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UPPER BOUND FOR THE SPAN OF (S,1)-TOTAL LABELLING OF GRAPHS

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Upper bound for the span of (s,1)-total labelling of graphs

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Abstract

Let G be a graph. The degree of a vertex v is denoted by $d_G(v)$ or $d(v)$ if G is clearly understood. The maximum degree of G is denoted by $\Delta(G)$. An $(s,1)$-total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least s in absolute value. The *span* of a $(s,1)$-total labelling is the maximum difference between two labels. The minimum span of a $(s,1)$-total labelling of G is denoted by $\lambda_s^T(G)$.

In [4], it is conjectured that $\lambda_s^T \leq \Delta + 2s - 1$, where Δ is the maximum degree of a vertex in a graph. This is an extension of the Total Colouring Conjecture. It is also shown that $\lambda_s^T \leq 2\Delta + s - 1$ and $\lambda_2^T(G) \leq 6$ if $\Delta(G) \leq 3$ and $\lambda_2^T(G) \leq 8$ if $\Delta(G) \leq 4$.

In this paper, we prove that $\lambda_s^T \leq 2\Delta - \log(\Delta+2) + s - 1 + 2\log(16s-10)$. The proof is an induction based on the maximal cut of a graph. We use the same technique to improve a little bit this result in the case of $(2,1)$-total labelling. We prove that if $\Delta(G) \geq 3$, then $\lambda_2^T(G) \leq 2\Delta(G)$ and that if $\Delta(G) \geq 5$ is odd then $\lambda_2^T(G) \leq 2\Delta(G) - 1$.

1 Introduction

An $(s,1)$-total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least s in absolute value. The *span* of an $(s,1)$-total labelling is the maximum difference between two labels. The minimum span of a $(s,1)$-total labelling of G is denoted by $\lambda_s^T(G)$. Note that a $(1,1)$-total labelling is a total colouring and that $\lambda_1^T(G) = \chi^T - 1$ where χ^T is the total colouring number.

An $(s,1)$-total labelling of a graph G corresponds to an $L(s,1)$-labelling of its incidence graph $I(G)$ which is the bipartite graph defined as follows : $V(I(G)) = V(G) \cup E(G)$ and $ve \in E(I(G))$ if and only if $v \in V(G)$, $e \in E(G)$ and v and e are incident. $L(2,1)$-labellings were first introduced in Griggs and Yeh [3] and $L(s,1)$-labelling have been studied for several class of graphs, for example chordal graphs [1] or planar graphs [5]. The $(2,1)$-total labellings of graphs were first studied by Whittlesey, Georges and Mauro [7] as $L(2,1)$-labellings of incidence graphs. In [4], Havet and Yu investigate $(s,1)$-total labelling for any s. They derive from Brooks and Vizing's Theorems that $\lambda_s^T \leq 2\Delta + s - 1$. Generalizing the Total Colouring Conjecture, they conjecture the following :

Conjecture 1 (Havet and Yu [4])

$$\lambda_s^T \leq \Delta + 2s - 1$$

By the previous result, it suffices to prove the conjecture for $s < \Delta$. Rosenfeld [6] established that $\lambda_1^T(G) \leq 4$ if $\Delta(G) \leq 3$. Havet and Yu completed the proof of the conjecture for $\Delta \leq 3$ by proving that $\lambda_2^T(G) \leq 6$ if $\Delta(G) \leq 3$.

In this paper, we improve Havet and Yu's upper bound by showing $\lambda_s^T \leq 2\Delta - \log(\Delta+2) + s - 1 + 2\log(16s-8)$. The proof is an induction based on the maximal cut of a graph. The idea and tools are presented Section 2 and the proof is given Section 3. Finally using the same technique to improve a little bit the result in the case of $(2,1)$-total labelling. We prove that if $\Delta(G) \geq 3$, then $\lambda_2^T(G) \leq 2\Delta(G)$ this generalizes results of Havet and Yu [4] who proved it for $\Delta \in \{3,4\}$. Furthermore, we show that if $\Delta(G) \geq 5$ is odd then $\lambda_2^T(G) \leq 2\Delta(G) - 1$.

2 The tools and the idea

Definition 1 A *cut* $[A,B]$ of a graph G is a set of two induced subgraphs A and B of G such that $(V(A), V(B))$ is a partition of $V(G)$. The bipartite graph (A,B) is the graph with vertex set $V(G)$ and edge set $E(G) \setminus (E(A) \cup E(B))$. The edges of (A,B) are called the *cut edges.* A *maximum cut* $[A,B]$ is a cut with maximum number of cut edges.

