LABORATOIRE

INFORMATIQUE, SIGNAUX ET SYSTÈMES
DE SOPHIA ANTIPOLIS
UMR 6070

ALGEBRAIC SUCCESSION RULES

Jean-Marc Fédou, Christine Garcia

Projet RECIF

Rapport de recherche
I3S/RR–2002-37–FR

Septembre 2002

LABORATOIRE I3S: Les Algorithmes / Euclide B – 2000 route des Lucioles – B.P. 121 –
06903 Sophia-Antipolis Cedex, France – Tél. (33) 492 942 701 – Télécopie : (33) 492 942 898
`http://www.i3s.unice.fr/I3S/FR/`

RÉSUMÉ :

Nous étudions dans cet article une famille de règles de succession et nous montrons qu'elles ont une série génératrice algébrique pour des suites d'exposants rationnelles. Nous décomposons algébriquement les chemins dans l'arbre de génération correspondant et nous en déduisons une équation algébrique satisfaite par la série génératrice non commutative.

MOTS CLÉS :

combinatoire énumérative, série génératrice algébrique, série génératrice rationnelle, règles de succession

ABSTRACT:

In this paper, we study a familly of succession rules and we show that they have an algebraic generating function for a rational sequence of exponents. We decompose algebraically the paths in the corresponding generating tree and deduce an algebraic equation satisfied by the noncomutative generating function.

KEY WORDS :

enumerative combinatorics, algebraic generating function, rational generating function, succession rules

Algebraic Succession Rules

Jean-Marc Fedou, Christine Garcia

I3S, URA 1376 du CNRS Sophia-Antipolis, FRANCE

Abstract

In this paper, we show that succession rules

$$(k) \leadsto (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p},$$

have an algebraic generating function when the sequence (α_i) is rational. We decompose algebraically the paths in the corresponding generating tree and deduce an algebraic equation satisfied by the noncommutative generating function.

1 Introduction

The succession rules were first introduced by Chung, Graham, Hoggatt and Kleimann in [3] to study Baxter permutations. The method was later successfully used by West [11], Dulucq, Gire and Guibert [5], [6],[7] for the enumeration of permutations with forbidden sequences. The concept has more recently been exploited by Barcucci, Del Lungo, Pergola and Pinzani [2] as the ECO method for the enumeration and recursive construction of various classes of combinatorial objects. The succession rule approach has several equivalent interpretations, ECO rules, random paths, infinite automata or Riordan arrays and deals with different kinds of generating functions (rational, algebraic or exponential). The problem of classifying successions rules according to the type of their generating functions has been proposed by R.Pinzani [2] in the area of ECO systems. A classical and easy result is that finite succession rules have rational generating function since they correspond to a regular language. It is shown in [1] that every finite transformation of Catalan succession rule,

$$(k) \leadsto (1)(2) \dots (k)(k+1),$$

Email address: `fedou, cgarcia@unice.fr` (Jean-Marc Fedou, Christine Garcia).

Preprint submitted to Elsevier Science *3 July 2002*

is algebraic. In the same paper are also described succession rules leading to exponential generating functions which have been more extensively studied by S.Corteel in [4].

Our paper is devoted to the study of algebraic system of succession rules having algebraic generating function. Our approach is closely related with the Schutzenberger methodology, which consists in finding first a bijection between the objects and the words of an algebraic language, second a non ambiguous grammar for the language and finally take the commutative image and deduce an algebraic system for the generating function. For a succession rule, we define its noncommutative formal power series using the infinite alphabet of positive integers. We use a new operation $\oplus$ which allows us to get a non ambigous decomposition of the formal power series associated to the generating tree. We deduce algebraic equation by taking the commutative image of the formal power series. This method allows us to get an algebraic decomposition of the general succession rule,

$$(k) \rightsquigarrow (1)\dots(k-1)(k)^{\lambda_0}\dots(k+p)^{\lambda_p},$$

for any finite sequence (λ_i), and more generally for the succession rule,

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}}\dots(k-1)^{\alpha_1}(k)^{\lambda_0}\dots(k+p)^{\lambda_p},$$

for any sequence $(\alpha_i)_{i=1}^{\infty}$ proving thereby that their generating function is *algebraic* when the sequence (α_i) is *rational*. Finally we generalize the results of Flajolet and al [1].

