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MULTI SCATTERING IN CONSTANT TIME

N. Lichiardopol

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LABORATOIRE I3S: Les Algorithmes / Euclide B – 2000 route des Lucioles – B.P. 121 –
06903 Sophia-Antipolis Cedex, France – Tél. (33) 492 942 701 – Télécopie : (33) 492 942 898
http://www.i3s.unice.fr/I3S/FR/

RÉSUMÉ :

Nous étudions le problème de la multi-diffusion en temps constant. Nous donnons des bornes inférieures sur le temps minimum de multi -diffusion. Nous introduisons les notions de multi arborescence et de multi arborescence compact. Nous donnons deux conjectures et certains résultats partiels

MOTS CLÉS :

temps mimimum de multi diffusion, multi arborescence, graphes de Cayley

ABSTRACT:

In this paper, we study the multi scattering in constant time. More precisely, we give lower bound on the minimum multiscattering time of a graph. We introduce the notion of multi arborescence and compact multi arborescence. We give two conjectures and some partial results.

KEY WORDS :

minimum multi scattering time, multi arborescence, Cayleuygraphs

Multi-scattering in constant time

Nicolas Lichiardopol*
E.S.S.I, Nice Sophia-Antipolis, France

Abstract

In following, we will study the multi-scattering in constant time, a communication problem not much known. More precisely, we will give unpublished results on the minimum multi-scattering time of a graph. The new notions of multi-arborescence and compact multi-arborescence will be very useful. Two conjectures with several partial results will be given.

Keywords : Minimum multi-scattering time; multi-arborescences; scheme of a multi-scattering process; Cayley graphs

1. Introduction, basic notions and notation

We consider networks where initially each node has a distinct message. A global communication (gossiping) in such a network is the spreading process of the information of each node to all other nodes. Of course, a natural goal is to gossip as fast as possible.

According to physical constraints, several models are possible :
If a node x can simultaneously send a message to y and receive an other message from y, the model is said to be full-duplex. Conversely, if a link can be used in one direction only, the model is said to be half-duplex.

When any node x can communicate simultaneously with at most k neighbours the model is said to be k-port and when x can send mesages simultaneously to all its neighbours, the model is said to be D-port.

If each node can send simultaneously at most p messages to a same neighbour, we will have a model p k-port

The combination of these conditions yields a lot of gossiping models.

As far as time of transmission is concerned, two principal models are considered :

- In the constant time model, the transmission of a message frome a node to one of its neighbours « costs » one step, whatever the length of the message.
- In the linear time model, the transmission time of a message depends linearly on the length of the message.

In this paper, we consider a not much known variant of the full-duplex *1* D-port model called multi-scattering in [4] or total exchange in [2] or all to all personalised communication in [5] We will use the first designation and the study will be made in constant time.
As usual the network is modelized by a connected graph.
In the second section, we define more precisely our multi-scattering model by describing its protocol. We define the essential notion of minimum multi-scattering time (*M.M.S.T)* and we recall some known results (in particular two lower bounds of the *M.M.S.T* of a graph).

* *E-mail address :* lichiar@club-internet.fr

In section 3 we give a new lower bound which is sometimes the most accurate one.
In section 4 we give the exact value of the *M.M.S.T* of a graph of minimal degree *1*. We also give two upper bounds in the general case.
In section 5 we introduce two new notions that we call multi-arborescence and compact multi-arborescence.
In section 6 we show how to use the notions defined in section 5 to represent a multi-scattering process.
In section 7 we give two conjectures for new upper bounds of the *M.M.S.T* of a graph. We prove these conjectures in some particular cases, for instance for Cayley graphs.

We consider only connected graphs (unless otherly stated).
For a graph G we note $V(G)$ and $E(G)$ the vertex set and the edge set of G. We note $v(G)=|V(G)|$ and $e(G)=|E(G)|$.
We note $deg_G(x)$ the degree of a vertice x and $d(G)$ is the minimal degree of G.
The distance between two vertices x et y of G is noted $d_G(x,y)$. For $x\in V(G)$ and $W\subseteq V(G)$ we note : $d(x,W)=min(d(x,y),\quad y\in W)$.
We note $D(G)$ the diameter of G.
We note $k(G)$ (resp. $l(G)$) the vertex connectivity of G (resp. edge connectivity).
We will omit the subsccript G when it is clear from the context.
When it is possible we keep the same notations for a digraph.

If G is an undirected graph, $\vec{G}$ is the symetric digraph obtained by replacing each edge xy by the couples (x,y) and (y,x).
For a digraph G, the underlying graph is the undirected graph G' obtained from G by removing all orientations.

If T is an arborescence of root x, the height of T is $h(T)=max(d_T(x,y),\ y\in V(T))$.
For every $i\in\{0,\dots,h(T)\}$, the level $L_i(T)$ is the set of vertices y such that $d_T(x,y)=i$.
For every $i\in\{1,\dots,h(T)\}$, the rank $R_i(T)$ is the set of arcs (y,z) of T such that $y\in L_{i-1}(T)$.

If $\mathcal{G}$ is a group and if S is a generating subset of $\mathcal{G}$ no containing the identity *1* and such that : $x\in S\Rightarrow x^{-1}\in S$, the Cayley graph $G=Cay(\mathcal{G},S)$ is the undirected graph defined by : $V(G)=\mathcal{G}$ and $E(G)=\{\{x,y\};\quad x^{-1}y\in S\}$.
G is a connected regular graph of degree $d=|S|$ and it is known that we have : $k(G)\geq\left\lceil\frac{2(d+1)}{3}\right\rceil$ (see .. .).
For an integer $n\geq 2$, considering the additive group Z_2^n, the hypercube $H(n)$ is the Cayley graph $Cay(Z_2^n,S)$ where $S=\{(1,0,\dots,0),\dots,(0,\dots,0,1)\}$. Clearly $H(n)$ is a regular graph of degree n, having 2^n vertices.

2. Multi-scattering, description, known results

By multi-scattering, we mean the global communication model whose protocol is the following :

G is a connected graph. Initially each vertex of G knows a message distinct from all other messeges. At each step a vertex x may transmet to each of its neighbours, either its original message or a message previously received. We specify that x may send the same message to several neighbours or send distinct messages.

The goal of the process is that each node receive all the messages.
Any way of realizing such a dissemination is called a multi-scattering process (*M.S.P)*.
The realization time $t(M)$ of a *M.S.P M* is the number of steps necessary to fully inform all the vertices.
The minimum multi-scattering time (*M.M.S.T)* of a graph *G* is the smallest of the $t(M)$, where *M* is any *M.S.P* on *G*.
As in paper … , we note $gf_*(1,G)$ the *M.M.S.T* of a graph *G*.

The first obvious result is that for the complete graph K_n, we have : $gf_*(1,K_n)=1$.
Indeed during the first step each vertex being linked to all other vertices can receive all the messages and so after this first step all the vertices are fully informed.

We will sometimes restrict our attention to protocol respecting additional rules :

Rule $\mathbf{R_1}$: Each vertex must receive exactly once each message.

Rule $\mathbf{R_2}$: If initially or after a given step, a neighbour *x* of a vertex *y*, knows some messages unknown by *y*, then at the following step, either all these messages are sent to *y* by other neighbours of *y* or *x* sends one of these messages to *y*.

We can state that for a *M.S.P* on *G* respecting rule R_1, the total number of transmissions of messages is $v(v-1)$.
Indeed, each vertex receive exactly $v-1$ messages and since they are *v* vertices the result is obvious.
It's also obvious that for a *M.S.P* respecting rule R_2, each vertex *x* sends at the first step its original message to all its neighbours.
We can state also that for any graph *G*, there exists a *M.S.P M* respecting rule R_2, such that $t(M)=gf_*(1,G)$.

In . . . Bermond J.C, Perennes S. and Kodate T. give the following results :

Proposition 2.1 *For any graph G, we have :*
$gf_*(1,G)\geq\left\lceil\frac{v(G)-1}{d(G)}\right\rceil$.

Proposition 2.2 *For any graph G, we have :*
$gf_*(1,G)\geq D(G)$.

The proofs of these propositions are easy.
Thus, for a graph *G* of minimal degree *1*, proposition 2.1 yields : $gf_*(1,G)\geq v-1$.
The above propositions yields two lower bounds that we call lower bound 1 and lower bound 2.

In the same paper the authors also prove that lower bound 1 is achieved for a hypercube $H(n)$ and a star-graph $S(k)$.
Supposing that some conjecture is true, they prove that lower bound 1 is also achieved for a toroidal mesh $TM(p)^k$. Recently this conjecture has been proved by the author (see …) and so their assertion is valid.

We finish this section by noting that for a graph *G,* if G_1 is a spanning subgraph of *G* we have : $gf_*(1,G)\leq gf_*(1,G_1)$. Knowing that un arc of $\overrightarrow{G_1}$ is also an arc of $\overrightarrow{G}$, a *M.S.P* on $\overrightarrow{G_1}$ is also a *M.S.P* on *G* and then the result is obvious.

