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# BROADCAST TIME AND CONNECTIVITY

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RÉSUMÉ :

Nous donnons une borne superieure sur le temps de diffusion d'un graphe en fonction de sa connectivité, puis une nouvelle condition suffisante pour qu'un graphe ait un temps de diffusion minimal.

MOTS CLÉS :

temps de diffusion minimal, connectivité

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ABSTRACT:

First we give an upper bound on the broadcast time of a graph, then a new sufficient condition to have a minimal broadcast graph.

KEY WORDS :

connectivity minimum broadcast graph

# Broadcast time and connectivity

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## Abstract

First, we give an upper bound of the broadcast time of a graph, then a new sufficient condition to have a broadcast graph.
This condition will yield numerous broadcast graphs.

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## 1. Introduction, basic results

In this paper we deal with the classical problem of broadcasting.
We recall that the protocol is the following :

At step *0* some node $x$ of a connected graph $G$ knows a message. At step $i$, any node having already received this message, may send it to one of its neighbours with the condition that all calls must use independent edges.

Then the broadcast time $b(x)$ of $x$ is the minimum number of steps necessary to complete broadcasting from $x$.
The broadcast time $b(G)$ of $G$ is the greatest of numbers $b(x)$, where $x$ travels the set $V(G)$ of vertices of $G$.

Let $v$ be the order of $G$. We know that for every $x \in V(G)$ we have $b(x) \geq \lceil log_2 v \rceil$ and so $\lceil log_2 v \rceil$ is a lower bound of the broadcast time of $G$.

A broadcast graph is a graph $G$ such that $b(G) = \lceil log_2 v \rceil$.
We know that the complete graph is a broadcast graph and that they are broadcast graphs with fewer edges (minimum broadcast graphs).
However we don't know a general characterization of broadcast graphs.

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In this paper we obtain a sufficient condition for a graph to be a broadcast graph.This will follow from a new upper bound of the broadcast time of a graph which involves the connectivity of the graph.

Clearly, in a graph $G$, if we dispose of $r+1$ subsets $A_0,\ldots,A_r$ of $V(G)$, with $A_0=\{x\}$, $A_r=V(G)$, $A_i\subset A_{i+1}$ for any $i\in\{0,\ldots,r-1\}$, and such that for each $i\in\{0,\ldots,r-1\}$ there are exactly $|A_{i+1}|-|A_i|$ independent edges between $A_i$ and $A_{i+1}\setminus A_i$, we can define a broadcasting from $x$ using $r$ steps.

We will also need of the following known result :

**Proposition 1.** *Let G be a k-connected graph and let A be a subset of* $V(G)$, *with* $|A|\le|V(G)\setminus A|$. *Then :*

**a)** *If* $|A|\ge k$, *there exist* $k$ *independent edges between A and* $V\setminus A$.

**b)** *If* $|A|<k$, *there exist* $|A|$ *independent edges between A and* $V\setminus A$.

For a proof see [2].

## 2. An upper bouund for the broadcast time.

**Proposition 1.** *Let G be a connected graph and let k be its connectivity. Then :*

$$b(G)\le[\log_2 k]+1+\left\lceil\frac{v-2^{[\log_2 k]+1}}{k}\right\rceil.$$

**Proof.** Let us put $a_{v,k}=[\log_2 k]+1+\left\lceil\frac{v-2^{[\log_2 v]+1}}{k}\right\rceil$.

Consider at first the case $k=1$.
As $G$ is connected, for a proper no empty subset of $V(G)$, there exists a vertex in $V(G)\setminus A$ having a neighbour in $A$. This implies that from any $x\in V(G)$ we can construct a broadcasting scheme using $v-1$ steps. Consequently $b(G)\le v-1$ and as the right hand member of the above inequality is $v-1$, the proof is given.