Lemma 1 *Let G be a graph with maximum degree $2k+1$. Then a maximum cut $[A,B]$ satisfies $\Delta(A) \leq k$ and $\Delta(B) \leq k$.*

Proof. Consider a maximum cut $[A,B]$. B contains no vertex b of degree greater than k otherwise $[A+b, B-b]$ is a cut with strictly more cut edges. Analogously A has no vertex of degree greater than k. □

Lemma 2 *Let G be a graph with maximum degree $2k$. Then G has a cut $[A,B]$ such that $\Delta(A) \leq k$ and $\Delta(B) \leq k$.*

Proof. Consider a maximum cut $[A,B]$ wich minimizes the number of vertices with degree k in A. As in the proof of Lemma 1, A and B contain no vertex of degree greater than k Moreover A has no vertex a of degree k otherwise $[A-a, B+a]$ is a cut with the same number of cut edges as $[A,B]$ and one vertex less of degree k in the first subgraph. □

Lemma 3 *Let G be a bipartite graph with maximum degree Δ. There is an edge colouring c of G in $[1,\Delta]$ such that $c(e) \geq i$ only if it is incident to a vertex of degree at least i.*

Proof. Let us prove it by induction on Δ, the result holding trivially when $\Delta = 0$. Consider now a graph with maximum degree $\Delta \geq 1$. By König's theorem, it admits an edge colouring c_1 in $[1,\Delta]$. Let M be the set of edges coloured Δ incident to a vertex of degree Δ. Consider G' the graph obtained from G by removing M. Since every vertex of degree Δ is adjacent to an edge of M, $\Delta(G') = \Delta - 1$. Then by induction G' has an edge colouring c of G in $[1, \Delta-1]$ such that $c(e) \geq i$ only if it is incident to a vertex of degree at least i. Extending c into an edge colouring of G in $[1,\Delta]$ by colouring the edges of M with Δ, we obtain the result. □

Definition 2 Let G be a graph. A *list assignment* L is an assignment of a set $L(v)$ of integers to every vertex v of G. The graph G is *L-colourable* if it admits an apllication c called *L-colouring* from its vertex set into the set of integers such that for any vertex v, $c(v) \in L(v)$ and for any edge (u, v), $c(u) \neq c(v)$.

Let k be a non-negative integer. A *k-list assignment* is an assignment L such that $|L(v)| = k$ for every vertex v. A graph is *k-choosable* if it is L-colourable for any k-list assignment L.

Let v be a vertex of G. A *(d, v)-list assignment* of G is a list assignement L such that $|L(u)| = d(u)$ if $u \neq v$ and $|L(v)| = d(v) + 1$. We say that G is *(d, v)-choosable* if it is L-colourable for any (d, v)-list assignment L.

Proposition 1 *Let G be a connected graph and $v \in V(G)$. Then G is (d, v)-choosable.*

Proof. There is an ordering $v_1, v_2, \dots, v_n$ of the vertices of the graph such for $i < n$ the vertex v_i has a neighbour in $\{v_j, i < j \leq n\}$. Hence by a greedy algorithm, one can find an L-colouring of G for any (d, v)-list assignment L. □

Using this proposition, we strengthen Havet and Yu result [4] stating that $\lambda_s^T(G) \leq 2\Delta + s - 1$.

Lemma 4 *Let G be a graph with maximum degree $\Delta \leq k$. G admits a $(s, 1)$-total labelling in $[0, 2k + s - 1]$ such that a vertex v is assigned a label in $[0, d(v)]$ and an edge a label in $[k + s - 1, 2k + s - 1]$.*

Proof. Obviously it suffices to prove it when G is connected.

By Vizing's Theorem, there is an edge colouring c' of G with colours in $[k + s - 1, 2k + s - 1]$. Let v be a vertex of G. Free to permute the colours of c', we may assume that for every edge incident to v, $c'(v) \geq k + s$. Let L be the (d, v)-list assignement with $L(u) = [0, d(u) - 1]$ if $u \neq v$ and $L(v) = [0, d(v)]$. By Proposition 1, G has an L-colouring. The union of c and c' is an $(s, 1)$ total labelling of G.