3. Lower bounds

We begin with a general result and its corollaries :

Proposition 3.1. *Let* (V_1,V_2) *be a partition of* $V(G)$ *and let* n *be the number of edges between* V_1 *and* V_2. *Let* V_1', *be the set of vertices of* V_1 *having neighbours in* V_2 *and let* V_2', *be the set of vertices of* V_2 *having neighbours in* V_1. *Then :*

$$gf_*(1,G) \geq max\left(\left\lceil\frac{|V_1|}{n}\right\rceil + max_{x\in V_2-V_2'}\, d(x,V_2'),\ \left\lceil\frac{|V_2|}{n}\right\rceil + max_{x\in V_1-V_1'}\, d(x,V_1')\right)$$

Proof. Consider a multi-scattering process reqiring $gf_*(1,G)$ steps. The $|V_1|$ original messages of V_1, must reach the vertices of V_2 by using the n edges linking the vertices of V_1 and V_2. Since at each step, at most n messages of V_1 can cross the n edges, at least $\left\lceil\frac{|V_1|}{n}\right\rceil$ steps are necessary and some message m, reach V_2 at the step $\left\lceil\frac{|V_1|}{n}\right\rceil$.

This message m is in fact received by vertices of V_2' and since the others vertices of V_2 must receive m, at least $max_{x\in V_2'-V_2}\, d(x,V_2')$ additional steps are necessary and consequently :

$$gf_*(1,G) \geq \left\lceil\frac{|V_1|}{n}\right\rceil + max_{x\in V_2'-V_2}\, d(x,V_2').$$

Similarly we obtain : $gf_*(1,G) \geq \left\lceil\frac{|V_2|}{n}\right\rceil + max_{x\in V_1'-V_1}\, d(x,V_1')$ and the result follows. ¤

It's easy to deduce $gf_*(1,G) \geq \left\lceil\frac{max(|V_1|,|V_2|)}{n}\right\rceil$, which is easer to use but less tight.

Remark that by taking each time V_1 equals to a single vertex, we obtain propositions 2.1 and 2.2 as corollaries.

Another consequence is :

Corollary 3.2. *For every graph* G, *we have :*

$$gf_*(1,G) \geq \left\lceil\frac{v}{2l}\right\rceil.$$

Proof. There is a subset A of E, having l edges, such that $G-A$ is not connected.
It is known that $G-A$ admits two connected compounents V_1 and V_2. Of course, (V_1,V_2) is a partition of V and the number of edges between V_1 and V_2 is exactly l (the edges of A).

We have seen that $gf_*(1,G) \geq \left\lceil\frac{max(|V_1|,|V_2|)}{l}\right\rceil$ and since $max(|V_1|,|V_2|) \geq \frac{v}{2}$, we deduce the announced result. ¤

The lower bound of this corollary, will be called : lower bound 3.

We will now compare lower bounds 1, 2 and 3
In …, F. Harary proves that for any given integgers k, l and d verifying $1 \le k \le l \le d$, there exist graphs G such that: $k(G) = k$, $l(G) = l$ and $d(G) = d$.
In particular there exist graphs G such that $2l(G) < d(G)$, and in this case, lower bound 3 is better than lower bound 1. There are also graphs G such that $d(G) < 2l(G)$ and then lower bound 1 is better than lower bound 3.

There exist graphs G, such that $D(G) < \left\lceil \frac{v(G)}{2l(G)} \right\rceil$. The hypercubes $H(n)$ with $n \ge 7$ are such examples. In this case lower bound 3 is better than bound 2.

On the other hand, there exist graphs G verifying $\left\lceil \frac{v(G)}{2l(G)} \right\rceil < D(G)$. The hypercubes $H(n)$ with $n \le 5$ are such examples and then lower bound 2 is better than lower bound 3.

We give now an example where bound 3 is better that both bounds 1 and 2 :
Let G_n, be the graph of order $2n$, $n \ge 4$ obtained by adding a single edge between two copies of the complete graph K_n.
We have $d(G_n) = n-1$, $D(G_n) = 3$ and $l(G_n) = 1$. Then : $\left\lceil \frac{v(G_n)-1}{d(G_n)} \right\rceil = 3$, $\left\lceil \frac{v(G_n)}{2l(G_n)} \right\rceil = n$ and that prove that lower bound 3 is the best one. In fact, by proposition 3.1 we have $gf_*(1, G_n) \ge n+1$. But it's easy to describe a multiscattering process using exactly $n+1$ steps and then we can state $gf_*(1, G_n) = n+1$.

Consider the graph G_1 of fig. 1

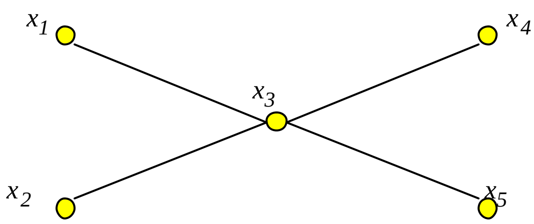

Fig. 1 Graph G_1

We can easily verify that $\left\lceil \frac{v(G_1)-1}{d(G_1)} \right\rceil > D(G_1)$ and therefore in this case, lower bound 1 is better than lower bound 2 . In fact we have $gf_*(1, G_1) = \left\lceil \frac{v-1}{d} \right\rceil = 4$.
For most of the classical networks lower bound 1 is reached.

Let's consider also the graph G_2 of fig. 2 .
It is easy to see that $D(G_2) = 4$ and $\left\lceil \frac{v(G_2)-1}{d(G_2)} \right\rceil = 3$ and consequently lower bound 2 is better that lower bound 1.

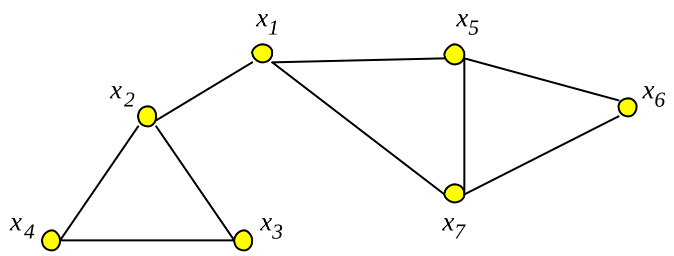

Fig. 2 Graph G_2

Let's consider now the graph G_3 with 10 vertices of fig. 3 .

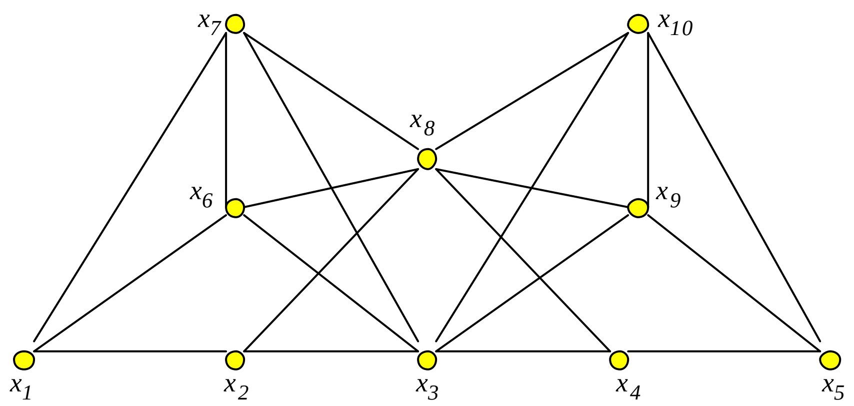

Fig. 3 Graph G_3

Here we have $D(G_3) = 4$, $\left\lceil \frac{v(G_3)-1}{d(G_3)} \right\rceil = 3$ and again bound 2 is sharper that bound 1.

We will see later that for G_2, the lower bound 2 is not reached, while for G_3 this lower bound is reached.

4. M.M.S.T of a graph of minimal degree 1, upper bounds.

We start with two lemmas :

Lemma 4.1. *Let G, be a tree and M be a multiscattering process using T steps.*
Let y be a vertex of G and let x be a neighbour of y. Suppose that at the end of a step t, x knows a message m.
Then, at the end of step t, either y knows m, or x is the only neighbour of y knowing m.

Proof. Suppose that at the end of step t, y doesn't know the message m, but that a neighbour x_1 of y, distinct of x knows this mesage.
Let w , be the originator of m. During the $M.S.P$, m follows two paths starting from$w$, one path ending at x, the other path at x_1 and none using y. But as xy and $x_1 y$ are edges of G, assembling them with both paths, we obtain a cycle, which is impossible in a tree.

Consequently, if y doesn't know m, x is the only neighbour of y knowing m and the result is proved. ¤

We need also :

Lemma 4.2. *Let M, be a moltiscattering process on a tree G, using T steps and respecting rule* R_2
For some edge xy of G, if y receives a message from x at step t, where $2 \leq t \leq T$, then y has also a message from x at step $t-1$.

