dx \\ &+ \int_{d}^{d+\alpha} \left(\varphi_m'^+(d)x + q^+ - \varphi_m(x)\right)^2 dx + \left(\varphi_m'^+(d) - \varphi_m'(d+\alpha)\right)^2 + \left(\varphi_m'^-(d) - \varphi_m'(d-\alpha)\right)^2 \\ &+ \alpha^2 \left(\varphi_m'^+(d) + \varphi_m'^-(d) - \varphi_m'(d+\alpha) - \varphi_m'(d-\alpha)\right)^2 . \end{aligned} \tag{34}$$

By taking the limit as α tends to 0, we find that all the integrals, in the expression of $E_\alpha(\varphi)$, tend to 0. It is clear that the last terms by hypothesis tend also to zero.
Moreover since

$$0 \le E_\alpha(\varphi_\alpha) \le E_\alpha(\varphi)$$

we deduce that $E_\alpha(\varphi_\alpha)$ tends to zero as $\alpha \to 0$. So we have:

$$\begin{cases} \lim\limits_{\alpha\to 0} \varphi_\alpha' = \varphi_m', \text{ in } L^2(I) \\ \lim\limits_{\alpha\to 0} \varphi_\alpha = \varphi_m, \text{ in } L^2(I_0) \\ \lim\limits_{\alpha\to 0} \left(\dfrac{\varphi_\alpha(d+\alpha) - \varphi_\alpha(d)}{\alpha}\right) = \varphi_m'^+(d) \\ \lim\limits_{\alpha\to 0} \left(\dfrac{\varphi_\alpha(d) - \varphi_\alpha(d-\alpha)}{\alpha}\right) = \varphi_m'^-(d) \\ \lim\limits_{\alpha\to 0} \varphi_\alpha(d+\alpha) - \varphi_\alpha(d-\alpha) = 0 \end{cases} \tag{35}$$

Now taking into account that $\lim\limits_{\alpha\to 0} E_\alpha(\varphi) = 0$ we obtain that $E_\alpha(\varphi_\alpha)$ is bounded independently of α, from which it follows that φ_α is bounded in $H^1(I)$. Hence up to a subsequence there exists a function $\tilde{\varphi} \in H^1(I)$ such as:

$$\varphi_\alpha \underset{\alpha\to 0}{\rightharpoonup} \tilde{\varphi}, \text{ in } H^1(I)\text{-weak}$$

And by using (35) we get:

$$\begin{cases} \tilde{\varphi}' = \varphi_m', \text{ a.e on } I \\ \tilde{\varphi} = \varphi_m, \text{ a.e on } I_0 \end{cases} \tag{36}$$

The continuity condition for $\tilde{\varphi}$, $\tilde{\varphi}^+(d_k) = \tilde{\varphi}^-(d_k)$ is easily deduced from the Sobolev injection of $H^1(I)$ in $\mathcal{C}(\overline{I})$.

Now, we want to prove that $\tilde{\varphi}$ satisfies the regularity conditions:

$$\begin{cases} \tilde{\varphi}'^{+}(d_k) = \varphi'^{+}_m(d_k) \\ \tilde{\varphi}'^{-}(d_k) = \varphi'^{-}_m(d_k) \end{cases} \tag{37}$$

Starting from $\tilde{\varphi}$ and using the integral formulation in $H^1(I)$ of $\tilde{\varphi}(d_k+\alpha)-\tilde{\varphi}(d_k)$, we have:

$$\frac{\tilde{\varphi}(d_k+\alpha)-\tilde{\varphi}(d_k)}{\alpha} = \frac{1}{\alpha}\int_{d_k}^{d_k+\alpha} \tilde{\varphi}'(t)dt$$

But remind that $\tilde{\varphi}' = \varphi'_m$, a.e on I:

$$\frac{\tilde{\varphi}(d_k+\alpha)-\tilde{\varphi}(d_k)}{\alpha} = \frac{1}{\alpha}\int_{d_k}^{d_k+\alpha} \tilde{\varphi_m}'(t)dt$$

hence

$$\frac{\tilde{\varphi}(d_k+\alpha)-\tilde{\varphi}(d_k)}{\alpha} = \frac{\varphi_m(d_k+\alpha)-\varphi_m(d_k)}{\alpha} \tag{38}$$

By the same way we get

$$\frac{\tilde{\varphi}(d_k)-\tilde{\varphi}(d_k-\alpha)}{\alpha} = \frac{\varphi_m(d_k)-\varphi_m(d_k-\alpha)}{\alpha} \tag{39}$$

Equations (37) are obtained by taking the limit of equations (38) and (39) as α tends to 0.