Indeed for every edge $e = xy$, if $x \neq v$ then $c(x) \leq k - 1 \leq c'(e) - s$ and if $x = v$ then $c(v) \leq k \leq c'(e) - s$. □

Analogously, we have the following lemma:

Lemma 5 *Let G be a graph with maximum degree $\Delta \leq k$. G admits a $(s, 1)$-total labelling in $[0, 2k + s - 1]$ such that an edge is assigned a label in $[0, k]$ and a vertex v a label in $[k + s - 1, k + s - 1 + d(v)]$.*

Theorem 1 (Galvin [2]) *Every bipartite graph G is $\Delta(G)$-edge choosable.*

The idea of the results is to consider a suitable maximum cut of G given by Lemma 1 or 2 and to label edge and vertices of A and B with Lemma 4 or induction hypothesis and Lemma 5 respectively and then to label the bipartite graph (A, B) using Lemma 3. Some few relabellings are then necessary to obtain the desired $(s, 1)$-total labelling. Theorem 1 is used for some of the relabellings.

3 Main result

The aim of this section is to prove the following theorem :

Theorem 2 *For any $s \geq 1$,*

$$\lambda_s^T \leq 2\Delta - 2\log(\Delta+2) + 2\log(16s-8) + s - 1$$

In order to prove this theorem, we prove by induction a stronger result.

Let G be a graph with maximal degree Δ. An $(s,1)$-total labelling in $[0,p]$ is a *s-good labelling* if each vertex is assigned a label in $[0, \Delta+s-1]$.

Theorem 3 *Let G be a graph with maximal degree Δ. Then G has a s-good labelling in $[0, 2\Delta - 2\log(\Delta+2) + 2\log(16s-8) + s - 1]$.*

The idea is to prove this result by induction. Note that Lemma 5 give the result for small value of Δ. We will now give two Lemmas allowing us to do an induction step.

Lemma 6 *Let $k \geq \max(i+2s-1, 2i+6s-5)$. If every graph of maximal degree k admits a s-good labelling in $[0, 2k-i]$ then every graph G of maximal degree $\Delta = 2k+2$ admits a s-good labelling in $[0, 2\Delta - i - 2]$.*

Proof. According to Lemma 2 there is a cut $[A, B]$ of G such that $\Delta(A) \leq k$ and $\Delta(B) \leq k+1$. Thus by hypothesis, there is a s-good labelling of A in $[0, 2k-i]$. And by Lemma 5, there is an $(s,1)$-total labelling of B such that vertices are labelled in $[k+s, k+s+d_B(v)]$ and edges in $[0, k+1]$.

By Lemma 3, label the edges of (A, B) with $[2k-i+1, 4k-i+2]$ so that an edge is labelled $4k-i+3-l$ only if it is incident to a vertex of degree at least l in (A, B).

The obtained labelling is not yet an $(s,1)$-total labelling. Indeed for $j \in [0, i+2s-1]$, edges (a,b) labelled $2k-i+1+j$ when b is labelled in $[2k-i+j-s+2, 2k-i+j+s]$ violate the constraints. Hence they must be relabelled.

Let us consider the bipartite graph induced by such edges. It has degree at most $i+2s$. We want to relabel the edges with labels in $[k+2s-2, 2k-i]$. According to Theorem 1, we need to find a list of $i+2s$ available labels for each edge. Let (a,b) be an edge labelled $2k-i+1+j$ with b labelled in $[2k-i+j-s+2, 2k-i+j+s]$. Then $d_B(b) \geq k-i+j-2s+2$. So b has degree at most $k+i-j+2s$ in (A, B). But by construction (a,b) is incident to a vertex of degree at least $2k+2-j$ in (A, B). Since $k \geq i+2s-1$ then this vertex is a and $d_A(a) \leq j$. So at most j labels of $[k+2s-2, 2k-i]$ are forbidden because of the edges of A incident to a. Moreover at most $2s-j-2$ labels of $[k+2s-2, 2k-i]$ are forbidden because of b (those of $[2k-i+j-2s+3, 2k-i]$). Hence at most $2s-2$ labels of $[k+2s-2, 2k-i]$ are forbidden. So because $k \geq 2i+6s-5$, at least $k-i-2s+3-(2s-2) \geq i+2s$ labels available on (a,b).