Now, let us suppose that $k\ge 2$. Let $x$ be a vertex of $G$.
There exists an integer $m$ such that $2^m\le k<2^{m+1}$ and then we have $m=[\log_2 k]$.
For every $i\in\{0,\ldots,m-1\}$ we have $2^i<k$ and $2^i\le k-2^i$ hence $2^i\le v-2^i$. Then by Proposition 1.1 for any set $A$ having $2^i$ vertices, there exists $2^i$ independent edges between $A$ and $V(G)\setminus A$.
Then, it is easy to see that we can construct $m+1$ vertex sets $A_0,\ldots,A_m$ such that $A_0=\{x\}$, $A_0\subset\ldots\subset A_m$, $|A_i|=2^i$ for every $i\in\{0,\ldots,m-1\}$, and such that for each

$i \in \{0,\dots,m-1\}$ there are exactly $|A_{i+1}| - |A_i| = 2^i$ independent edges between $A_i$ and $A_{i+1} \setminus A_i$.

Now, let us suppose at first that $2^m \geq v - 2^m$.
Since $2^m \leq k$, we have $v - 2^m \leq k$ and then by Proposition 1.1, there is $v - 2^m$ independent edges between $A_m$ and $V(G) \setminus A_m$.
So, we have a broadcasting from $x$ using $m+1$ steps. Therefore for any vertex $x$ of $G$ we have $b(x) \leq m+1$ and consequently $b(G) \leq m+1$.
It easy to verify that $-1 < \frac{v - 2^{m+1}}{k} \leq 0$ and consequently $\left\lceil \frac{v - 2^{m+1}}{k} \right\rceil = 0$. We get $a_{v,k} = m+1$ and so the assertion holds.

Let us consider the case $2^m < v - 2^m$.
Since $2^m \leq k$, by Proposition 1.1, there is $2^m$ independent edges linking $A_m$ to $V(G) \setminus A_m$. Let $A'_{m+1}$ be the set of end points of all these edges and let us put $A_{m+1} = A_m \cup A'_{m+1}$.
Then $|V(G) \setminus A_{m+1}| = v - 2^{m+1}$.

Let us put $s = \left\lceil \frac{v - 2^{m+1}}{k} \right\rceil$.
For a vertex set $A$ with $|A| \geq k$, Proposition 1.1 says that if $|V(G) \setminus A| \geq k$ there exists $k$ independent edges between $A$ and $V(G) \setminus A$, and if $|V(G) \setminus A| < k$ there exists $|V(G) \setminus A|$ independent edges between $A$ and $V(G) \setminus A$.
This implies that we can construct $s$ sets $A_{m+2}, \dots, A_{m+1+s}$ with , $A_{m+2} \subset \dots \subset A_{m+1+s}$, $A_{m+1+s} = V(G)$ and such that for $m+1 \leq i \leq m+s$ they are $|A_{i+1} \setminus A_i|$ independent edges between $A_i$ and $A_{i+1} \setminus A_i$.
So we have constructed $a_{v,k} + 1$ vertex sets $A_0, \dots, A_{m+1+s}$ defining a broadcasting from $x$ using $a_{v,k}$ steps. So for any vertex $x$ of $G$ we have $b(x) \leq a_{v,k}$ and consequently $b(G) \leq a_{v,k}$.
Thus, the assertion is proved. ¤

**Remark.** This result implies $D(G) \leq a_{v,k}$ where $D(G)$ is the diameter of $G$.

We will give two examples of graphs for which our upper bound is better that other known bounds.
Let $D(G)$ be the maximum degree of $G$. It was shown that $(D(G)-1)D(G)+1$ is an upper bound for $b(G)$ (see [5]).

For integers $r \geq 3$ and $k \geq 2$ it is easy to prove that there exists a graph $G_{r,k}$ of order *rk* and of connectivity *k*.

It is clear that $\left[log_2 k\right] + 1 + \left\lceil \frac{rk - 2^{[log_2 k]+1}}{k} \right\rceil \leq log_2 k + 1 + r.$

It is clear also that $\left(D(G_{r,k}) - 1\right)D(G_{r,k}) + 1 \geq 2(k-1) + 1.$ Easily we see that for $k \geq r+2$ we have $log_2 k + 1 + r < 2(k-1) + 1.$

This implies $a_{rk,k} < \left(D(\mathrm{G}_{r,k}) - 1\right)D(G_{r,k}) + 1$ and consequently, for the graphs $G_{r,k}$ with $k \geq r+2$, $a_{rk,k}$ is a better upper bound than $\left(D(\mathrm{G}_{r,k}) - 1\right)D(G_{r,k}) + 1$.