$\diamondsuit$

Remark: The method is the same when φ_m has several phase discontinuities. Let P_{φ_m} be the set of the points referred as phase jumps, as denoted in Section 2. As shown for one discontinuity, we construct a function piecewise-linear over a neighboorhood of each discontinuity d_k. **Remark:** The function $\tilde{\varphi}$ satisfies the same Euler equations as φ_α:

$$\begin{cases} -(\tilde{\varphi}' - \varphi'_m)' + \lambda \chi_{I_0}(\tilde{\varphi} - \varphi_m) = 0, \quad I =]a,b[\\ \\ \tilde{\varphi}'(a) = \varphi'_m(a), \quad \tilde{\varphi}'(b) = \varphi'_m(b). \end{cases}$$

But since equations (27) admits only one solution, it follows that:

$$\tilde{\varphi}(x) = \varphi_\alpha(x), \quad \forall \alpha \in \mathbb{R}^+, \quad \text{a.e. } x \in I.$$

We may also remark that from equation (27) we only deduce that $\tilde{\varphi} \in H^1(I)$ and no more regularity. Further regularity (37) has been obtained by imposing some contraints on φ_α. **Remark:** It remains to proof that

- $\tilde{\varphi}(x) = \varphi_m(x) + c_k, \quad \forall x \in]d_k, d_{k+1}[$
- $c_k \in \mathbb{R}$ is proportional to 2π

From the regularity of $\tilde{\varphi}$ and from equation (36) we have easily

$$\tilde{\varphi}(x) = \varphi_m(x) + c_k, \quad \forall x \in]d_k, d_{k+1}[$$

$$\tilde{\varphi}(x) = \varphi_m(x), \quad \forall x \in I_0 \tag{40}$$

Hence for β sufficiently large we have $(d_{k_0} + \beta) \in I_0$ and we get:

$$\tilde{\varphi}(d_{k_0} + \beta) = \varphi_m(d_{k_0} + \beta) + c_{k_0}$$

Comparing this with (40) we obtain $c_{k_0} = 0$.
We have now to consider what happens both on the left side and on the right side of the interval of reference I_0.

- For all $k > k_0$:

  $$\begin{aligned} \tilde{\varphi}(d_k - \alpha) &= \varphi_m(d_k - \alpha) + c_k \\ \tilde{\varphi}(d_k + \alpha) &= \varphi_m(d_k + \alpha) + c_{k+1} \end{aligned}$$

  So taking the limit when α tends to 0, and since $\varphi_m^-(d_k) - \varphi_m^+(d_k) = 2\pi$, we obtain:

  $$\begin{cases} c_{k+1} = c_k + 2\pi, \quad \forall k > k_0 \\ c_{k_0} = 0 \end{cases}$$

  and by induction:

  $$c_{k+1} = 2\pi(k - k_0), \quad \forall k > k_0 \tag{41}$$

- For all $k \leq k_0$:

  $$\begin{aligned} \tilde{\varphi}(d_{k-1} - \alpha) &= \varphi_m(d_{k-1} - \alpha) + c_{k-1} \\ \tilde{\varphi}(d_{k-1} + \alpha) &= \varphi_m(d_{k-1} + \alpha) + c_k \end{aligned}$$

  We deduce by the same way as for the right side of I_0

  $$c_{k-1} = -2\pi(k - k_0 + 1), \quad \forall k \leq k_0 \tag{42}$$

3.5 Discretization and numerical results

The equation (12) is a linear parabolic equation and thus there is no real difficulty to implement it. We use a standard finite difference discretization and an iterative method such as Jacobi or Gauss-Seidel can be used to solve the resulting linear system.