Since the labels of the vertices are in $[0, 2k+1+s]$, we have a s-good labelling of G in $[0, 4k-i+2]$. □

Lemma 7 *Let $k \geq \max(i+4s-1, 2i+6s-3)$. If every graph of maximal degree k admits a s-good labelling in $[0, 2k-i]$ then every graph G of maximal degree $\Delta = 2k+1$ admits a s-good labelling in $[0, 2\Delta - i - 2]$.*

Proof. Let $[A, B]$ be a maximum cut of G. Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$. Thus by hypothesis, there is a s-good labelling of A in $[0, 2k-i]$. And by Lemma 5, there is an $(s,1)$-total labelling of B such that vertices are labelled in $[k+s, k+s+d_B(v)]$ and edges in $[1, k]$.

By Lemma 3, label the edges of (A, B) with $[2k-i, 4k-i]$ so that an edge is labelled $4k-i+1-l$ only if it is incident to a vertex of degree at least l in (A, B).

There are two types of edges of (A, B) violating a constraint of an $(s,1)$-total labelling :

(1) edges (a, b) labelled $2k-i+j$ while b is labelled in $[2k-i+j-s+1, 2k-i+j+s-1]$ for some $j \in [0, i+2s-1]$;

(2) edges (a, b) labelled $2k-i$ with a incident to an edge (of A) labelled $2k-i$.

Let us first relabel the edges of type (1) with labels in $[k+2s-1, 2k-i-1]$. Let us consider the bipartite graph induced by them. It has degree at most $i+2s$. According to Theorem 1, we need to find a list of $i+2s$ available labels for each edge. Let (a, b) be an edge labelled $2k-i+j$ with b labelled in $[2k-i+j-s+1, 2k-i+j+s-1]$. Then $d_B(b) \geq k-i+j-2s+1$. So b has degree at most $k+i-j+2s$ in (A, B). But by construction (a, b) is incident to a vertex of degree at least $2k+1-j$ in (A, B). Since $k \geq i+2s$, this vertex is a and $d_A(a) \leq j$. So at most j labels are forbidden because of the edges of A incident to a and at most $2s-j-2$ are forbidden because of b (those of $[2k-i+j-2s+2, 2k-i-1]$. Hence at most $2s-2$ labels of $[k+2s-1, 2k-i-1]$ are forbidden. So since $k \geq 2i+6s-3$, there are at least $k-i-2s+1-(2s-2) \geq i+2s$ labels available on (a, b).

Let us now relabel the edges of type (2). Since a is incident to an edge of A, it has degree less than $2k+1$ in (A, B). Hence b has degree $2k+1$ in (A, B) and thus is isolated in B. In particular b was not incident to an edge of type (1). Let $l(a)$ be the label of a. There is a label in $[0, k+2s-1] \setminus [l(a)-s+1, l(a)+s-1]$ that is not assigned to any edge of A incident to a. Relabel (a, b) with l. Since $l+s \leq k+3s-1 \leq 2k-i-s$, we can relabel b with $k+3s-1$.

Since the labels of the vertices are in $[0, 2k+s]$ we have a good labelling of G in $[0, 4k-i]$. □

Proof of Theorem 3. Set $c_s = 2\log(16s-8)+s-1$. If $\Delta \leq 16s-10$, then we have the result by Lemma 4. Suppose now that G is a graph with maximal degree $\Delta \geq 16s-9$.

If Δ is even, set $\Delta = 2k+2$. By induction hypothesis $\lambda_s^T(H) \leq 2k - 2\log(k+2) + c_s$. And setting $i = 2\log(k+2) - c_s$, we have $k \geq \max(i+2s-2, 2i+6s-5)$. Hence by Lemma 6, $\lambda_s^T(G) \leq 2\Delta - 2\log(k+2) + c_s - 2$. Since $\log(k+2)+1 = \log(2k+4) = \log(\Delta+2)$. We obtain $\lambda_s^T(G) \leq 2\Delta - 2\log(\Delta+2) + c_s$.

In the same way, we have the result if Δ is odd. □

4 Better bounds when $s = 2$

4.1 Upper bound 2Δ

Theorem 4 *If $\Delta(G)$ is odd and at least 5 then G has a 2-good labelling in $[0, 2\Delta(G)]$.*

Proof. Set $\Delta(G) = 2k+1$. Consider a maximum cut $[A, B]$ of G. Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$.