Proof. The assertion is obvious for $t = 2$.
Suppose that the assertion is true up to step $t-1$, $3 \leq t \leq T$ and let's study for t.
Suppose then, that the vertex y receives a message a from x at step t. Then, x has received a at a step $t_1 \leq t-1$.
If $t_1 < t-1$, x has a at the end of step $t-2$, y doesn't has a at the end of the same step, then by the previous lemma, x is the only neighbour of y, knowing a at the end of step $t-2$. But as this *M.S.P* works according to rule, x sends a message to y at step $t-1$, and consequently the assertion is verified for t.
If $t_1 = t-1$, x has received a from a neighbour $z \neq y$, at the step $t-1$. By induction hypothesis x has received a message b from z at the step $t-2$. Let w, be the originator of b. It is clear that $w \neq y$.
Suppose at first , that y receives b from a neighbour distinct of x. The itinerary of b from w to y and the itinerary from w to x assembled with the edge xy, will give two distincts paths from w to y, which is impossible in a tree.
Therefore y receives b from x and consequently, at the end of the step $t-2$, x knows the message b but no y. Then, as previously, at the step $t-1$, y receives a message from x (not necessaarily b).
The assertion is again verifyed for t and consequently the assertion is true for every t verifying $2 \leq t \leq T$. ¤

An immediate consequence is :

Corollary 4.3. *If xy is an edge of G and if y receives a message from x at the step $t \geq 2$, y receives a message from x at each of the previous steps.*

Proof. Induction from the previous lemma. ¤

We can now state :

Proposition 4.4. *If G is a tree we have : $gf_*(1, \mathrm{G}) = v-1$.*

Proof. Let M, be a *M.S.P* using $gf_*(1,G)$ steps. During the step $gf_*(1,G)$, a vertex x sends a message to one of its neighbours y.
By the previous corollary, y has received a message from x, at each of the previous steps, and therefore, y has received at least $gf_*(1,G)$ messages of x. As, obviously, y must receive exactly $v-1$ messages we have : $gf_*(1,G) \leq v-1$
On the other hand , since $d = 1$, by proposition 2.1, we have : $gf_*(1,G) \geq v-1$ and so, the result follows. ¤

We can give now an upper bound of the M.M.S.T of any graph :

Proposition 4.5. *For every connected graph we have :* $gf_*(1,\mathrm{G}) \le v(G)-1$.

Proof. We know that any connected graph G admits a spanning tree H, and since $gf_*(1,\mathrm{H}) = v-1$, there is a *M.S.P* using $v-1$ steps. This M.S.P on H, is also a M.S.P on G and consequently we have : $gf_*(1,\mathrm{G}) \le v(G)-1$. ¤

Proposition 4.4, may be generalised as it follows :

Proposition 4.6. *For every graph G of minimal degree 1, we have :* $gf_*(1,G) = v-1$.

Proof. This is an immediate consequence of propositions 4.5 and 2.1 ¤

M being a multi-scattering process on a graph G, for a vertex x and for t verifying $0 \le t \le t(M)$, $M_t(\mathrm{x})$ will be the set of messages known by x after the step t. Now we can give a new upper bound :

Proposition 4.7. *For every graph G, we have :*

$$gf_*(1,G) \le \left\lceil \frac{v(v-1)-2e}{k} \right\rceil + 1.$$

Proof. Let M, be a M.S.P using $T = gf_*(1,G)$ steps and respecting rule R_2.

Suppose at first, that at each step, there are at least k *transmissions of messages*. Since at the first step there are $2e$ transmissions and since the total number of transmissions is $v(v-1)$, we deduce : $2e + (T-1)k \le v(v-1)$, which will imply the result.

Suppose now that there is a step which requires less k transmissions. Let t, be the first of these steps. Let m *be* the number of transmissions during step t and let l be the number of vertices receiving messages during this step.

Then we have : $t \ge 2$ and $1 \le l \le m \le k-1$. Let A, be the set of vertices receiving messages during the step t.

First, we state that for $x, y \in V-A$, we have: $M_{t-1}(x) = M_{t-1}(y)$.

Indeed, since $|A| = l < k$, $G-A$ is a connected graph and so, there is a path : $(z_0 = x, \ldots, z_j = y)$ containing only verices of V/A.

For $0 \le i \le j-1$, since the vertice z_{i+1} doesn't receive message at step t, we have then : $M_{t-1}(z_i) \subseteq M_{t-1}(z_{i+1})$ (otherwise, by rule R_2, z_{i+1} would receive at least a message).

With the same reasoning we obtain : $M_{t-1}(z_{i+1}) \subseteq M_{t-1}(z_i)$ and from both inclusions, we deduce : $M_{t-1}(z_i) = M_{t-1}(z_{i+1})$ and this for $i \in \{0, \ldots, j-1\}$, which implyies $M_{t-1}(x) = M_{t-1}(y)$.

We state now, that for every $x \in V/A$, we have $M_{t-1}(x) = V$.

Indeed, for every vertex $y \in V/A$, we have $y \in M_{t-1}(y)$, hence $y \in M_{t-1}(x)$ and consequently : $V/A \subseteq M_{t-1}(x)$.

Every $z \in A$, has at least $d-l+1$ neighbours contained in V/A and since $d-l+1 \ge 2$, there is a neighbour $y \in V/A$ of z ; since $t \ge 2$, we have $z \in M_{t-1}(y)$, hence $z \in M_{t-1}(x)$ and consequently we have : $A \subseteq M_{t-1}(x)$.

Both inclusions imply : $M_{t-1}(x) = V$.

Suppose that there is a vertex z of A lacking at least $d-l+1$ messages at the end of step $t-1$.

As z has at least $d-l+1$ neighbours belonging to V/A and as at the end of the step $t-1$ all these neighbours had all messages , during the step t, the vertex z will receive at least $d-l+1$ messages .

But as each other elements of A receives at least a message during step t, we have at least $d-l+1+l-1 = d$ transmissions and since $d > m$, this is impossible.

Consequently at the end of step $t-1$, each y of A lacks at most $d-l$ messages. As y has at least $d-l+1$ neighbours in V/A, y receives all the missing messages during the step t. Therefore $T=t$ and then :

$2e+(T-2)k \le v(v-1)-m$, hence $T \le \frac{v(v-1)-2e}{k}+1+1-\frac{m}{k}$, and since $0<1-\frac{m}{k}<1$, we obtain the announced result. ¤

In the following, we call upper bound 1 the bound of proposition 4.6 and upper bound 2 that of proposition 4.7.

For a graph G with $k \ge \left[\frac{v}{2}\right]+1$, we have $\left\lceil\frac{v(v-1)-2e}{k}\right\rceil+1<v-1$, which means that upper bound 2 is better than upper bound 1, but if $k \le \left[\frac{v}{2}\right]-1$, upper bound 1 is better than upper bound 2.

Let's also note that for a complete graph and even for others graphs having a great size, the upper bound 2 is reached.

5. Multi-arborescences, compact multi-arborescences

In order to visualize multiscattering processes, let us introduce a new notion :

Definition 5.1. *A multi-arborescence (M.A) is a triplet $W=(V,\mathsf{T},q)$ where V is a finite set called vertex set of W, T is an arborescence and q is a surjective map from $V(\mathsf{T})$ to V. The root of W is $q(r)$ where r is the root of*

It's obvious that if q is a one to one map, we have a classical arborescence.

For instance, the set $V=\{x_1,x_2,x_3,x_4\}$, the arborescence T of fig. 4, the surjective map q defined by : $q(A_1)=x_1$, $q(A_2)=q(A_6)=q(A_8)=x_2$, $q(A_3)=q(A_7)=x_3$ and $q(A_4)=q(A_5)=x_4$ define a *M.A* W of root x_1.

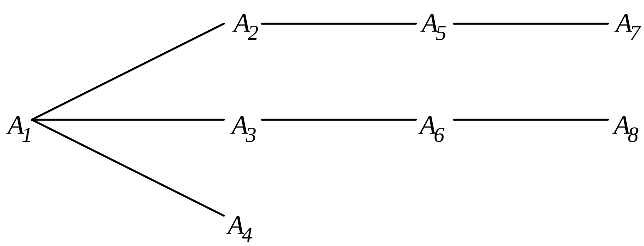

Fig. 4 Arborescence T

In order to simplifynotation and drawing, in the following we represent such a multi-arborescence W as in fig. 5, that is, we will directly replace every vertex $u \in V(\mathsf{T})$ by $q(u)$.

We will now intrduce some notations and definitions for a multi-arborescence $W=(V,T,q)$.
Every arc (u,v) of T, yields an arc of W. If $q(u)=q(v)$ the corresponding arc is said to be trivial otherwise non trivial.
The height $h(W)$ of W will be the height of T.
For every $i \in \{0,\dots,h(W)\}$ the level $L_i(W)$ of W will be $q(L_i(\mathsf{T}))$. Remark that $i \ne j$ doesn't imply

$L_i(W) \cap L_j(W) = \varnothing$.
Similarly the rank $R_i(W)$ of W will be the set of arcs $(q(u), q(v))$ with $(u,v) \in R_i(\mathrm{T})$. Once again $i \neq j$ doesn't imply $R_i(W) \cap R_j(W) = \varnothing$. We will note $R_i'(W)$ the subset of non trivial arcs of $R_i(W)$. At last we will note $R(W)$ the set of all arcs of W and $R'(W)$ will be the set of non trivial arcs of W.

