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Stability of Planar Nonlinear Switched Systems

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***Abstract* —– We study the global asymptotic stability of the time dependent nonlinear system $\dot{x}(t) = u(t)F(x(t)) + (1-u(t))G(x(t))$, where $x \in \mathbb{R}^2$, $F(x)$ and $G(x)$ are two C^∞ vector fields, globally asymptotically stable at the origin and $u(.) : [0, \infty[\to [0, 1]$ is a completely random measurable function. We give a sufficient and a necessary condition for global asymptotic stability. This result extend some of our previous results obtained in the linear case.**

***Keywords* —– Stabilization, Switched Systems.**

1 Introduction

By a switched system we mean a family of continuous–time dynamical systems and a rule that determines at any time which dynamical system is responsible for the time evolution. More precisely let $\{f_u : u \in U\}$ be a (finite or infinite) set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{1}$$

The rule is given assigning the so called switching function $u(.) : [0, \infty[\to U$. Here we consider the situation in which the switching function cannot be predicted a priori; it is given from outside and represents some phenomena (e.g. a disturbance) that it is not possible to control or include in the dynamical system model. In the following we use the notation $u \in U$ to indicate a fixed individual system and $u(.)$ to indicate the switching function.

Suppose now that all the f_u have a given property for every $u \in U$. A typical problem is to study under which conditions this property holds for the system (1) for arbitrary switching functions. For a discussion of various issues related to switched system we refer the reader to [6]. In this paper we consider the problem of the global asymptotic stability of a single input convexified nonlinear problem.

$$\dot{x}(t) = u(t)F(x(t)) + (1-u(t))G(x(t)), \tag{2}$$

where $x \in \mathbb{R}^2$, $F(.)$, $G(.)$ are $\mathcal{C}^\infty$ functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $F(0) = 0$, $G(0) = 0$ and the two dynamical systems $\dot{x} = F(x)$, $\dot{x} = G(x)$ **are globally asymptotically stable at the origin**. We are interested to study under which conditions on $F(.)$ and $G(.)$ the origin of the system (2) is globally asymptotically stable for every measurable function taking values on the close interval $[0, 1]$ i.e. $u(.) : [0, \infty[\to [0, 1]$. The linear version of this problem was studied in [3] (and in [1, 5] in higher dimension, see also [8]):

$$\dot{x}(t) = u(t)Ax(t) + (1-u(t))Bx(t), \tag{3}$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbb{R}^2$ and $u(.) : [0, \infty[\to [0, 1]$ is an arbitrary measurable switching function. More precisely in the paper [3] a necessary and sufficient condition for asymptotic stability were given in terms of three relevant parameters: one depending on the eigenvalues of A, one depending on the eigenvalues of B, and the last containing the interrelation among the two systems and it is the cross ratio of the four eigenvectors of A and B in the projective line $\mathbb{C}P^1$. About the nonlinear problem, of course we have:

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Proposition 1 (necessary condition) *If the linearized system (3), where* $A := \nabla F|_{x=0}$ *and* $B := \nabla G|_{x=0}$*, has eigenvalues with non null real part, then a necessary condition for global asymptotic stability of the non linear system (2) is that the linearized system (3) is asymptotically stable.*

The conditions for stability of the linear system (3) are quite technical and can be found in [3]. Notice that, for linear systems, asymptotic stability is equivalent to global asymptotic stability, that is equivalent to the more often quoted property of GUES (global exponential stability, uniform with respect to switching), see for example [2] and references therein.

In this paper, we give a necessary and a sufficient condition for global asymptotic stability for the nonlinear system (2) when F and G are globally asymptotically stable at 0.

A key ingredient is the set on which the two vector fields are parallel. This is $Q^{-1}(0)$, the set of zeros of the function:

$$Q(x) = det(F(x), G(x)) \tag{4}$$

The set $Q^{-1}(0)$ is used, similarly to [3], to build the "worst" trajectory. The idea is that the system is globally asymptotically stable if and only if for each point the worst trajectory passing through that point goes to the origin (and of course does not reach points arbitrarily far from the origin). We recall that in the linear case (3), excepted for some degenerate cases, the set $Q^{-1}(0)$ is equal to $\{0\}$, or it is a couple of straight lines passing through the origin. In the non linear case, the situation is more complicated. The set $Q^{-1}(0)$ can be expressed as:

$$Q^{-1}(0) = \{0\} \cup \bigcup_{j,k} \Gamma_j^k, \quad j \in J \subset \mathbb{R} \text{ and } k = 0, 1, \tag{5}$$

where the Γ_j^0 are the connected components of $\Gamma^0 - \{0\}$, Γ^0 being the subset of $Q^{-1}(0)$ connected to the origin, while the Γ_j^1 are the other connected components of $Q^{-1}(0)$.
Let Γ_j^k one of these components. We say that at $x \in \Gamma_j^k$, Γ_j^k is "direct" (resp "inverse") if $F(x)$ and $G(x)$ have the same (resp. opposite) direction.