2 Definitions

A succession rule is a function $(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k))$ which associates to each positive integer k a finite multiset of integers, called successors of k. A generating tree is a succession rule with a particular integer a, called axiom. We suppose in the following that a equals 1. A generating tree can be viewed as the infinite tree constructed with a root labelled by the axiom and where each node labelled k has sons labelled according to the succession rule. Hence, ECO systems are those generating trees where each integer has exactly k successors.

For a generating tree Ω,

$$\Omega \begin{cases} (1) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k)), \end{cases}$$

we define the language L_Ω as the set of words over N, begining by the axiom 1 and satisfying the succession rule, $(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k))$. Each word w of L_Ω corresponds to at least one path of Ω. For each word $w \in L_\Omega$, we note $m(w)$ the number of paths in the generating tree Ω corresponding to the word w. We denote by $S(\Omega)$ the noncommutative formal power series,

$$S(\Omega) = \sum_{w \in L_\Omega} m(w)w.$$

By construction, the generating tree Ω and the noncommutative formal power series $S(\Omega)$ have the same generating function,

$$F_\Omega(z) = \sum_n f_n z^n,$$

where

$$f_n = \sum_{w \in L_\Omega, |w|=n} m(w).$$

For simplifying the notation, we write F_Ω for $F_\Omega(z)$. We use standard external product and concatenation, by an integer n, over the noncommutative formal power series $S(\Omega)$. We write,

$$nS(\Omega) = \sum_{w \in L_\Omega} (nm(w))w,$$

$$(n).S(\Omega) = \sum_{w \in L_\Omega} m(w)(n.w).$$

We define the operation $\oplus$ as follows,

Definition 1 *For $i \in N^+$, we define by $i^\oplus = i + 1$. By extension if $w = w_1w_2\dots w_i$ is a word of L_Ω then $w^\oplus = w_1^\oplus w_2^\oplus \dots w_i^\oplus$ and $S(\Omega)^\oplus = \sum_{w \in L_\Omega} m(w)w^\oplus$.*

Example 2 *The partial generating tree of Θ is illustrated in Fig. 1,*

$$\Theta \begin{cases} (1) \\ (k) \rightsquigarrow (1)(2)(3)(3)(4)\dots(k)(k+1), \end{cases}$$