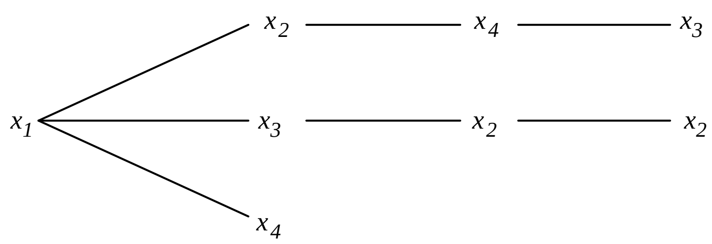

Fig. 5 Multi-arborescence W

For any multi-arborescence $W = (V, \mathrm{T}, q)$, we note $U(W)$ the undirected graph with vertex set V and such that there is an edge between x and y iff either (x,y) or (y,x) is a non trivial arc of W. Note that $U(W)$ is a connected graph but not necessarly a tree. For that we give :

Definition 5.2. *A compact multi-arborescence is a multi-arborescence* $W = (V, \mathrm{T}, q)$ *such that for every* $x \in V$, *the elements of* $q^{-1}(x)$ *form a directed path in* T *(possibly reduced at a single vertex).*

A compact multi-arborescence will be often abbreviated by *C.M.A* . We can state :

Proposition 5.3. *If* $W = (V, \mathrm{T}, q)$ *is a compact multi-arborescence, then* $U(W)$ *is a tree.*

Proof. Suppose that $U(W)$ is not a tree. Then there is a cycle $(z_1, \ldots, z_l, z_{l+1} = z_1)$ in $U(W)$ and consequently there is a sequence : $s = u_1', u_2, u_2', \ldots, u_l, u_l', u_{l+1}$ of vetices of T such that $u_i \in q^{-1}(z_i)$ for each integer i verifying $2 \le i \le l+1$, $u_i' \in q^{-1}(z_i)$ for $1 \le i \le l$ and such that $u_i' u_{i+1}$ is an edge of T' for $1 \le i \le l$.
Since W is a *C.M.A*, $u_i, u_i' \in q^{-1}(z_i)$ for $2 \le i \le l$ and $u_{l+1}, u_1' \in q^{-1}(z_1)$, there is in T' a path between u_i and u_i' (for each $i \in \{2, \ldots, l\}$) and a path between u_{l+1} and u_1' . Inserting all these paths in the sequence *s*, we obtain a cycle in T' which is impossible as T' is a tree.
Consequently $U(W)$ is a tree. ¤

A spanning multi-arborescence of a graph *G*, is a multi-arborescence with the same vertex set as *G* and whose non trivial arcs are arcs of $\vec{G}$. It's easy to see that in any connected graph *G*, for any $x \in V(G)$, there is a spanning *C.M.A* of *G* rooted at x.

If G is a graph, f an automorphism of G and W a multi-arborescence such that $U(W)$ is a subgraph of G, we define the multi-arborescence $f(W)$ with $V(f(W)) = f(V(W))$, $h(f(W)) = h(W)$ and $R_i(f(W)) = \{(f(a), f(b));\ (a,b) \in R_i(W)\}$ for $1 \le i \le h(W)$. It's clear that if W is rooted at *x*, $f(W)$ is rooted at $f(x)$ and that if W is a *C.M.A*, $f(W)$ will be a *C.M.A*. It 's clear also that if W is a spanning multi-arborescence, $f(W)$ is also a spanning multi-arborescence.

6. Scheme of a multiscattering process. Examples

We will now see how to use multi-arborescences to describe multi-scattering processes.
Let M be a $M.S.P$ on a graph G. For each x of $V(G)$ we define a multi-arborescence W_x rooted at x, in the following way :

- The height of W_x is the number T_x of steps necessary to spread the original message m_x of x.
- The vertices of W_x are those of G.
- For $1 \le i \le T_x$, the non trivial arcs of W_x of rank i are the arcs (u,v) of $\vec{G}$ such that u transmit the
- message m_x to v at step i.
- The trivial arcs of W_x of rank i are the couples (u,u) such that u sends m_x at a step $i_1 \le i$ and sends it again at a step $i_2 \ge i+1$.

Obviously for every $x \in V(G)$ W_x is a spanning $M.A$ of G. It is also obvious that for each given $i \in \{1,..., T(M)\}$, all the $R'_i(W_x)$, $x \in V(G)$ are disjoints.
We will call scheme of the multi-scatering process M, the family of the $M.A$ W_x, $x \in V(G)$. It's easy to see that the maximum height of the W_x, $x \in V(G)$ gives us the time $t(M)$ of M. It' clear that if M respects the rule R_1 this scheme is formed by $C.M.A.$.
Conversely, for a graph G, the construction of a family W_x, $x \in V(G)$ of spanning $M.A$ of G rooted at x such that for each i verifying $1 \le i \le max_{x \in V(G)} h(W_x)$, the sets $R'_i(W_x)$, $x \in V(G)$ are disjoints, yields the scheme of a multi-scattering process M. It's clear also that if all the W_x are $C.M.A$, the $M.S.P$ M respects the rule R_1.

For a graph G, the construction of such a schema gives us an upper bound of the $M.M.S.T$ of G and it is sometimes possible to prove that this upper bound is the $M.M.S.T$ of the graph by using some additional arguments.
For instance fig. 6 describe the scheme of a multi-scattering M in G_2 (see fig. 2) with $t(M) = 5$.
So we have $gf_*(1,G_2) \le 5$. Since $D(G_2) = 4$, we have: $gf_*(1,G_2) \ge 4$.
For $1 \le i \le 7$ let's note m_i the original message of x_i.
We have $d(x_3,x_6) = d(x_4,x_6) = 4$.
Let's consider a minimal multi-scattering process M'. Messages m_3 and m_4 must reach x_6. Either one of these messages follows a path of length > 4 and then $gf_*(1,G_2) > 4$, or both messages follow paths of length 4. In this case, both paths use arc (x_2,x_1) and the vertex x_2 must transmit m_3 and m_4 to x_1. As x_2 may transmit only one message to x_1 per step, again we will have $gf_*(1,G_2) > 4$. Consequently $gf_*(1,G_2) > 4$ and $gf_*(1,G_2) = 5$ follows.

In Fig. 7, we give the scheme of a multiscattering process M in G_3 (see fig. 2) with $t(M) = 4$. Since $D(G_3) = 4$ is a lower bound of $gf_*(1,G_3)$ we deduce $gf_*(1,G_3) = 4$.

Let's consider now the cycle $C_n = (x_0, x_1, \dots, x_{n-1}, x_0)$ of length n and let's put for $j \in N$: $x_j = x_i$ where i is the remainder of the division of j by i.
For $i \in \{0, \dots, n-1\}$, we define the $C.M.A$ W_i of height $\left\lceil \frac{n-1}{2} \right\rceil$ rooted at x_i, in the following way :

For $1 \le m \le \left\lceil \frac{n-1}{2} \right\rceil - 1$, the arcs of W_i of rank m will be the couples $(i+m-1, i+m)$ and $(i-m+1, i-m)$. If n is an odd integer, the arcs of W_i of rank $\left\lceil \frac{n-1}{2} \right\rceil$ will be the couples

$\left(i+\left\lceil\frac{n-1}{2}\right\rceil-1, i+\left[\left\lceil\frac{n-1}{2}\right\rceil\right]\right)$ and $\left(i-\left\lceil\frac{n-1}{2}\right\rceil+1, i-\left\lceil\frac{n-1}{2}\right\rceil\right)$ and if n is even, the only arc of rank $\left\lceil\frac{n-1}{2}\right\rceil=\frac{n}{2}$ will be the couple $\left(i+\frac{n}{2}-1, i+\frac{n}{2}\right)$.

It is no difficult to prove that we have thus defined the scheme of a multiscattering process using exactly $\left\lceil\frac{n-1}{2}\right\rceil$ steps. Then, we can state $gf_*(1,C_n)\leq\left\lceil\frac{n-1}{2}\right\rceil$ and using proposition 2.1, we deduce $gf_*(1,C_n)=\left\lceil\frac{n-1}{2}\right\rceil=\left[\frac{n}{2}\right]$.