Lemma 1 *Let* Γ_j^k *be "direct" (resp. "inverse") at* $x \in \Gamma_j^k$*. Then, for every* $y \in \Gamma_j^k$, Γ_j^k *is "direct" (resp. "inverse").*

An example is depicted in Figure 1. Our main result is the following Theorem:

Theorem 1 (sufficient condition) *If F and G are globally asymptotically stable at 0 and if the set* $Q^{-1}(0) = \{0\}$*, then the system (2) is globally asymptotically stable at 0 for every switching function* $t \mapsto u(t)$*.*

This is a generalization of the results proved in the linear case. If the hypothesis of this Theorem applies, then there exists also a common Lyapunov function [4, 7]. Moreover the result of Theorem 1 can be extend to the following case: if a set V is open and simply connected, and if the two vector fields point inside V along its boundary, then under the assumptions of the Theorem 1 the system (2) is globally asymptotically stable on V.

The straightforward following Theorem says that the "inverse components" are bad for stability. The reason is clear: if Γ_j is "inverse" and $\bar{x} \in \Gamma_j^k$, then there exists a constant switching function u such that the corresponding trajectory of (2) satisfying $\gamma(0) = \bar{x}$ is constantly equal to $\bar{x}$.

Theorem 2 (necessary condition) *If the system (2) is globally asymptotically stable for every switching function, then, at each point of* $Q^{-1}(0) \setminus \{0\}$*, the fields F and G point in the same direction.*

If all the components are "direct", but $Q^{-1}(0)$ does not contain only the origin, then the fact that the system is stable (or not) depends on the topology and on the way the (direct) components are "connected" by the trajectories.
Remark The results of Theorems 1 and 2 are true also if $u(.)$ take values only on the boundary of the interval $[0, 1]$, i.e. $u(.) : [0, \infty[\to \{0, 1\}$.

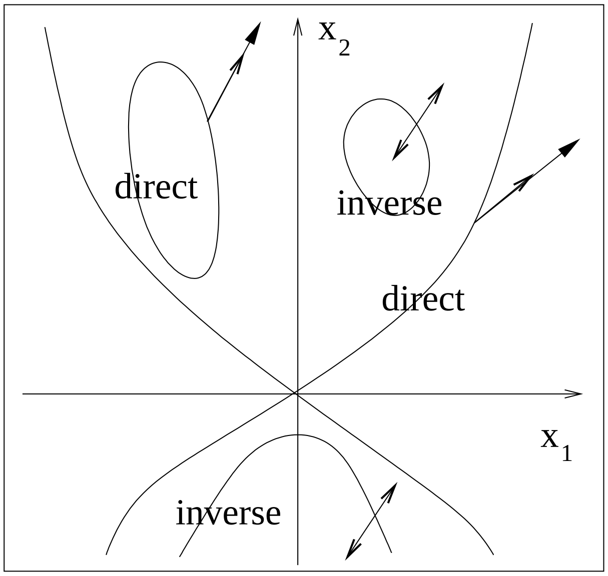

Figure 1: An example of set $Q^{-1}(0)$. Notice that some of the components can be compact

In the following section, we present the proof of Theorem 1.

2 Proof of the main result: Theorem 1

First, we prove that, given $u()$ and q, the solution of (2), starting from q, converges to 0. Second, we prove the stability of (2).

Because F and G are never parallel, we can assume that G is always pointing on the left of F, i.e. that (F, G) is direct.

In the sequel we denote $t \mapsto \gamma_X(q,t)$ the integral curve of the vector field X issued from q.

2.1 Proof of the convergence

In order to prove the convergence, we first prove that the accessible set from q, denoted $\mathcal{A}_q$, is bounded. Hence we show that the only possible accumulation point of an admissible trajectory is 0. These two facts clearly imply that any admissible trajectory converges to 0.

2.1.1 Proof that $\mathcal{A}_q$ is bounded

Let us first prove that the accessible set from a point q is bounded. The idea is to prove first that the two curves $\gamma_F(q,.)$ and $\gamma_G(q,.)$ allow to separate $\mathbb{R}^2$ in two parts, one being bounded, and to prove that the accessible set is included in the bounded part.

In order to do this we consider two cases.