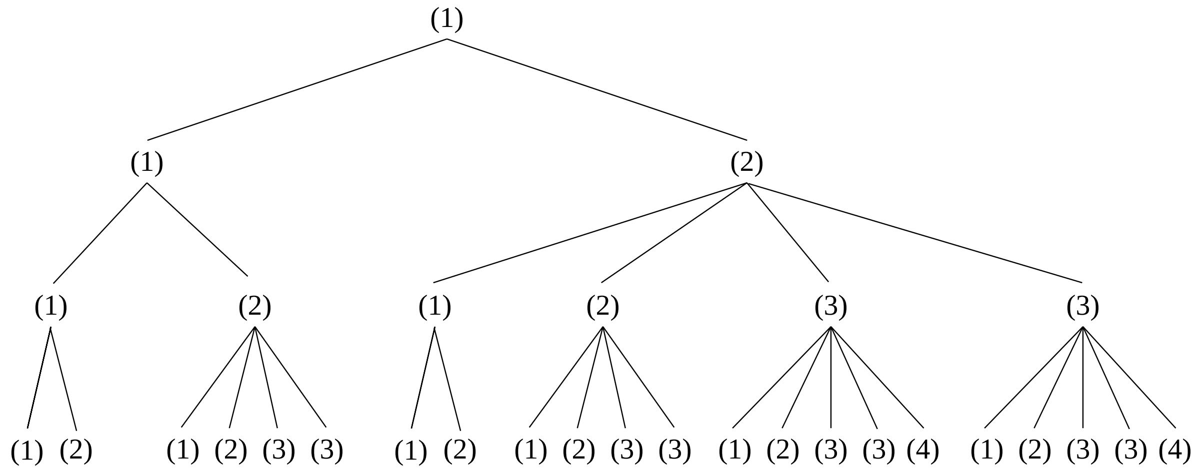

Fig. 1. Partial generating tree of Θ

$$\begin{aligned} S(\Theta) &= (1) + (11) + (12) + (111) + (112) + (121) + (122) + 2(123) + \dots \\ F_\Theta &= z + 2z^2 + 6z^3 + 22z^4 + \dots \end{aligned}$$

3 Algebraicity and rationality

In this section, we study generating trees having the following general form,

$$\Upsilon \begin{cases} (1) \\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}. \end{cases} \tag{1}$$

The words occuring in $S(\Upsilon)$ have λ_i kind of rises from (k) to $(k+i)$ and α_i kind of descents from (k) to $(k-i)$. Using decomposition of paths, we prove the algebraicity of $F(\Upsilon)$, first when *the sequence* (α_i) *is constant* equal to one (Theorem 3), i.e only one kind of descents, and then when *the sequence* (α_i) *is rational* (Theorem 5).

Theorem 3 *The formal power series associated to the generating tree,*

$$\Lambda \begin{cases} (1) \\ (k) \rightsquigarrow (1) \dots (k-1)(k)^{\lambda_0} \dots (k+p)^{\lambda_p}, \end{cases}$$

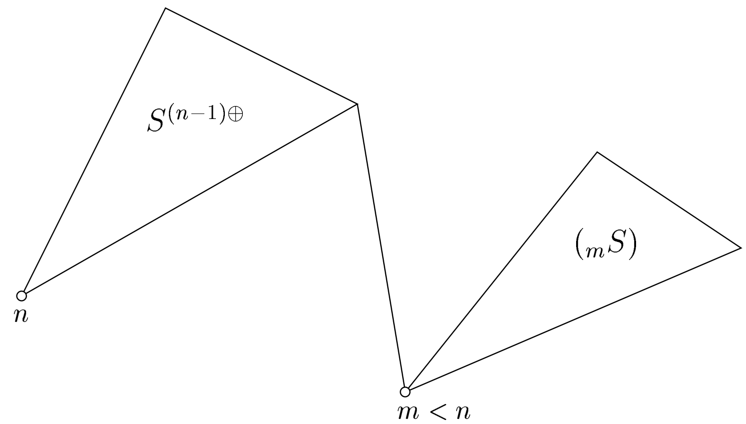

Fig. 2. Decomposition of $({}_nS)$

satisfies the following equation,

$$S(\Lambda) = (1) + (1)\sum_{i=0}^{p} \lambda_i S(\Lambda)^{i\oplus} \prod_{j=0}^{i-1} (\epsilon + S(\Lambda)^{j\oplus}),$$

where $S(\Lambda)^{i\oplus} = (S(\Lambda)^{(i-1)\oplus})^{\oplus}$ *and* $S(\Lambda)^{0\oplus} = S(\Lambda)$.

Remark The Catalan generating tree Γ corresponds to the sequence $\lambda_i = (1, 1, 0, \ldots)$ and $S(\Gamma) = (1) + S(\Gamma) + S(\Gamma)^{\oplus}(\epsilon + S(\Gamma))$.