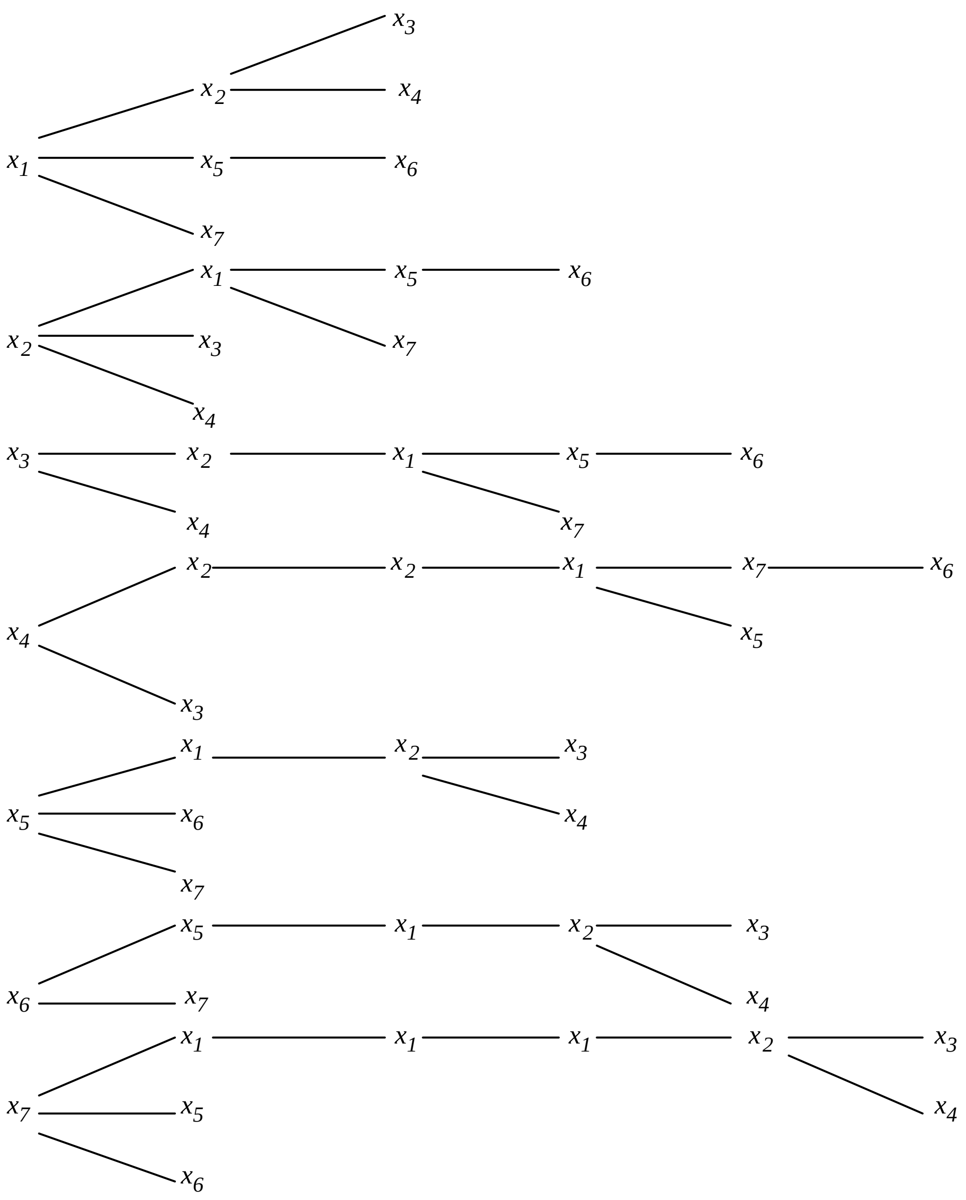

Fig. 6 Scheme of a multiscattering in G_2

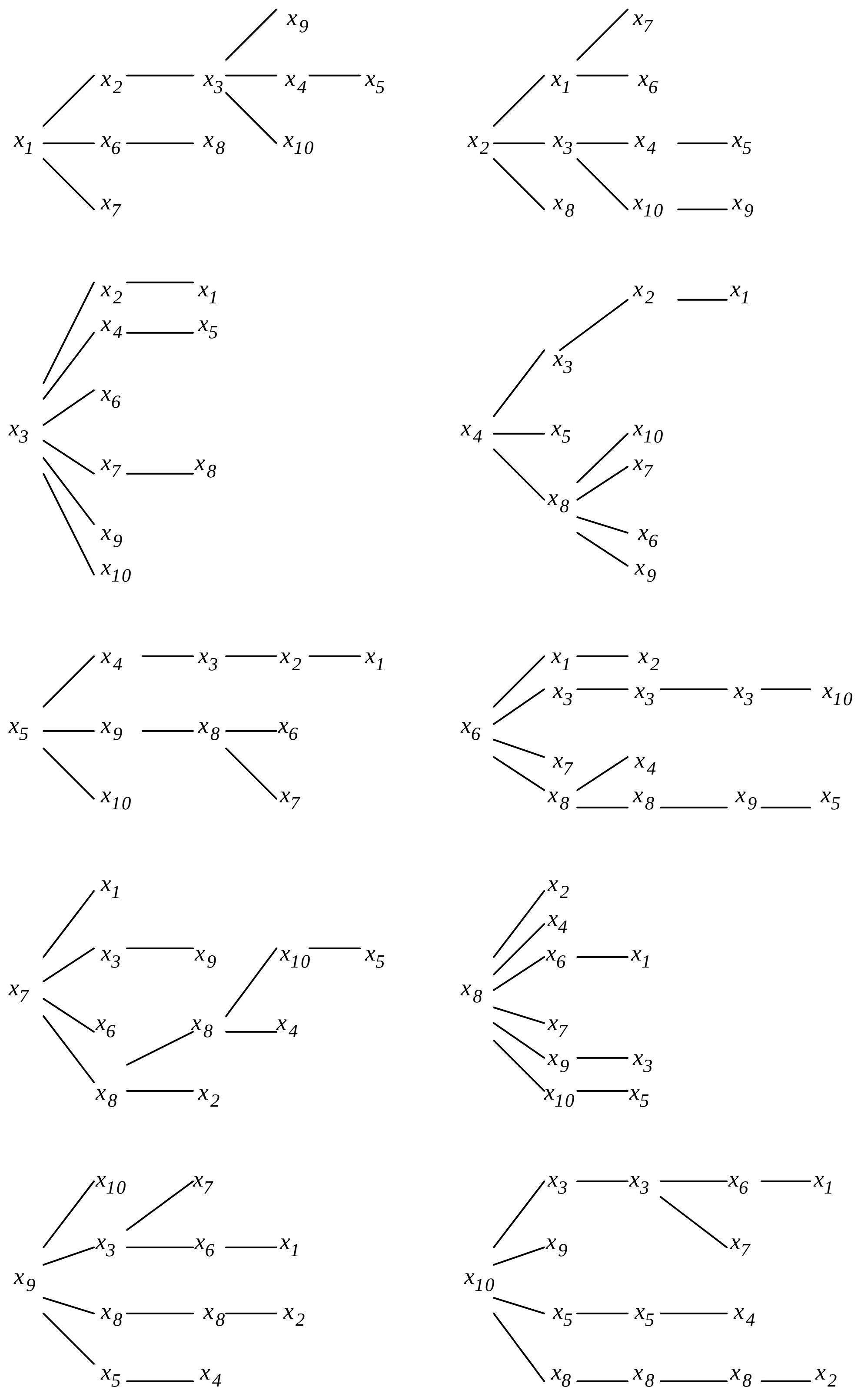

Fig. 7 Scheme of a multiscattering in G_3

7. Two conjectures and partial results

From the known results on classical graphs and multiple computations, we state at first the following conjecture :

Conjecture 7.1 *For any connected graph G we have :* $gf_*(1,G) \le v(G) - d(G)$.

Remark that this conjecture is compatible with lower bounds 1, 2 and 3. That is obvious for lower bounds 1 and 2, let's prove it for lower bound 3, that is, let's prove that for a graph G we have *:* $v(G) - d(G) \ge \left\lceil \frac{v(G)}{2l(G)} \right\rceil$. We will use the following lemma.

Lemma 7.2 *For a graph G verifying* $d \ge \left\lfloor \frac{v}{2} \right\rfloor$, *we have* $l = d$.

Proof. Let A, B be a partition of V with $a \le b$ where $a = |A|$ and $b = |B|$. Clearly $1 \le a \le d$.
Each $x \in A$ has at least $d - a + 1$ neighbours in B and consequently there are at least $a(d - a + 1)$ edges linking A and B. But since $a(d - a + 1) \ge d$ for $1 \le a \le d$, there are at least d edges between A and. B That means $l \ge d$ and since $l \le d$ the result follows. ¤

Now we can prove the above inequality. If $d \le \left\lfloor \frac{v}{2} \right\rfloor - 1$ we have $v - d \ge v - \left\lfloor \frac{v}{2} \right\rfloor + 1 \ge \left\lceil \frac{v}{2} \right\rceil$ and since $\left\lceil \frac{v}{2} \right\rceil \ge \left\lceil \frac{v}{2l} \right\rceil$ the result follows.

If $d \ge \left\lfloor \frac{v}{2} \right\rfloor$, it's easy to prove at firrst that $v - d \ge \frac{v}{2d}$ and that implies $v - d \ge \left\lceil \frac{v}{2d} \right\rceil$ and by the previous lemma $v - d \ge \left\lceil \frac{v}{2l} \right\rceil$.

Obviously this conjecture is true for a graph of minimal degree one and for a complete graph (in both cases we even have equality).

We now prove the conjecture in the particular case of Cayley graphs.
So, let $G = Cay(G\ ,\ S)$ be a Cayley graph with $|S| = d$.
For every $a \in G$, let's note $t_a : G \to G$ defined by $t_a(x) = ax$.
We can easily verify that for each $a \in G$, t_a is a graph automorphism.
We will say that an arc (x, y) of $\vec{G}$ has $s \in S$ as generator if we have $y = xs$.