for $x_i \in I \backslash I_0$:

$$\begin{aligned} \varphi_\alpha^{n+1}(x_i) = \varphi_\alpha^n(x_i) + \Delta t \quad & (\varphi_\alpha^n(x_{i-1}) - \varphi_m(x_{i-1}) + 2\varphi_\alpha^n(x_i) - 2\varphi_m(x_i) \\ & + \varphi_\alpha^n(x_{i+1}) - \varphi_m(x_{i+1})) \end{aligned}$$

for $x_i \in I_0$:

$$\begin{aligned} \varphi_\alpha^{n+1}(x_i) = \varphi_\alpha^n(x_i) + \Delta t \quad & (\varphi_\alpha^n(x_{i-1}) - \varphi_m(x_{i-1}) + 2\varphi_\alpha^n(x_i) - 2\varphi_m(x_i) \\ & + \varphi_\alpha^n(x_{i+1}) - \varphi_m(x_{i+1})) + \lambda(\varphi_\alpha^n(x_i) - \varphi_m(x_i)) \end{aligned}$$

with the boundary conditions:

$$\begin{aligned} \varphi_\alpha^{n+1}(a) &= \varphi_\alpha^n(a + \Delta x) - \varphi_m(a + \Delta x) + \varphi_m(a) \\ \varphi_\alpha^{n+1}(b) &= \varphi_\alpha^n(b - \Delta x) + \varphi_m(b) - \varphi_m(b - \Delta x) \end{aligned}$$

The difficulty comes from the approximation of the condition (14) for which we need to take into account a direction of propagation with respect to the reference interval I_0. This is intuitive since the values on I_0 are constrained and influence both side in a certain direction.

So the following discretization is proposed for (14).

For φ_α^n given, we have:

$$\begin{cases} \varphi_\alpha^{n+1}(d_k+\widetilde{\alpha}) = \varphi_\alpha^n(d_k+2\widetilde{\alpha}) - \varphi_m(d_k+2\widetilde{\alpha}) + \varphi_m(d_k+\widetilde{\alpha}) \\ \varphi_\alpha^{n+1}(d_k) = \varphi_\alpha^n(d_k-\widetilde{\alpha}) - \varphi_m(d_k-2\widetilde{\alpha}) + \varphi_m(d_k-\widetilde{\alpha}) \\ \varphi_\alpha^{n+1}(d_k-\widetilde{\alpha}) = \varphi_\alpha^n(d_k+\widetilde{\alpha}) + \varphi_m'(d_k+\widetilde{\alpha}) + \varphi_m'(d_k-\widetilde{\alpha}) \end{cases} \tag{43}$$

where $I_0 =]a_0, b_0[\subset]d_{k_0}, d_{k_0+1}[$

$$\widetilde{\alpha} = \begin{cases} \alpha & \text{if } d_k > b_0 \\ -\alpha & \text{if } d_k < a_0. \end{cases}$$

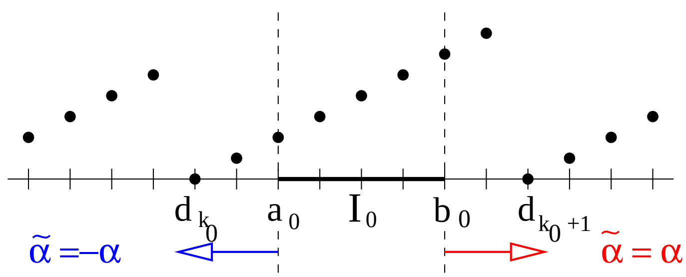

Figure 3: Direction of propagation with respect to the reference interval.

Notice that the discretization of the derivatives $\varphi_m'(d_k+\widetilde{\alpha})$ and $\varphi_m'(d_k-\widetilde{\alpha})$ depends on which side of I_0 they are estimated (jumps have to be avoided). As suggested by a discrete viewpoint, and shown in the Figure 3, α is chosen to be equal to grid spacing.

The interferometric phase image only contains information related to the topography. In the case of repeat-pass interferometry, temporal changes degrade the detection, and introduce high noise level in the wrapped phase of the interferogram. In this paper we do not take into account the influence of noise on the phase unwrapping process. We focus our attention on the reconstruction of the absolute phase. Hence an example of result is shown in Figure 4, where we have considered the signal φ_m, with a small additive Gaussian noise. Several iterations are displayed to see the reconstruction with respect to the interval of reference I_0. Note that the noise is not removed but does not disturb the unwrapping process. Besides, because of the convexity of the problem, changes in the values of the parameters have little influence on the solution.

In the two, dimensional real data case, φ_m will be filtered before retrieving the absolute phase φ.