PROOF. The proof is deduced from the recursive decomposition of the paths in the generating tree Λ. We need to define $({}_nS)$ as the formal sum of the paths in the generating tree obtained by replacing the axiom of Λ by n,

$$\begin{cases} (n) \\ (k) \rightsquigarrow (1)\ldots(k-1)(k)^{\lambda_0}\ldots(k+p)^{\lambda_p}. \end{cases}$$

$({}_1S) = S(\Lambda)$ is noted S for simplifing. We can write recursively $({}_nS)$ using the following non ambigous decomposition (see Fig. 2).
Let $w \neq n$ be a non trivial path of $({}_nS)$, then w can be written $w = nu$,

- if each letter of u is $\geq n$ then $u = v^{(n-1)\oplus}$ where v is a path of Λ,
- if not, let m the first letter $< n$ in u, so u can be writen $v_1^{(n-1)\oplus} v_2$ where v_1 is a path of Λ and v_2 is a path of $({}_mS)$, v_2 being the longest suffix of u beginning by m.

$$({}_nS) = S^{(n-1)\oplus}(\epsilon + \sum_{m=1}^{n-1} ({}_mS)).$$

It is easy to see that,

$$({}_{n+1}S) = S^{n\oplus} \prod_{m=0}^{n-1} (\epsilon + S^{m\oplus}).$$

The following equality concludes the proof,

$$S = (1) + (1) \sum_{n=0}^{p} \lambda_n ({}_{n+1}S). \qquad \square$$

Corollary 4 *The generating function F_Λ of the generating tree Λ is algebraic and satisfies,*

$$F_\Lambda = z(1 + \sum_{n=0}^{p} \lambda_n F_\Lambda (1 + F_\Lambda)^n).$$

All the algebraic generating function given in the small catalog of ECO-systems of [1] can be deduced from the previous theorem. For instance, Motzkin numbers correspond to the sequence $\lambda_i = (0, 1, 0, \ldots)$, Schröder numbers correspond to the sequence $\lambda_i = (1, 2, 0, \ldots)$ and Ternary trees correspond to the sequence $\lambda_i = (1, 1, 1, 0, \ldots)$.

For the general case, the difficulty is to deal with the α_i kind of descents from (k) to $(k - i)$.

Theorem 5 *The rationality of the sequence (α_i) implies the algebraicity of F_Υ.*

PROOF. We begin by giving the different equations obtained from the recursive decomposition of the paths in the generating tree Υ. As in the proof of Theorem 3, we need to define $({}_nS_i)$ as the formal sum of the paths ending by i in the following generating tree,

$$\begin{cases} (n) \\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \ldots (k-1)^{\alpha_1} (k)^{\lambda_0} \ldots (k+p)^{\lambda_p}, \end{cases}$$

and we note (S_i) for $({}_1S_i)$. Applying the same non ambigous decomposition as in the proof of Theorem 3 and considering the last letter of each factor (see Fig. 3), we get,

$$({}_{n+1}S_i) = (S_{i-n})^{n\oplus} + \sum_{m=1}^{n} \sum_{j \geq 1} \alpha_{j+n-m} (S_j)^{n\oplus} ({}_mS_i). \tag{2}$$