Lemma 7.3 *Let* W_1 *be a spanning M.A of* $G = Cay(G, S)$*, rooted at 1, such that for every* $i \in \{1, \ldots, h(W_1)\}$ *the arcs of* $R_i'(W_1)$ *have distinct generators.*
Then we have : $gf_*(1,G) \le h(W_1)$.

Proof Let's consider the family $W_a = t_a(W_1)$, $a \in G$.
For each $a \in G$, W_a is a spanning multi-arborescence of G rooted at a and of height $h(W_1)$.
Let's suppose that there exists $i \in \{1, \ldots, h(W_1)\}$ and distinct elements u and v of G such that $R_i'(W_u) \cap R_i'(W_v) \ne \emptyset$.

That means that there exist arcs (x_1, y_1) and (x_2, y_2) of $R_i'(W_1)$ such that :
$(ux_1, uy_1) = (vx_2, vy_2)$ which implies : $x_1^{-1} y_1 = x_2^{-1} y_2$.
That means that both arcs (x_1, y_1) and (x_2, y_2) have the same generator and then by definition of W_1, we must have $x_1 = x_2$ and $y_1 = y_2$ but then we deduce $u = v$, a contradiction.
Consequently for each $i \in \{1, \dots, h(W_1)\}$ the arcs sets $R_i'(W_a)$, $a \in G$ are disjoints.
Then the family W_a, $a \in G$ is the scheme of a multi-scattering process M and we have $t(M) = h(W_1)$ hence $gf_*(1, G) \le h(W_1)$. ¤

We continue by :

Lemma 7.4 *Let* $G = Cay(G, S)$ *be a cayley graph.*
There is a spanning $C.M.A$ W_1 *of* G*, rooted at 1 and verifying :*
(i) $R_1'(W_1) = \{(1,s)\ ;\ s \in S\}$
(ii) For every $i \in \{1, \dots, h(W_1)\}$ *we have :* $R_i'(W_1) \ne \emptyset$
(iii) For every $i \in \{1, \dots, h(W_1)\}$*, the arcs of* $R_i'(W_1)$ *have distincts generators.*

Proof. We first remark that there exist $C.M.A$ verifying conditions (i), (ii) and (iii) and such that $U(W)$ is a subgraph of G.
Indeed, the compact multi-arborescence of height 1 whose arcs are the couples $(1, s)$ with $s \in S$ is such an example.
Let W_1 be such a $C.M.A$ with as many vertices as possible.
Suppose that $V(W_1) \ne G$.
Since G is a connected graph, there is a vertex $y \in G / V(W_1)$ having a neighbour $x \in V(W_1)$.
Then let W_1', be the $C.M.A$ obtained from W_1 by possibly adding copies of x to the copy of x which is the farest from the root 1, until level $h(W_1)$ is reached and by creating a new rank by linking the last copy of x (copy of level $h(W_1)$) with y.
This new $C.M.A$ verifies conditions (i) (ii) and (iii) and since $|V(W_1')| = |V(W_1)| + 1$ this is contradictory with the definition of W_1.
Consequently, we have $V(W_1) = G$ and the assertion is proved since the $C.M.A$ W_1 verifies all requiredconditions. ¤

Now we can state :

Proposition 7.5 *For a Cayley graph* $G = Cay(G, S)$ *we have* $gf_*(1, G) \le v(G) - d(G)$.

Proof. If $d = v - 1$ that is obvious as G is complete.
Suppose $d < v - 1$. Let W_1 be a spanning $C.M.A$ of G rooted at 1 and verifying conditions (i), (ii) and (iii) of the previous lemma. Then $h(W_1) \ge 2$ and since $|R_i'(W_1)| \ge 1$ for $2 \le i \le h(W_1)$, $|R_1(W_1)| = d$ and $|R'(W_1)| = v - 1$ and since the $R_i'(W_1)$, $1 \le i \le h(W_1)$ are disjoint, we deduce : $h(W_1) - 1 \le v - 1 - d$.
Therefore $h(W_1) \le v(G) - d(G)$ and the result follows by lemma 7.3. ¤

We will now prove that this conjecture is true for a graph G such that $d \ge \left[\frac{v}{2}\right] + 2$ (remark that such a graph is always connected).
Several intermediate results are necessary and we start by :

Lemma 7.6 *Let G be a graph with* $d \geq \left[\frac{v}{2}\right] + 2$. *Let* $\{B, A\}$ *be a partition of* V *with* $|A| \leq |B|$

If $|A| \geq 2$, *there exist at least* $v+1$ *edges between* A *and* B.

If $|A| = 1$, *there exist at least* $\left[\frac{v}{2}\right] + 2$ *edges between* A *and* B.

Proof. Let's note : $|A| = a$ and $|B| = b$. It's easy to see that $a \leq d - 1$.
With the same reasonning used for lemmma 7.2, we conclude that there are at least $a(d - a + 1)$ edges between A and $B = V/A$.
If $a = 1$, the assertion is obvious.
For $2 \leq a \leq d - 1$ it's easy to prove that: $a(d - a + 1) \geq 2(d - 1) \geq v + 1$ and so the assertion is proved also for $|A| \geq 2$. ¤

The essential result is :

Lemma 7.7 *Let* G *be a graph with* $d \geq \left[\frac{v}{2}\right] + 2$.
Let $(V_x,\ x \in V)$, *be a family of subsets of* V *not all equal to V, such that for each* $x \in V$, V_x *contains* x *and all its neighbours.*
Let W *be the non empty subset of* V *defined by :* $W = \{x \in V\ ;\ V_x \neq V\}$.
Then there exists a family $(u_x,\ x \in W)$ *of distincts edges of* G *such that for each* $x \in W$, u_x *joins an element of* V_x *to an element of* V/V_x .

Proof. We will use Hall's theorem.
For each $x \in W$, let F_x be the no empty set of edges between V_x and V/V_x .
For a no empty set $S \subseteq W$, let's study the cardinality of $G(S) = \bigcup_{x \in S} F_x$.

If for some element y of S we have $|V/V_y| \geq 2$, by the previous lemma we have $|F_y| \geq v + 1$ and since $F_y \subseteq G(S)$ and $|S| \leq v$, we deduce : $|G(S)| \geq |S|$.
If for every $x \in S$ we have $|V/V_x| = 1$, for each $x \in S$, V/V_x is a singleton $\{a_x\}$, which means $V_x = V/\{a_x\}$.
If the a_x, $x \in S$ are all equal to an element a of V, each vertex $x \in S$ is not a neighbour of a. That implies : $|S| \leq v - deg_G(a)$ and since $v - deg_G(a) \leq deg_G(a)$, we deduce : $|S| \leq deg_G(a)$, that is $|G(S)| \geq |S|$.

If there are vertices y and z such that $a_y \neq a_z$, we have : $|F_y \cup F_z| = |F_y| + |F_z| - |F_y \cap F_z|$ and since $|F_y| \geq \left[\frac{v}{2}\right] + 1$, $|F_z| \geq \left[\frac{v}{2}\right] + 1$ and $|F_y \cap F_z| \leq 1$ (because $F_y \cap F_z = \emptyset$ if yz is not an edge of G and $F_y \cap F_z = \{yz\}$ if yz is an edge of G) we deduce : $|F_y \cup F_z| \geq 2\left[\frac{v}{2}\right] + 1$ which implies $|F_y \cup F_z| \geq v$ and since $F_y \cup F_z \subseteq G(S)$ and $|S| \leq v$, again we deduce : $|G(S)| \geq |S|$.
In conclusion, for every no empty set $S \subseteq W$, we have $|G(S)| \geq |S|$ and then by Halls theorem there is a family $(u_x,\ x \in W)$ of distinct edges with $u_x \in F_x$ for each $x \in W$ and so the result is proved. ¤

We continue by :

Lemma 7.8 *Let* G *be a graph with :* $d(G) \geq \left[\frac{v}{2}\right] + 2.$

There exists a family $(W_x\,,\ \ x \in V)$ *of spanning* $C.M.A$ *of* G *such that :*

(i) *For each* $x \in V$, $R_1(W_x) = \{(x, y)\ ;\ \ xy \in E(G)\}$

(ii) *For each* $x \in V$ *such that* $h(W_x) \geq 2$, we have $|R_i'(W_x)| = 1$ *for every* $i \in \{2, \ldots, h(W_x)\}$.

(iii) *For every integer* i *verifying* $1 \leq i \leq max(h(W_x),\ x \in V)$, *if* y *and* z *are distinct vertices such that* $R_i'(W_y)$ *and* $R_i'(W_z)$ *are defined, we have :* $R_i'(W_y) \cap R_i'(W_x) = \varnothing$.

Proof. There are families $L = (L_x\,,\ \ x \in V)$ of $C.M.A$ such that $U(L_x) = G$ for $x \in V$, verifying conditions (i), (ii) and (iii) and verifying also the condition :

(iv) For every y such that $h(L_y) < max(h(L_x),\ \ x \in V)$, we have $V(L_y) = V$.

Indeed, the family $L' = (L_x'\,,\ \ x \in V)$ where L_x' is the $C.M.A$ of root x and of height 1 with $R_1(L_x') = \{(x, y);\ \ xy \in E(G)\}$ is such an example.