The second example suggered by C. Peyrat concern the undirected de Brujin's graphs $UB(d, D)$.

It was shown that $b(UB(d, D)) \leq \frac{(d+1)(D+1)}{2}$ (Bermond & Peyrat's theorem).

It was also proved that for $D \geq 2$ we have $k(UB(d, D)) = 2d - 2.$

Then, by Proposition 2.1, we get $b(UB(3{,}2)) \leq 4$, $b(UB(5{,}2)) \leq 6$, $b(UB(6{,}2)) \leq 6$, and $b(UB(7{,}2)) \leq 7$ and by Bermond & Peyrat's theorem we have $b(UB(3{,}2)) \leq 6$, $b(UB(5{,}2)) \leq 9$, $b(UB(6{,}2)) \leq 10{,}5$ and $b(UB(7{,}2)) \leq 12.$

So, for all these graphs, our upper bound is better than the other, and it is even reached for $UB(3{,}2)$ and $UB(6{,}2)$.

In fact, one can prove that for all the graphs $UB(d, 2)$, our upper bound is better that the one of Bermond and Peyrat., but generally it is not the best (see 4)

## 3. A sufficient condition to have a broadcast graph

For integers $n$ and $r$ with $n \geq 4$ and $1 \leq r \leq n-1$, we consider :

$$a_{n,r} = \left[log_2 r\right] + 1 + \left\lceil \frac{n - 2^{[log_2 r]+1}}{r} \right\rceil.$$

To prove the main theorem we need several results .

**Lemma 1.** If $n \leq 2^{[log_2 r]+1}$ we have $\left\lceil \frac{n - 2^{[log_2 r]+1}}{r} \right\rceil = 0.$

**Proof.** We have $log_2 r \geq \left[log_2 r\right]$, hence $r \geq 2^{[log_2 r]}$. As $n > r$, we have $n > 2^{[log_2 r]}$. Then $n + r > 2 \times 2^{[log_2 r]}$, that is $n + r > 2^{[log_2 r]+1}$ hence $\frac{n - 2^{[log_2 r]+1}}{r} > -1$ and since $\frac{n - 2^{[log_2 r]+1}}{r} \leq 0$, the result follows. ¤

Let $y_n : \{1,\ldots,n-1\} \to N$ be the fonction defined by $y_n(r) = a_{n,r}$. Then :

**Lemma 2.** *For each* $n \geq 4$, $y_n$ is a decreasing fonction.

**Proof.** For $r \in \{1,\ldots,n-2\}$, two cases are possible :

**Case 1 :** $[log_2(r+1)] = [log_2 r]$.
Then we have :

$$y_n(r) = [log_2 r] + 1 + \left\lceil \frac{n - 2^{[log_2 r]+1}}{r} \right\rceil \text{ and } y_n(r+1) = [log_2 r] + 1 + \left\lceil \frac{n - 2^{[log_2 r]+1}}{r+1} \right\rceil .$$

If $n - 2^{[log_2 r]+1} > 0$, easily we deduce $y_n(r+1) \leq y_n(r)$.

If $n - 2^{[log_2 r]+1} \leq 0$, by Lemma 3.1 we have $\left\lceil \frac{n - 2^{[log_2 r]+1}}{r} \right\rceil = 0$ and consequently we have $y_n(r+1) = y_n(r)$.

**Case 2 :** $[log_2(r+1)] = [log_2 r] + 1$.
Then, there exists an integer $s > 0$ such that $r + 1 = 2^s$. We get :

$$y_n(r) = s + \left\lceil \frac{n-2^s}{2^s - 1} \right\rceil = s - 1 + \left\lceil \frac{n-1}{2^s - 1} \right\rceil, \; y_n(r+1) = s + 1 + \left\lceil \frac{n - 2^{s+1}}{2^s} \right\rceil = s - 1 + \left\lceil \frac{n}{2^s} \right\rceil .$$

As $n > 2^s$, we have $\frac{n}{2^s} < \frac{n-1}{2^s - 1}$ and this implies $y_n(r+1) \leq y_n(r)$.