Thus by Lemmas 4 and 5, one may label A and B in $[0, 2k+1]$ such that a vertex v in A receives a label in $[0, d_A(v)]$ (resp. $[k+1, k+1+d_B(v)]$) and edges labels in $[k+1, 2k+1]$ (resp. $[0, k]$).

Now by Lemma 3, label the edges of (A, B) in $[2k+2, 4k+2]$ such that an edge is assigned $2k+2$ only if it is adjacent to a vertex with degree $2k+1$ in (A, B) and so an isolated vertex in A or B.

The label of an edge (a, b) of (A, B) fullfill the constraints of a $(2,1)$-total labelling unless it is labelled $2k+2$ and b is labelled $2k+1$. But in this case, a is an isolated vertex of A and thus labelled 0. So we may relabel (a, b) with $k+1$. This is possible since $k \geq 2$ so $(2k+1)-(k+1) \geq 2$.

Since the vertices are labelled in $[0, 2k+1]$ we have a 2-good labelling. □

The proof of Theorem 4 does not work when $\Delta = 3$. However, we give an alternative proof of a result of Havet and Yu [4] asserting that a graph with maximum degree 3 has a $(2,1)$-total labelling in $[0, 6]$.

Theorem 5 *If* $\Delta(G) \leq 3$ *then* $\lambda_2^T(G) \leq 6$.

Proof. Let $[V_1, V_2]$ a maximal cut of G. Easily $\Delta(V_i) \leq 1$.

For $i = 1, 2$, let S_i (resp. T_i) be the set of isolated vertices (resp. vertices with degree 1) in G_i.

Label the edges of V_1 (resp. V_2) with 3 (resp. 0) and their endvertices with 0 and 1 (resp. 2 and 3). Label the vertices of S_2 with 2.

By König's Theorem, there is a 3-edge colouring of (V_1, V_2) with colours a, b and c. For each a-coloured edge (u, v) with $u \in G_1$ do the following :

- If $u \in S_1$ and $v \in S_2$, assign 4 to (u, v) and 0 to u.
- If $u \in T_1$ and $v \in S_2$, assign 4 to (u, v).
- If $u \in S_1$, $v \in T_2$ and v is labelled 2 then assign 4 to (u, v) and 0 to u.

At this stage the vertices of S_1 whose incident a-coloured edge has an end in T_2 labelled 3 are not yet coloured. We will label them one after another doing the following algorithm :

(1) If there is a vertex $y \in T_2$ that is adjacent to two non labelled vertices x and z (of S_1), assign 0 to x and z, 3 to (x, y), 4 to (y, z) and relabel y with 6. Go to (1).

(2) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex x and a labelled vertex $z \in S_1$, then z is labelled 0 and there is an integer l in $\{2, 3, 4\}$ that label no edge incident to z. Then assign 0 to x, l to (y, z), an integer of $\{2, 3, 4\} \setminus \{l\}$ to (x, y) and relabel y with 6. Go to (2).

(3) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex x and a vertex $z \in T_1$. Let e be the edge of B incident to z and distinct from (y, z).

If e is not labelled yet then assign 4 to (y, z), 3 to (x, y) and 0 to x. Relabel y with 6. Go to (3).

Otherwise e is already labelled with 4. Let a be the label of z. Assign 6 to to (y, z), 4 to (x, y) and a to x. Relabel y with the integer of $\{0, 1\} \setminus \{a\}$. Go to (3).

Let E' be the set of non labelled edges. It induces a bipartite graph with maximum degree 2. And the vertices incident to edges of E' are labelled in $[0, 3]$.

By König's theorem, E' can be two coloured with label 5 and 6. It is easy to see that we have a $(2, 1)$-total labelling of G. □

Remark 1 The $(2, 1)$-total labelling obtained by such a proof is really different from the one obtained by the proof of Havet and Yu [4].

Theorem 6 *If $\Delta(G)$ is even and at least 6 then G has a 2-good labelling in* $[0, 2\Delta(G)]$.

Proof. Set $\Delta = 2k$. Consider a cut $[A, B]$ as in Lemma 2. Following Lemma 5, label A such that a vertex v receives a label in $[k+1, k+1+d_A(v)]$ and an edge a label in $[1, k]$. Following Lemma 4, label B such that a vertex v receives a label in $[0, d_B(v)]$ and an edge a label in $[k+1, 2k+1]$.