For such a family $L = (L_x\,,\ \ x \in V)$ let's note : $h(L) = max\,(h(L_x);\ \ \ x \in V)$. There is $y \in V$ such that $h(L) = h(L_y)$.

Since $|R'(L_y)| = deg_G(y) + h(L) - 1$ and since $|R'(L_y)| \leq v - 1,$ we deduce :
$h(L) \leq v - deg_G(y)$ hence $h(L) \leq v - d$.

Consequently, there is a family $W = (W_x\,,\ \ x \in V)$ verifying (i), (ii), (iii) and (iv) and such that $h(W)$ is the greatest possible.

Let be $W = \{x \in V\ ;\ \ \ V(W_x) \neq V\}$. By definition of W (condition (iv)) we have :
$W \subseteq \{x \in V\ ;\ \ \ h(W_x) = h(W)\}$.

If $W = \varnothing$, each W_x is a spanning $C.M.A$ verifying (i), (ii) and (iii) and then the lemma is proved.

Suppose $W \neq \varnothing$. By lemma 7.6, there are distinct edges $u_x = a_x b_x$, $x \in W$ with $a_x \in V(W_x)$ and $b_x \in V/V(W_x)$ for each $x \in W$. Of course the couples (a_x, b_x), $x \in W$ are also distincts.

Keeping all the $C.M.A$ W_x with $x \notin W$, and taking for each $x \in W$ the $C.M.A$ W_x' obtained from W_x by adding the couple (a_x, b_x) at rank $h(W_x) + 1 = h(W) + 1$ (see also the proof of lemma 7.4), we obtain a new family $W' = (W_x'\,,\ \ x \in V)$ of $C.M.A$ verifying conditions (i), (ii), (iii) and (iv) and such that $h(W') = h(W) + 1$. That is contradictory with the definition of $h(W)$.

Consequently $W = \varnothing$ and the lemma is proved. ¤

Now we can state :

Proposition 7.9 *For every graph* G *verifying* $d \geq \left[\frac{v}{2}\right] + 2$ *we have :*

$gf_*(1, G) \leq v - d$.

Proof. There is a family $W = (W_x\,,\ \ \ x \in V)$ defined as in lemma 7.8.

Clearly this family is the scheme of a multiscattering process M using $h(W)$ steps and since $h(W) \leq v - d$, the result follows. ¤

For similar reasons, we give also a second conjecture :

Conjecture 7.10 *For every connected graph G, we have :* $gf_*(1,G) \le \left\lceil \frac{v(G)-1}{k(G)} \right\rceil$.

We first remark that here also the conjecture is compatible with lower bounds 1, 2 and 3. This is obvious for lower bounds 1 and 3 and it's a little more difficult for lower bound 2.
Proposition 4.5 shows that this conjecture is true for a graph with vertex connectivity equal to 1 , It is also true for a cycle and for a complete graph (in both cases we even have equality)
This conjecture is probabily more difficult to prove or disprove that conjecture 7.1, but it is also more interesting. If true, this conjecture implies that for any graph G having identical connectivity and minimal degree, we have $gf_*(1,G) = \left\lceil \frac{v-1}{d} \right\rceil$ (which would be a strong result).

We have seen that for a Cayley graph $G = Cay(G,S)$ we have $k(G) \ge \left\lceil \frac{2d+2}{3} \right\rceil$ and therefore G is r-connected for any non zero integer r verifying $3r \le d+3$. We can state :

Proposition 7.11 *Let* $G = Cay(G,S)$ *be a Cayley graph with* $d \ge 3$.

Then : $gf_*(1,G) \le \left\lceil \frac{v+3}{\left\lceil \frac{d+3}{3} \right\rceil} \right\rceil - 2.$

Proof. Let's note $r = \left\lceil \frac{d+3}{3} \right\rceil$. Clearly we have : $3r \le d+3$.
There are $C.M.A$ W of root 1, satisfying :
(i) $U(W)$ is a subgraph of G.
(ii) For every $i \in \{1,\ldots,h(W)\}$, the arcs of $R_i'(W)$ have distincts generators.
(iii) $h(W) \le \left\lceil \frac{v(W)+3}{r} \right\rceil - 3.$
Indeed, the $C.M.A$ L of height 1 with $R_1(L) = \{(1,s);\ s \in S\}$ is such an example.
Let W_1 be such a $C.M.A$ with as many vertices as possible.
If $V(W_1) = G$, W_1 is a spanning $C.M.A$ of G and the result follows by using lemma 7.3.
Let's now consider the case $V(W_1) \ne G$. Since G is connected, there exist arcs of $\vec{G}$ with starting points in $V(W_1)$ and ending points in $G/V(W_1)$.
Let m be the maximum number of arcs of $\vec{G}$ having distinct generators, starting points in $V(W_1)$ and distincts ending points in $G/V(W_1)$. Let $(x_i, x_i s_i)$, $1 \le i \le m$ be m arcs realizing these conditions.
By possibly extending the vertices x_i of $V(W_1)$ and adding the arcs $(x_i, x_i s_i)$ at rank $h(W_1)+1$, we obtain a $C.M.A$ W_1' verifying (i) and (ii) with $h(W_1') = h(W_1)+1$ and $v(W_1') = v(W_1)+m$.
If $m \ge r$, condition (iii) is also verified, which is contradictory with the definition of $v(W_1)$.
Consequently we have $m < r$.
Suppose $v(W_1') < v$. Then $G/V(W_1') \ne \emptyset$ and for any $y \in G/V(W_1')$ and any $s \in S/\{s_1,\ldots,s_m\}$, we have $ys^{-1} \in G/V(W_1)$ (otherwise we obtain a contradiction with the definition of m).
Since $m < r$ and G is r-connected, $G - \{x_1 s_1,\ldots,x_m s_m\}$ is a connected graph and consequently the set W of elements of $G/V(W')$ having at least a neighbour in $V(W_1)$ is non empty.
It's clear also that for arcs (x,xa) with $x \in V(W_1)$ and $xa \in W$, we have $a \in \{s_1,\ldots,s_m\}$.

Let y, be an element of W. Let $x \in V(W_1)$, be a neighbour of y.

Suppose that there exists $b \in \{s_1,\dots,s_m\}$ such that $yb^{-1} \notin V(W_1)$.

By adding to W_1 the arc (x,y) at rank $h(W_1)+1$, and the arcs (y, ys^{-1}), $s \in S/\{s_1,\dots,s_m\} \cup \{b\}$ at rank $h(W_1)+2$, we obtain a $C.M.A$ of height $h(W_1)+2$, having $v(W_1)+d-m+2$ vertices and verifying conditions (i) and (ii). Since $d-m+2 \geq 2r$, condition (iii) is also verified, which is contradictory with the definition of $v(W_1)$.

Furthermore if $m=1$, by adding to W_1 the arc (x,y) at rank $h(W_1)+1$, and the arcs (y, ys^{-1}), $s \in S/\{s_1,\dots,s_m\}$ at rank $h(W_1)+2$, we obtain a $C.M.A$ of height $h(W_1)+2$, having $v(W_1)+d$ vertices, verifying (i), (ii) and since $d \geq 2r$, condition (iii) will be also satisfyied, again contradictory.

Consequently for any $y \in W$ and any $s \in \{s_1,\dots,s_m\}$, we have $ys^{-1} \in V(W_1)$ and moreover $m \geq 2$.

For each $i \in \{1,\dots,m\}$, by replacing the arc $(x_i, x_i s_i)$ by an arc (ys_i^{-1}, y) with $y \in W$ and repeating the previous reasonning, we deduce that $x_i s_i s^{-1} \in G/V(W_1)$ for any $s \in S/\{s_1,\dots,s_m\}$ and that $x_i s_i s^{-1} \in V(W_1)$ for any $s \in \{s_1,\dots,s_m\}$.

Consider now the set $X = W \cup \{x_1 s_1, \dots x_m s_m\}$ of elements of $G/V(W_1)$ having a neighbour at least in $V(W_1)$.

Suppose that for some $y_1 \in X$ there is $z_1 \in X/\{y_1\}$ such that $z_1 \notin \{y_1 s^{-1}\ ;\ s \in S/\{s_1,\dots,s_m\}\}$. Then by adding to the $C.MA$ W_1 the arcs $(y_1 s_1^{-1}, y_1)$, $(z_1 s_2^{-1}, z_1)$ at rank $h(W_1)+1$ and the arcs $(y_1, y_1 s^{-1})$, $s \in S/\{s_1,\dots,s_m\}$ at rank $h(W_1)+2$, we obtain a $C.M.A$ of height $h(W_1)+2$, having $v(W_1)+d-m+2$ vertices and verifying (i), (ii) and (iii), which is a contradiction.

Consequently for every $y \in X$, we have: $X \subseteq \{y\} \cup \{ys^{-1}\ ;\ s \in S/\{s_1,\dots,s_m\}\}$, which implies that two distinct elements of X are linked.