In both cases we have $y_n(r+1) \leq y_n(r)$ and so the assertion holds. ¤

Considering the complete graph $K_n$, by Proposition 2.1 we have $b(K_n) \leq y_n(n-1)$, that is $\lceil log_2 n \rceil \leq y_n(n-1)$. Consequently for $1 \leq r \leq n-1$ we have $\lceil log_2 n \rceil \leq y_n(r)$.

For $n \geq 4$ there exist unique integers $m$ and $a$ with $m \geq 1$ and $1 \leq a \leq 2^m$ such that $n = 2^m + a$. Then, we define $q(n)$ as follows :

- $q(n) = 2^{m-2} + \left[ \frac{a+1}{2} \right]$ if $1 \leq a \leq 2^{m-1}$
- $q(n) = a$ if $2^{m-1} < a \leq 2^m$.

It is easy to see that $1 \leq q(n) \leq n-1$ and we can state :

**Proposition 3.** *For* $q(n) \leq r \leq n-1$, *we have* $a_{n,r} = \lceil log_2 n \rceil$.
*For* $1 \leq r < q(n)$, *we have* $a_{n,r} > \lceil log_2 n \rceil$.

**Proof.** We need only to prove that $a_{n,q(n)} = \lceil log_2 n \rceil$ and $a_{n,q(n)-1} > \lceil log_2 n \rceil$.
We can put $n = 2^m + a$ with $m \geq 1$ and $1 \leq a \leq 2^m$. Severals cases must be studied :

**Case 1** : $1 \leq a \leq 2^{m-1} - 2$. Then, $\lceil log_2 n \rceil = m + 1$ and $q(n) = 2^{m-2} + \left[ \dfrac{a+1}{2} \right]$.

It is easy to prove that $2^{m-2} < q(n) < 2^{m-1}$. Then $[log_2 q(n)] = m - 2,$ hence

$$a_{n,q(n)} = m - 1 + \left\lceil \frac{2^m + a - 2^{m-1}}{2^{m-2} + \left[ \frac{a+1}{2} \right]} \right\rceil \text{ that is } a_{n,q(n)} = m - 1 + \left\lceil \frac{2^{m-1} + a}{2^{m-2} + \left[ \frac{a+1}{2} \right]} \right\rceil .$$

It is easy to prove that $1 < \dfrac{2^{m-1} + a}{2^{m-2} + \left[ \frac{a+1}{2} \right]} \leq 2,$ and then $a_{n,q(n)} = \lceil log_2 n \rceil$.

Elsewhere, we have $q(n) - 1 = 2^{m-2} + \left[ \dfrac{a-1}{2} \right]$ and then :

$$a_{n,q(n)-1} = m - 1 + \left\lceil \frac{2^m + a - 2^{m-1}}{2^{m-2} + \left[ \frac{a-1}{2} \right]} \right\rceil = m - 1 + \left\lceil \frac{2^{m-1} + a}{2^{m-2} + \left[ \frac{a-1}{2} \right]} \right\rceil .$$

It is easy to prove that $\dfrac{2^{m-1} + a}{2^{m-2} + \left[ \frac{a-1}{2} \right]} > 2,$ hence $a_{n,q(n)-1} > m + 1$ that is $a_{n,q(n)-1} > \lceil log_2 n \rceil$.

**Case 2** : $a = 2^{m-1} - 1$. Then, $\lceil log_2 n \rceil = m + 1$ and $q(n) = 2^{m-2} + \left[ \dfrac{a+1}{2} \right] = 2^{m-1}$. We get :

$$a_{n,q(n)} = m + \left\lceil \frac{2^m + 2^{m-1} - 1 - 2^m}{2^{m-1}} \right\rceil = m + 1 = \lceil log_2 n \rceil .$$

Elsewhere we have :

$$a_{n,q(n)-1} = m - 1 + \left\lceil \frac{2^m + 2^{m-1} - 1 - 2^{m-1}}{2^{m-1} - 1} \right\rceil = m - 1 + \left\lceil 2 + \frac{1}{2^{m-1} - 1} \right\rceil = m + 2$$

and then clearly we have $a_{n,q(n)-1} > \lceil log_2 n \rceil$.