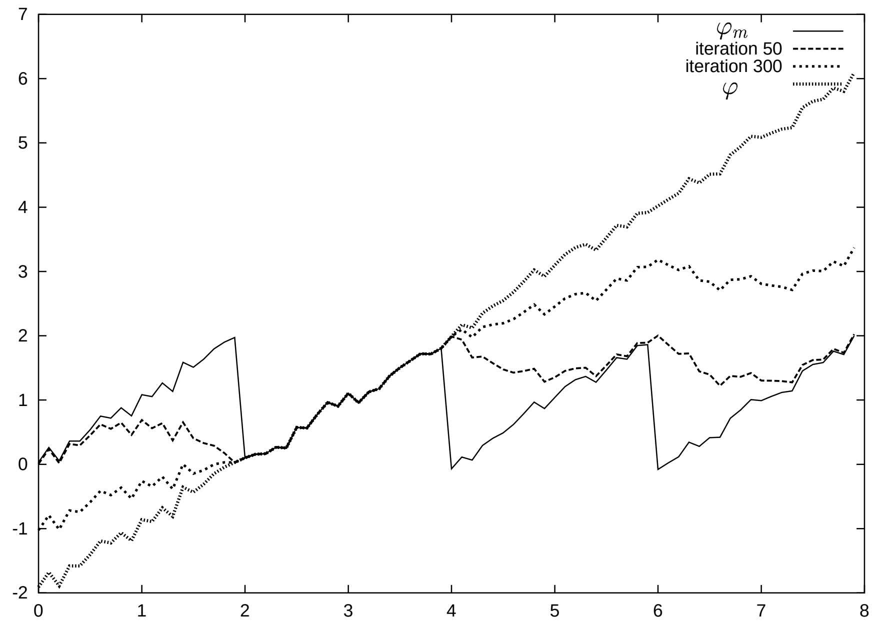

Figure 4: Phase unwrapping without terrain discontinuities. Intermediary stages are displayed. φ_m was created with additive Gaussian noise ($\sigma = 0.08$). $I_0 =]2.5, 3.5[$

4 Case of phase unwrapping in $BV(I)$

4.1 Description of the functional

In this section we assume that φ and φ_m have ground discontinuities (denoted by $T_{\varphi_m} = \{t_j\}_j$) due to geological faults. We recall that d_k is known but not t_j and $t_j \neq d_k$ for all j and k (see Section 2). Therefore we need to recover the unknown discontinuities T_{φ_m}. Unfortunately, this is not possible with Sobolev spaces. When a function is discontinuous its gradient has to be understood as a measure and the space of functions of bounded variations, $BV(I)$ is then suitable (see [1] and [2]). We recall that $BV(I)$ is the set:

$$\Big\{ f \in L^1(I) : \sup_{\substack{g \in L^\infty(I) \\ ||g||_{L^\infty} \leq 1}} \int_I f(x) g'(x) dx < \infty \Big\}.$$

The most important property is that the distributional derivative (which is a measure) can be decomposed in three terms:

$$Df = \underbrace{f' dx}_{\text{regular part}} + \underbrace{J_f}_{\text{Jump part}} + \underbrace{C_f}_{\text{Cantor part}} . \tag{44}$$

The total variation of Df is given by:

$$\int_I |Df| = \int_I |f'(x)|dx + \sum_{t\in S_f} |f(t_+) - f(t_-)| + |C_f|$$

where

- f' is the approximate derivative of f ($f'dx$ is absolutely continuous with respect to dx)
- $f(t_+)$ (respectively $f(t_-)$) is the approximate upper limit (resp. lower limit) of f at t defined by

$$f(t_+) = \lim_{\rho\to 0} \frac{1}{\rho} \int_t^{t+\rho} f(s)ds.$$

- S_f is the set of discontinuities of f, that is $\{t;\ f(t^+) \neq f(t^-)\}$.
- C_f which is a diffuse singular measure is called the Cantor part of Df.

Here we will suppose that $\varphi_m \in SBV(I) = \{f \in BV(I) : C_f = 0\}$, the set of special functions of bounded variations. We define S_{φ_m} by $S_{\varphi_m} = P_{\varphi_m} \cup T_{\varphi_m}$. So, in this case we search for φ_α in $BV(I)$ which minimizes the functional:

$$\widetilde{E}_\alpha(\varphi) = \int_I |D(\varphi - \varphi_m)| + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi). \qquad (45)$$

where E_2 and $E_{3,\alpha}$ are defined by (7) and (8), and

$$\int_I |D(\varphi - \varphi_m)| = \int_I |\varphi' - \varphi_m'|dx + \sum_{t\in S_{\varphi-\varphi_m}} |(\varphi - \varphi_m)(t_+) - (\varphi - \varphi_m)(t_-)| + |C_{\varphi-\varphi_m}|.$$

4.2 Existence and uniqueness of a minimizer φ_α

The existence and uniqueness of a minimizer for (45) is straightforward. It follows from the definition of $\widetilde{E}_\alpha(\varphi)$.