Let's fix a vertex $y \in X$ et supppose that : $X \neq \{y\} \cup \{ys^{-1}\ ;\ s \in S/\{s_1,\dots,s_m\}\}$

It's obvious that the set X disconnects the graph G, consequently we have : $|F| \geq k$ which implies : $\left|\{y\} \cup \{ys^{-1}\ ;\ s \in S/\{s_1,\dots,s_m\}\}\right| \geq k+1$.

By adding to W_1 the arc (ys_1^{-1}, y) at rank $h(W_1)+1$ and the arcs (y, ys^{-1}), $s \in S/\{s_1,\dots,s_m\}$ at rank $h(W_1)+2$, we obtain a $C.M.A$ W_1'' of height $h(W_1)+2$, having at least $v(W_1)+k+1$ vertices and verifying (i) and (ii). It's no difficult to prove that $k+1 \geq 2r$ and so, condition (iii) is also verified, which is contradictory with the definition of $h(W_1)$.

Consequently : $F = \{y\} \cup \{ys^{-1};\ s \in S/\{s_1,\dots,s_m\}\}$.

Let G_1 be the subgraph of G generated by $V(W_1'') = V(W_1) \cup F$.

Clearly for every $x \in V(W_1'')$, wehave $deg_{G_1}(x) = deg_G(x)$, which implies $V(W_1'') = G$ and since $h(W_1) \leq \left\lceil \frac{v(W_1)+3}{r} \right\rceil - 3$, $h(W_1') = h(W_1)+2$ and $d-m+1 \geq r$, we deduce without difficulty :

$h(W_1'') \leq \left\lceil \frac{v+3}{r} \right\rceil - 2$. The result follows by lemma 7.3. ¤

This proposition doesn't prove our conjecture, but it gives an interesting upper bound for a Cayley graph.

We will prove now that conjecture 7.10 is true for some Cayley graphs, more precisely :

Proposition 7.12 *Let* $G = Cay(G,S)$ *be a Caley graph, with* G *an abelian group and* $d = 3$.
Then : $gf_*(1,G) = \left\lceil \frac{v-1}{3} \right\rceil$.

Proof. It's clear that either the three elements of S are of order 2 in G, or exactly one element of S is of order 2 in G **.**
In the first alternative, as G is either the complete graph K_4, or the hypercube $H(3)$, the result is obvious.
Let's consider the second alternative.
We can note $S = \{a, b, b^{-1}\}$ with $a = a^{-1}$ and $b \neq b^{-1}$. $|G|$ being even, we can write $v = 2n$.
Let's note $r = \left\lceil \frac{2n-1}{3} \right\rceil$. We now dinstinguish two cases wether the family (a, b) is free or not.
Case 1. The family (a, b) is free.
Necessary b is of order n and we have : $G = \{1, b, \ldots, b^{n-1}, a, ab, \ldots, ab^{n-1}\}$.
First, suppose that 3 is a divisor of $2n-1$ (which means $r = \frac{2n-1}{3}$)
Let A_i, $0 \le i \le r$ be the sequence of vertex sets defined by :
- $A_0 = \{1\}$.
- $A_i = \left\{ ab^{i-1}, b^i, b^{-\frac{i+1}{2}} \right\}$ for every odd integer i verifying $1 \le i \le r-2$.
- $A_i = \left\{ ab^{i-1}, b^i, ab^{-\frac{i}{2}} \right\}$ for every even integer i verifying $2 \le i \le r-1$.
- $A_r = \left\{ ab^{r-1}, b^r, ab^{-\frac{r+1}{2}} \right\}$.

Let's prove that : $\bigcup_{0 \le i \le r} A_i = G$.
Let x be an element of G and first, suppose that we have $x = b^j$ with $0 \le j \le n-1$.
For $0 \le j \le r$, clearly we have: $x \in A_j$.

Suppose now $r+1 \le j \le n-1$. We have $b^j = b^{-(n-j)} = b^{-\frac{2(n-j)-1+1}{2}}$. We can easily prove that : $1 \le 2(n-j)-1 \le r-2$. Therefore $A_{2(n-j)-1}$ is well defined and clearly it contains x.
Suppose now that we have $x = ab^j$ with $0 \le j \le n-1$.
If $0 \le j \le r-1$, we have $x \in A_{j+1}$.

For $j = r$, we have $x = ab^{-(n-r)} = ab^{-\frac{r+1}{2}}$ and therefore $x \in A_r$.

For $r+1 \le j \le n-1$, we have $x = ab^{-\frac{2n-2j}{2}}$. It's easy to verify that : $2 \le 2n-2j \le r$. Therefore A_{2n-2j} is well defined and it contains x.
In all cases we have $x \in \bigcup_{0 \le i \le r} A_i$ and consequently $\bigcup_{0 \le i \le r} A_i = G$.
We now define sequence E_i, $1 \le i \le r$ of couples by :

- $E_1 = \{(1,a),(1,b),(1,b^{-1})\}$.
- For any odd integer i with $3 \le i \le r-2$, $E_i = \left\{ \left(b^{i-1}, ab^{i-1}\right), \left(b^{i-1}, b^{i}\right), \left(b^{-\frac{i-1}{2}}, b^{-\frac{i+1}{2}}\right) \right\}$.
- For an even integer i with $2 \le i \le r-1$, $E_i = \left\{ \left(b^{i-1}, ab^{i-1}\right), \left(b^{i-1}, b^{i}\right), \left(ab^{-\frac{i-2}{2}}, ab^{-\frac{i}{2}}\right) \right\}$.
- $E_r = \left\{ \left(b^{r-1}, ab^{r-1}\right), \left(b^{r-1}, b^{r}\right), \left(ab^{-\frac{r-1}{2}}, ab^{-\frac{r+1}{2}}\right) \right\}$.

Note that each E_i is a set of arcs of $\vec{G}$ having distinct generators and linking $A_0 \cup \ldots \cup A_{i-1}$ to A_i. Then we can easily construct a spanning multi-arborescence W_1 of G, rooted at 1 of height r such that $R'_i(W_1) = E_i$ for $1 \le i \le r$.

By lemma 7.3 and by proposition 2.1, we deduce : $gf_*(1,G) = \left\lceil \frac{v-1}{3} \right\rceil$.

Now, suppose that 3 is not a divisor of $2n-1$.
Let B_i, $0 \le i \le r$ be the sequence of vertex sets defined by :
- $B_0 = \{1\}$.
- $B_i = \left\{ ab^{i-1}, b^{i}, b^{-\frac{i+1}{2}} \right\}$ for every odd integer i verifying $1 \le i \le r$.
- $B_i = \left\{ ab^{i-1}, b^{i}, ab^{-\frac{i}{2}} \right\}$ for every even integer verifying $2 \le i \le r$.

As previously, let's prove that $\bigcup_{0 \le i \le r} B_i = G$.

Let x be an element of G and first, suppose that we have $x = b^{j}$ with $0 \le j \le n-1$.
If $0 \le j \le r$, clearly we have : $x \in B_j$.

Suppose now $r+1 \le j \le n-1$. We have $b^{j} = b^{-(n-j)} = b^{-\frac{2(n-j)-1+1}{2}}$. We can easily prove that : $1 \le 2(n-j)-1 \le r$. So, $B_{2(n-j)-1}$ is well defined and it contains x.

Suppose now that we have $x = ab^{j}$ with $0 \le j \le n-1$.
For $0 \le j \le r-1$, it's easy to see that $x \in B_{j+1}$.
For $r \le j \le n-1$, it's also easy to prove that $B_{2(n-j)}$ is well defined and that it contains x.
In all cases we have $x \in \bigcup_{0 \le i \le r} B_i$ and consequently $\bigcup_{0 \le i \le r} B_i = G$.

With analogous considerations we deduce again : $gf_*(1,G) = \left\lceil \frac{v-1}{3} \right\rceil$.

Case 2. (a,b) is not a free family.

Then G is a cyclic group having b as generator and so, $G = \{1, b, \ldots, b^{2n-1}\}$. Furthermore $a = b^{n}$.

We keep the same sequence of sets B_i, $0 \le i \le r$ previously defined.

Let's prove that here also we have : $\bigcup_{0\le i\le r} B_i = G$.

For $x\in G$, there is $j\in\{0,\ldots,2n-1\}$ such that $x=b^j$.

For $0\le j\le r$ we have $x\in B_j$.

For $r+1\le j\le n-1$ we have $x=ab^{-\frac{2n-2j}{2}}$. We can easily prove that $2\le 2n-2j\le r$. So B_{2n-2j} is well defined and clearly $x\in B_{2n-2j}$.

For $n\le j\le n+r-1$ we have $x=ab^{j-n}$. It's easy to prove that B_{j-n+1} is well defined and then clearly : $x\in B_{j-n+1}$.

For $n+r\le j\le 2n-1$ we have $x=b^{-\frac{2(2n-j)-1+1}{2}}$. We can prove that $B_{2(2n-j)-1}$ is well defined and then $x\in B_{2(2n-j)-1}$.

In all cases we have $x\in \bigcup_{0\le i\le r} B_i$ and consequently $\bigcup_{0\le i\le r} B_i = G$.

Now, with analogous reasonnings, we deduce $gf_*(1,G)=\left\lceil \frac{v-1}{3}\right\rceil$. ¤

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