**Case 3**: $a = 2^{m-1}$. Then, $\lceil log_2 n \rceil = m + 1$ and $q(n) = 2^{m-2} + \left[ \dfrac{a+1}{2} \right] = 2^{m-1}$. We get :

$$a_{n,q(n)} = m + \left\lceil \frac{2^m + 2^{m-1} - 2^m}{2^{m-1}} \right\rceil = m+1 = \lceil log_2 n \rceil.$$

Elsewhere we have :

$a_{n,q(n)-1} = m-1+\left\lceil \frac{2^m + 2^{m-1} - 2^{m-1}}{2^{m-1}-1} \right\rceil = m-1+\left\lceil \frac{2^m}{2^{m-1}-1} \right\rceil$ and as $\left\lceil \frac{2^m}{2^{m-1}-1} \right\rceil \geq 3$ we get $a_{n,q(n)-1} > \lceil log_2 n \rceil$.

**Case 4** : $2^{m-1} < a < 2^m$. Then, $\lceil log_2 n \rceil = m+1$ and $q(n) = a$. We get :

$$a_{n,q(n)} = m + \left\lceil \frac{2^m + a - 2^m}{a} \right\rceil = m+1 = \lceil log_2 n \rceil.$$

Elsewhere, we have $a_{n,q(n)-1} = m + \left\lceil \frac{2^m + a - 2^m}{a-1} \right\rceil = m + \left\lceil \frac{a}{a-1} \right\rceil$ and since $\frac{a}{a-1} > 1$ we deduce $a_{n,q(n)-1} > \lceil log_2 n \rceil$.

**Case 5** : $a = 2^m$. Then, $\lceil log_2 n \rceil = m+1$ and $q(n) = a = 2^m$. We get :

$$a_{n,q(n)} = m+1+\left\lceil \frac{2^{m+1} - 2^{m+1}}{2^m} \right\rceil = m+1 = \lceil log_2 n \rceil.$$

Elsewhere, we have $a_{n,q(n)-1} = m + \left\lceil \frac{2^{m+1} - 2^m}{2^m - 1} \right\rceil = m+2$ and so $a_{n,q(n)-1} > \lceil log_2 n \rceil$.

So, we have always $a_{n,q(n)} = \lceil log_2 n \rceil$ and $a_{n,q(n)-1} > \lceil log_2 n \rceil$ and consequently the assertion is proved. ¤

Now we can give the main result:

**Theorem 4.** *A graph G of order* $v$ *such that* $k(G) \geq q(v)$ *is a broadcast graph.*

**Proof.** By Proposition 2.1, we have $b(G) \leq a_{v,k(G)}$. By Proposition 3.3, we have $a_{v,k(G)} = \lceil log_2 v \rceil$ and since $b(G) \geq \lceil log_2 v \rceil$, the result follows. ¤

Below we give a table of values of $q(v)$ for $4 \leq v \leq 20.$

| $v$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q(v)$ | 2 | 2 | 2 | 3 | 4 | 3 | 3 | 4 | 4 | 5 | 6 | 7 | 8 | 5 | 5 | 6 | 6 |

It is clear that smaller $\frac{q(v)}{v}$ is, the more interesting values of $q(v)$ are. It is easy to prove that the best values of $q(v)$ are obtained when $v = 2^m + 2$, with $m \geq 2$. Then $q(v) = 2^{m-2} + 1 = \left\lceil \frac{v}{4} \right\rceil$ and so a connectivity $\geq \left\lceil \frac{v}{4} \right\rceil$ yields a broadcast graph.

Finally, we finish by :

**Conjecture 5.** *For any couple* $(v,k)$ *with* $v \geq 3$ *and* $1 \leq k \leq v-1$ *there exists a graph* $G_{v,k}$ *of order* $v$ *and of connectivity* $k$, *such that* $b(G_{v,k}) = a_{v,k}$.

This conjecture is true for the couples $(v,k)$ *with* $k \geq q(v)$.
It is easy to prove that this conjecture is also true for the couples $(v,1)$ (then $G_{v,1}$ will be a chain) and for the couples $(v,2)$ (then $G_{v,2}$ will be a cycle).

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