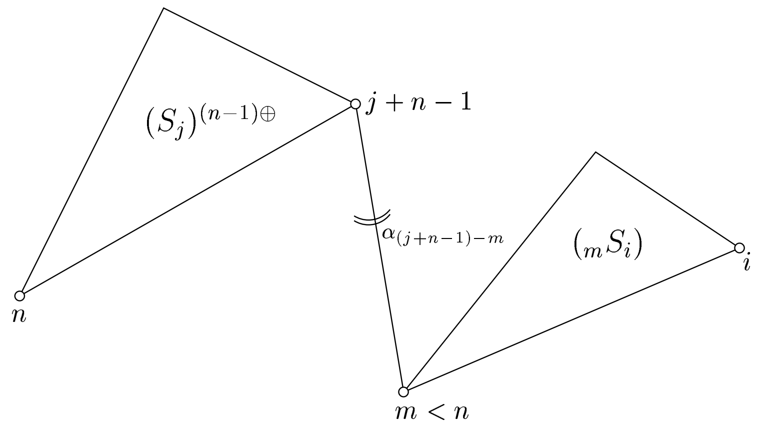

Fig. 3. Decomposition of $({}_nS_i)$

We note $F = F_\Upsilon$ for short, the generating function of Υ, $({}_nF_i)$ the generating functions of $({}_nS_i)$ and $({}_nF_i) = 0$ when $i \leq 0$ or $n \leq 0$. We define also $F_i = ({}_1F_i)$ for $i \geq 1$ and $F_i = 0$ for $i \leq 0$. We note $G_i = \sum_{j\geq 1} \alpha_{i+j-1} F_j$ for $1 \leq i \leq p$ and $PG(n) = \sum_{n=i_1+\ldots+i_h} G_{i_1} \ldots G_{i_h}$ for $n \geq 0$ with the convention that $PG(0) = 1$.

Remark $PG(n)$ is a polynomial in $G_1, \ldots, G_p$.

From the equation (2), we obtain,

$$\begin{aligned}({}_{n+1}F_i) &= F_{i-n} + \sum_{m=1}^{n} \sum_{j\geq 1} \alpha_{j+n-m} F_j ({}_mF_i) \\ &= F_{i-n} + \sum_{m=1}^{n} G_{n-m+1} ({}_mF_i)\end{aligned}$$

Lemma 6 $$({}_{n+1}F_i) = \sum_{m=0}^{n} PG(n-m) F_{i-m}$$

PROOF. By induction under n.

$$\begin{aligned}({}_{n+1}F_i) &= F_{i-n} + \sum_{m=1}^{n} G_{n-m+1} \sum_{j=0}^{m-1} PG(m-1-j) F_{i-j} \\ &= F_{i-n} + \sum_{m=0}^{n-1} F_{i-m} \sum_{j=0}^{n-m-1} G_{n-m-j} PG(j) \\ &= F_{i-n} + \sum_{m=0}^{n-1} F_{i-m} PG(n-m)\end{aligned}$$

$$= \sum_{m=0}^{n} PG(n-m) F_{i-m}. \qquad \square$$

For $t \geq 2$, decomposing F_t according to the first rise, gives,

$$F_t = z \sum_{m=0}^{p} \lambda_m ({}_{m+1}F_t),$$

which can be written using lemma 6,

$$\begin{aligned} F_t &= z \sum_{m=0}^{p} \lambda_m \sum_{i=0}^{m} PG(m-i) F_{t-i} \\ &= z \sum_{j=0}^{p} F_{t-j} \sum_{i=j}^{p} \lambda_i PG(i-j) \\ &= z F_t \sum_{i=0}^{p} \lambda_i PG(i) + z \sum_{j=1}^{p} F_{t-j} \sum_{i=j}^{p} \lambda_i PG(i-j). \end{aligned}$$

For $t = 1$, we obtain,

$$F_1 = z + z \sum_{i=0}^{p} \lambda_i ({}_{i+1}F_1).$$

Let $u_j = \sum_{i=j}^{p} \lambda_i PG(i-j)$, we have the following system,

$$\begin{cases} F_1 = \frac{z}{1-zu_0} \\ \\ F_t = \frac{z}{1-zu_0} \sum_{i=1}^{p} zu_i F_{t-i}. \end{cases}$$

Let M be the following p by p matrix (whose entries are rational functions in $G_1, \ldots, G_p$),

$$M = \begin{pmatrix} \frac{zu_1}{1-zu_0} & \frac{zu_2}{1-zu_0} & \ldots & \frac{zu_{p-1}}{1-zu_0} & \frac{zu_p}{1-zu_0} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix} = M \begin{pmatrix} F_{t-1} \\ F_{t-2} \\ \vdots \\ F_{t-p} \end{pmatrix} = M^{t-1} \begin{pmatrix} F_1 = \frac{z}{1-zu_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Lemma 7 *The rationality of the sequence (α_i) implies the algebraicity of the sequence (G_i).*