Theorem 4 *Let us assume that $\varphi_m \in SBV(I) \cap L^2(I)$ then for all $\alpha > 0$ the functional $\widetilde{E}_\alpha(\varphi)$ admits a unique minimizer in $BV(I)$.*

Proof: Thanks to the definition of $\widetilde{E}_\alpha$, if φ_n is a minimizing sequence in $BV(I)$ then for some constant $C(\alpha) > 0$ we have:

$$\int_I |D(\varphi_n - \varphi_m)| + \lambda \int_{I_0} |\varphi_n - \varphi_m|^2 dx \leq C(\alpha)$$

from which we deduce that

$$\int_I |D\varphi_n| + \int_{I_0} |\varphi_n|^2 dx \leq C(\alpha)$$

By the compactness embedding theorem [1], [2] and the semi-continuity of the total variation with respect to weak-* topology of BV, we conclude there exists $\varphi_\alpha \in BV(I)$ such that, up to a subsequence:

$$\varphi_n \xrightarrow[\alpha\to 0]{} \varphi_\alpha \text{ in BV weak-*}$$
$$\tilde{E}_\alpha(\varphi_\alpha) \leq \liminf_{n\to 0} \tilde{E}_\alpha(\varphi_n) \leq \sup_{\varphi\in BV(I)} \tilde{E}_\alpha(\varphi)$$

which means that φ_α is a minimizer of $\widetilde{E}_\alpha$. The uniqueness follows from the strict convexity of the term E_2.

◇

We do not pursue further this theoretical $BV-$based approach. The main reason is that it is very difficult to obtain efficient optimality conditions in BV. Numerical approximation is a real problem on BV (approximation on the discontinuity set, the cantor part...). So in the sequel we adapt the same formalism used for optical images, that is instead of the functional $\widetilde{E}_\alpha(\varphi)$ we consider the simpler functional:

$$\widetilde{J}_\alpha(\varphi) = \int_I |\varphi' - \varphi'_m| dx + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi) \tag{46}$$

In this case we ignore the singular part of the measure $D(\varphi - \varphi_m)$ and we search for an algorithm that compute a minimizer of $\widetilde{J}_\alpha$ while trying to numerically capture its discontinuities. This is the object of the following section.

4.3 Half-quadratic minimization

In order to avoid the non-differentiability of the absolute value function, we approximate it (see [4]) by an edge preserving Φ-function:

$$|u| \approx \Phi_\varepsilon(u) \text{ with } \Phi_\varepsilon(u) = \sqrt{\varepsilon^2 + u^2} - \varepsilon \text{ for small values of } \varepsilon.$$

The functional $\widetilde{J}_\alpha(\varphi)$ is then approximated by:

$$J_{\alpha,\varepsilon}(\varphi) = \int_I \Phi_\varepsilon(\varphi' - \varphi'_m) + \lambda E_2(\varphi) + E_{3,\alpha}(\varphi).$$

The term with the Φ_ε-function is non-quadratic, which implies a non-linear diffusion operator in the Euler-Lagrange equations. A way to overcome this difficulty is to propose an half quadratic algorithm based on duality results. Because Φ_ε is edge-preserving it is shown in [2, 4] that it is always possible to find a function Φ^*_ε of the form:

$$\Phi^*_\varepsilon(t, b) = bt^2 + G_\varepsilon(b)$$

such that $\Phi_\varepsilon(t) = \inf_{b\in[0,1]} \Phi^*_\varepsilon(t, b)$. Applying this transformation, we can rewrite the problem as:

$$\inf_\varphi J_{\alpha,\varepsilon}(\varphi) = \inf_{\varphi,b} J^*_{\alpha,\varepsilon}(\varphi, b)$$

where

$$J^*_{\alpha,\varepsilon}(\varphi, b) = \int b|\varphi' - \varphi'_m|^2 + G_\varepsilon(b) + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi)$$

There are two main advantages:

- $J^*_{\alpha,\varepsilon}$ is quadratic in φ when b is fixed
- for φ fixed, the minimizer in b can be found explicitly [2, 4].