Now by Lemma 3, label the edges of (A, B) in $[2k+1, 4k]$ such that an edge is assigned $2k+1$ only if it is adjacent to a vertex with degree $2k$ in (A, B) and so an isolated vertex in A or B.

The label of an edge (a, b) of (A, B) fullfill the constraints of a $(2, 1)$-total labelling unless (a, b) is labelled $2k+1$ and 1) a is labelled $2k$ or 2) b is incident to an edge of B labelled $2k+1$. Thus we need some relabelling.

1) If a is labelled $2k$, then a is not isolated in A. Thus b is isolated in B. Then relabel (a, b) with 0 and b with 2.

2) If b is incident to an edge (b, b') of B which is labelled $2k+1$, then b is not isolated in B. Thus a is isolated in A. In particular such an edge is disjoint from any edge of type 1). Let $l(b)$ be the label assigned to b. If $l(b) \geq 2$ then relabel (a, b) with 0. If $l(b) \leq 1$ then relabel (a, b) with 3 and a with 5 if $k = 3$. This is valid since $k \geq 3$.

In such a $(2, 1)$-total labelling a vertex is assigned an integer in $[0, 2k]$, so we have a 2-good labelling. □

One can extend Theorem 6 for $\Delta = 4$. This strengthen a result of Havet and Yu [4] stating that $\lambda_2(G) \leq 8$ if $\Delta \leq 4$.

Theorem 7 *If $\Delta(G) = 4$ then G has a 2-good labelling in* $[0, 8]$.

Proof. By Lemma 2, G has a cut $[A, B]$ such that $\Delta(A) \leq 1$ and $\Delta(B) \leq 2$.

Label the vertices of A with $\{0, 1\}$ and its edges with $\{3\}$ such that the isolated vertices of A receive 0.

Label the vertices and edges of B which do not lay on odd cycle of B as follows :

(i) the isolated vertices of B are labelled 3;

(ii) The vertices and edges of an even cycle or a path are labelled alternatively 3 and 4 and 0 and 1 respectively.

According to Lemma 3, label the edges of (A, B) with $[5, 8]$ so that an edge assigned 5 is incident to a vertex of degree 4 in (A, B) which are isolated vertices in A or B.

Some constraints are violated each time an edge (a, b) of (A, B) is labelled 5 and a is labelled 4. But in that case, a is not isolated in A thus b is isolated in B and so is labelled 0. Then relabel (a, b) with 2.

At this stage, it remains to assign labels to vertices and edges of odd cycles of G.

Let $C = (b_0, b_1, \ldots, b_{2p}, b_0)$ be an odd cycle of B. Then two consecutives vertices, say b_0 and b_1 are either both incident to an edge labelled 5 or both non incident to an edge labelled 5. Then for $1 \leq i \leq p$, label b_{2i-1} with 3, b_{2i} with 4, (b_{2i-1}, b_{2i}) with 1 and (b_{2i}, b_{2i+1}) with 0. And label b_0 with 2.
If b_0 and b_1 are non incident to an edge labelled 5 then label (b_0, b_1) with 5. Otherwise there is a label $l \in [6, 8]$ such that both b_0 and b_1 are incident to no edge labelled l. Label (b_0, b_1) with l.

Since the vertices are labelled in $[0, 4]$, we have a 2-good labelling of G in $[0, 8]$. □

Corollary 1 $\lambda_2^T(G) \leq 2\Delta - 2\log(\Delta + 2) + 8$

4.2 Upper bound $2\Delta - 1$ for odd Δ

Theorem 8 *If $\Delta(G)$ is odd and at least 7 then G has a 2-good labelling in $[0, 2\Delta(G) - 1]$.*

Proof. Set $\Delta(G) = 2k + 1$. Consider a maximum cut $[A, B]$ of G. Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$.

Following Lemma 4, label A such that each vertex v of A is assigned a label in $[0, d_A(v)]$ and each edge e a label in $[k + 1, 2k + 1]$.

Following Lemma 5, label B such that each vertex v of B is assigned a label in $[k + 1, k + 1 + d_B(v)]$ and each edge e a label in $[0, k]$.

By Lemma 3, label the edges of (A, B) with $[2k + 1, 4k + 1]$ so that an edge is labelled $4k + 2 - i$ only if it is incident to a vertex of degree i in (A, B).