PROOF. Supposing that the sequence (α_i) is rational means that there exists two polynomials P and Q such that $\sum_{i\geq 1} \alpha_i z^{i-1} = \frac{P(z)}{Q(z)}$, with $Q(0) \neq 0$. Thus we have $\sum_{i\geq 1} \alpha_i M^{i-1} = P(M)Q(M)^{-1}$, because $Q(M)$ is invertible. Indeed, decomposing $Q(z)$ in $\mathbb{C}$ leads to $Q(z) = c\prod_{i=1}^{deg(Q)}(z-a_i)$, so that $Det(Q(M)) = c\prod_{i=1}^{deg(Q)} Det(M - a_i I)$, which is obviously nonnull by computing,

$$Det(M - aI) = (-1)^{p+1}(-a^p + \frac{z}{1 - zu_0}\sum_{m=1}^{p} u_m a^{p-m}).$$

Thus we can write an algebraic system of p equations for $G_1, \dots, G_p$,

$$\begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_p \end{pmatrix} = \sum_i \alpha_i \begin{pmatrix} F_i \\ F_{i-1} \\ \vdots \\ F_{i-(p-1)} \end{pmatrix} = \sum_i \alpha_i M^{i-1} \begin{pmatrix} \frac{z}{1-zu_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = P(M)Q(M)^{-1} \begin{pmatrix} \frac{z}{1-zu_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The Jacobien of this system is equal to the identity for $z = 0$ so the G_i are algebraic functions of z. □

Moreover F_t is algebraic for all $t \geq 1$,

$$\begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p} \end{pmatrix} = M^{t-1} \begin{pmatrix} \frac{z}{1-zu_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Finally,

$$\sum_{t\geq 1}\begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p} \end{pmatrix} = (M-1)^{-1}\begin{pmatrix} \frac{z}{1-zu_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so $F = \sum_t F_t$ is algebraic. □

4 Conclusion

Theorem 5 allow us to generalize the results of Flajolet and al [1] concerning finite transformation of Catalan rule. A finite transformation of a rule consists in adding a fixed integer to one (resp. all) succession rule(s). The noncommutative formal power series approach allows us to interpret finite transformations and show that they do not change the algebraicity of the generating function. Moreover, the property of algebraicity does not depend on the choice of the axiom.

Theorem 8 *All finite transformations of the succession rule*

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p},$$

are algebraic when (α_i) *is rational.*

PROOF. For any fixed integer c, we note T_1 (resp. T_2) the following finite transformations of the generating tree Υ defined in section 3 by (1),

$$T_1(\Upsilon)\begin{cases}(1)\\ (k_0) \rightsquigarrow (1)^{\alpha_{k_0-1}} \dots (k_0-1)^{\alpha_1} (k_0)^{\lambda_0} \dots (k_0+p)^{\lambda_p}(c)\\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}, for \ \ k \neq k_0,\end{cases}$$

$$T_2(\Upsilon)\begin{cases}(1)\\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}(c).\end{cases}$$

As in the proofs of Theorem 3 and Theorem 5, we note (S_m) the formal sum of the paths ending by m in the generating tree Υ and $({}_cS)$ be the formal sum of the paths in the generating tree,

$$\begin{cases} (c) \\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}, \end{cases}$$

that is generating tree Υ where the axiom 1 has been replaced by c.
Let F_m is the generating function of (S_m). We have $S(T_1(\Upsilon)) = S(\Upsilon) + (S_m)S(T_1(\Upsilon))$, and deduce $F_{T_1(\Upsilon)} = F_\Upsilon + F_m F_{T_1(\Upsilon)}$, so $F_{T_1(\Upsilon)}$ is algebraic since F_Υ and F_m are algebraic.
Let ${}_cF$ the generating function of $({}_cS)$. We have $S(T_2(\Upsilon)) = S(\Upsilon)({}_cS)^* = \frac{S(\Upsilon)}{1-({}_cS)}$, which concludes the proof. □

A conjecture is to have a similar result when the sequence (α_i) is algebraic as discussed with Cyril Banderier during GASCOM'01.