A convergent algorithm [4] consists in minimizing $J^*_{\alpha,\varepsilon}$ alternately with respect to each variable, φ and b.

$$\left\{\begin{array}{ll} \text{Starting from } \varphi^0_{\alpha,\varepsilon} \equiv 0, & \\ \text{REPEAT} & \\ b^{n+1} = \dfrac{\Phi'_\varepsilon(|(\varphi^n_{\alpha,\varepsilon})' - \varphi'_m|)}{2|(\varphi^n_{\alpha,\varepsilon})' - \varphi'_m|} & (47) \\ -\left(b^{n+1}(\varphi^{n+1}_{\alpha,\varepsilon} - \varphi_m)'\right)' + \lambda(\varphi^{n+1}_{\alpha,\varepsilon} - \varphi_m)\chi_{I_0} = 0 & (48) \\ \text{UNTIL CONVERGENCE,} & \end{array}\right.$$

with the boundary conditions (13) and regularity conditions (14).
(48) is the Euler-Lagrange equation satisfied by $\varphi_{\alpha,\varepsilon}$, a minimizer of $J_{\alpha,\varepsilon}$.
Here we will not do the study of the behaviour of the solution $\varphi_{\alpha,\varepsilon}$ as α tends to 0 and ε tends to 0.

4.4 Discretization and numerical results

Let us comment on the discretization of the previous equations:

- (47) is an explicit formula.
- (48) is a linear equation which can be solved with an iterative method. For a simpler writting, we note $f = (\varphi_{\alpha,\varepsilon} - \varphi_m)$. Then the term $(bf')'$ is approximated by:

$$(bf')'\Big|_i \approx \frac{1}{h^2}\Big[b_{i-1}f_{i-1} + b_i f_{i+1} - (b_{i-1} + b_i)f_i\Big].$$

  with h the grid spacing.
- conditions (13) and (14) are discretized as in section 3.5.

The Figure 5 show the reconstruction of the absolute phase φ and we obseve that the two terrain discontinuities are preserved.

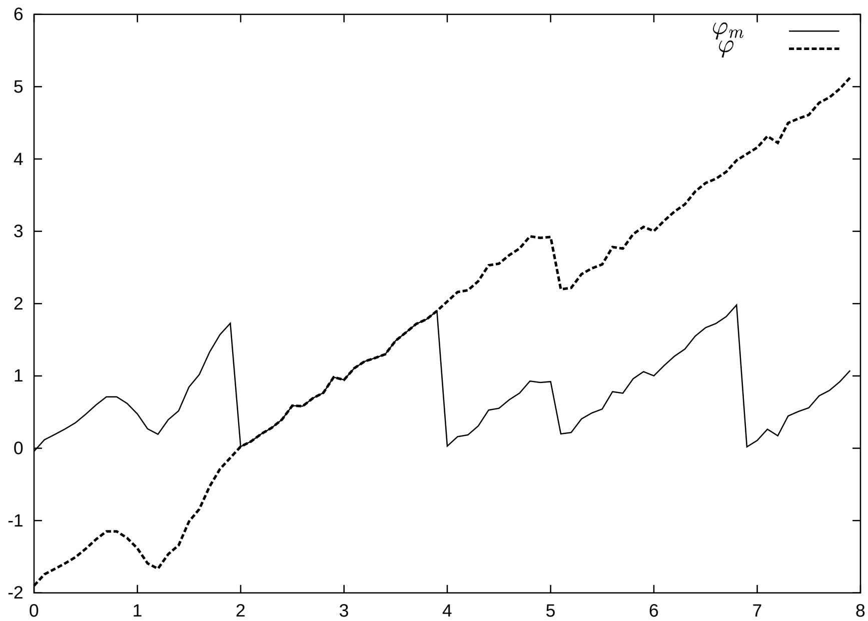

Figure 5: Phase unwrapping with terrain discontinuities. φ_m was created with two ground discontinuities $x = 1.1$ and $x = 5.1$ and with additive Gaussian noise ($\sigma = 0.08$). φ is computed with small values of ε. $I_0 =]2.5, 3.5[$.

5 Conclusion

This paper establishes the mathematical foundations of the 1D phase unwrapping problem in a continuous setting. A variational approach preserving terrain discontinuities was presented. Of course our 1D numerical results are similar to any standard unwrapping algorithm. Our approach is rigorous and promising. Future work will focus on the development of variational approaches to 2D-interferograms. We plan to estimate the phase jumps (curves of discontinuities) from the bidimensional interferogram using a level set approach, and then to apply the method from this paper in the 2D case.

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