This labelling may violate some constraints of a $(2, 1)$-total labelling in the following cases :

(1) a vertex $b \in B$ labelled $2k$ or $2k + 1$ is incident to an edge (a, b) of (A, B) labelled $2k + 1$;

(2) a vertex $b \in B$ labelled $2k + 1$ is incident to an edge (a, b) of (A, B) labelled $2k + 2$;

(3) a vertex $a \in A$ is incident to two edges labelled $2k + 1$ one (a, a') in A and one (a, b) in (A, B);

Therefore, we need to proceed to the following corresponding relabelling :

(1) Since $k \geq 2$, then $2k > k + 1$ so b is not isolated in B. Thus the vertex a is isolated in A and labelled 0. Then relabel (a, b) with k.

(2) The vertex b is labelled $2k + 1$ and so $d_B(b) = k \geq 2$. Hence b has degree less than $2k$ in (A, B) and a has degree at least $2k$ in (A, B). So a has degree at most 1 in A and thus is labelled 0 or 1. One of the two integers $k + 1$ and $k + 2$ does not label the (possible) edge incident to a in A. Then relabel (a, b) with l. This is valid since $k \geq 3$.

(3) Since a is not isolated in A, then b is isolated in B and thus labelled $k + 1$. If a is labelled 0 or 1 then relabel (a, b) with 3 and b with 5 if $k = 3$. Again this is valid since $k \geq 3$. If a is labelled in $[2, k + 1]$ then relabel (a, b) with 0.

□

The last two relabelling of the previous proof are not valid if $k = 2$. Hence, to get the result when $\Delta = 5$, we need some extra arguments :

Theorem 9 *If* $\Delta(G) = 5$ *then* $\lambda_2^T(G) \leq 9$.

Proof.

Let $[A, B]$ be a maximum cut of G. Then $\Delta(A) \leq 2$ and $\Delta(B) \leq 2$.

We need a more careful labelling of A than in Theorem 8. Let C be a component of A. If C is not an odd cycle then, following Lemma 4, label C such that each vertex v is assigned a label in $[0, d_A(v)]$ and each edge e a label in $[3, 4]$. If C is an odd cycle $(a_1, a_2, \dots, a_{2p+1}, a_1)$ then for $1 \leq i \leq p$, label a_{2i-1} with 0, (a_{2i-1}, a_{2i}) with 3, a_{2i-1} with 1, and (a_{2i}, a_{2i+1}) with 4. Label a_{2p+1} with 2 and (a_{2p+1}, a_1) with 5. Note that in that case a vertex labelled 1 in A is not incident to an edge labelled 5.

Following Lemma 5, label B such that each vertex v of B is assigned a label in $[3, 3 + d_B(v)]$ and each edge e a label in $[0, 2]$ with B_e.

By Lemma 3, label the edges of (A, B) with $[5, 9]$ so that an edge is labelled $10 - i$ only if it is incident to a vertex of degree at least i in (A, B).

This labelling may violate the constraints of a $(2, 1)$-total labelling in the same cases as in Theorem 8 :

(1) a vertex $b \in B$ labelled 4 or 5 is incident to an edge (a, b) of (A, B) labelled 5;

(2) a vertex $b \in B$ labelled 5 is incident to an edge (a, b) of (A, B) labelled 6;

(3) a vertex $a \in A$ is incident to two edges labelled 5 one (a, a') in A and one (a, b) in (a, b);

Therefore, we need to proceed to the following corresponding relabelling :

(1) As in Theorem 8, relabel (a, b) with 2.

(2) The vertex b is labelled 5 and so $d_B(b) = 2$. Hence b has degree less than 6 in (A, B) and a has degree at least 6 in (A, B). So a has degree at most 1 in A and thus is labelled 0 or 1. Relabel (a, b) with 3. This may violate a constraint if the edge (a, a') in A incident to a is also labelled 3. If a' is incident to no edge labelled 4 then relabel (a, a') with 4. Otherwise $d_{(A,B)}(a') \leq 2$. Thus there is a label $l \in [5, 7]$ that labels no edge incident to a or a' (since (a, b) is now labelled 3). Relabel (a, a') with l.

(3) Since a is not isolated in A, then b is isolated in B and thus labelled 3. Moreover, the vertex a is labelled either 0 or 2 because no vertex of A labelled 1 is incident to an edge of A labelled 5. If a is labelled 0 or then relabel (a, b) with 2 and b with 4. If a is labelled 2 then relabel (a, b) with 0.

□

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