We give some examples where such succession rules naturally appear (see Fig. 4).

- Diagonally directed convex polyominoes [14] (or fully directed compact animals) are known to be counted according to their number of diagonals by $\frac{1}{2n+1}\binom{3n}{n}$ which count the generating tree,

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)^{k+1}(2)^k \dots (k-1)^3 (k)^2 (k+1). \end{cases}$$

- Another example concerns a new generating tree for Catalan numbers [13].

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)^{2^{k-2}}(2)^{2^{k-3}} \dots (k-2)^2 (k-1)(k+1). \end{cases}$$

  This generating tree generates the partition $\{B_1, \dots B_p\}$ of $[n]$ such that the numbers $1, 2, \dots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \dots, B_p$ are pairwise disjoint. Indeed, let k be the number of isolated points around 1. The 2^{k-1} successors of this configuration are obtained by taking all the subset of $\{\alpha_1 = 1, \alpha_2, \dots, \alpha_k, n+1\}$ containing $n+1$.

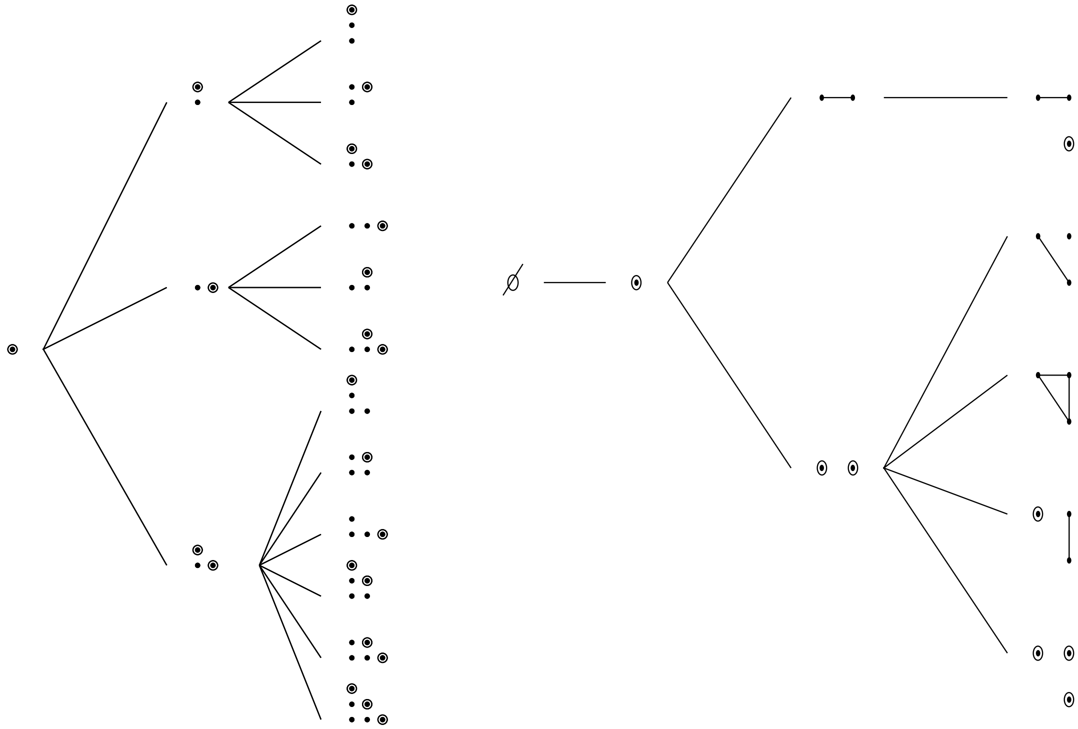

Fig. 4. Generating trees for FDC animals and Catalan blocks

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