First case: Let us first assume that the two curves $\gamma_F(q,.)$ and $\gamma_G(q,.)$ do not intersect. Then we can define a new closed curve, which is piecewise smooth :

$$\begin{array}{ll} \gamma_{F,G}(q,t) = \gamma_F(q, \tan(t\pi)) & \text{if } t \in [0, \frac{1}{2}], \\ \gamma_{F,G}(q,t) = \gamma_G(q, \tan((1-t)\pi)) & \text{if } t \in [\frac{1}{2}, 1]. \end{array}$$

This closed curve separates $\mathbb{R}^2$ in two sets, one being bounded. Let us call $\mathcal{B}(q)$ the interior of the bounded set and $\mathcal{D}(q)$ the interior of the other. We have $\mathbb{R}^2 = \mathcal{B}(q) \cup \mathcal{D}(q) \cup \gamma_{F,G}(q,[0,1])$.

Lemma: *$\mathcal{A}(q)$ is included in $\mathcal{B}(q) \cup \gamma_{F,G}(q,[0,1])$ or $\mathcal{D}(q) \cup \gamma_{F,G}(q,[0,1])$.*

Proof. There are only two possible situations: the first is that all along the curves $\gamma_F(q,.)$ and $\gamma_G(q,.)$, the vector $\frac{1}{2}(F+G)$ is pointing inside $\mathcal{B}$. The second is that all along the curves $\gamma_F(q,.)$ and $\gamma_G(q,.)$, the vector $\frac{1}{2}(F+G)$ is pointing outside $\mathcal{B}(q)$, that is inside $\mathcal{D}(q)$.

Actually, at q it is clear that $\frac{1}{2}(F+G)$ should point strictly inside or outside $\mathcal{B}(q)$. Hence close to q, it will be also pointing strictly in the same direction. But now, because the orientation defined by (F,G) does not change, it is also the case for $(F, \frac{1}{2}(F+G))$ and for $(G, \frac{1}{2}(F+G))$. Hence all along $\gamma_{F,G}(q,.)$, $\frac{1}{2}(F+G)$ is pointing or always inside, or always outside $\mathcal{B}(q)$ (except at 0 where it is 0).

Let us assume that $\frac{1}{2}(F+G)$ is always pointing inside $\mathcal{B}(q)$. Then at q all $uF+(1-u)G$ are pointing inside $\overline{\mathcal{B}}(q)$ for $u \in [0,1]$, strictly for $u \in]0,1[$. At $\gamma_F(q,t)$ $(t>0)$, all $uF+(1-u)G$ for $u \in [0,1[$ are pointing strictly inside $\mathcal{B}(q)$ and of course F is tangent to $\gamma_F(q,.)$. At $\gamma_G(q,t)$ $(t>0)$, all $uF+(1-u)G$ for $u \in]0,1]$ are pointing strictly inside $\mathcal{B}(q)$ and G is tangent to $\gamma_F(q,.)$. Hence all vectors of the dynamics are pointing inside $\overline{\mathcal{B}}(q)$, at a point q' of the boundary. Hence in this case ($\frac{1}{2}(F+G)$ pointing inside $\mathcal{B}(q)$ at q) the accessible set of q is included in $\overline{\mathcal{B}}(q)$.

One can do the same if $\frac{1}{2}(F+G)$ is pointing outside $\mathcal{B}(q)$ at q and find that in that case the accessible set of q is included in $\overline{\mathcal{D}}(q)$. ■

Let us prove now that this second case can not happen, that is that the accessible set $\mathcal{A}(q)$ is always included in $\overline{\mathcal{B}}(q)$:

Lemma: *$\mathcal{A}(q)$ is included in $\mathcal{B}(q) \cup \gamma_{F,G}(q,[0,1])$.*

Proof. In order to prove this, let us assume that $\mathcal{A}(q)$ is included in $\overline{\mathcal{D}}(q)$ and let us define the following family of curves:

$$\begin{array}{ll} \gamma^{F,G,t}(q,\tau) = \gamma_F(q,-\tau) & \text{if } \tau < -t \\ \gamma^{F,G,t}(q,\tau) = \gamma_G(\gamma_F(q,t), \tau + t) & \text{if } \tau > -t \end{array}$$

These curves, defined for τ in $]-\infty,+\infty[$, just consist in following $\gamma_F(q,.)$ from 0 until $\gamma_F(q,-\tau)$, or, if $-\tau$ is bigger than t, consist in following $\gamma_F(q,.)$ from 0 until $\gamma_F(q,t)$ and then $\gamma_G(\gamma_F(q,t),.)$ until $\gamma_G(\gamma_F(q,t),\tau)$ (see (2.1.1)).

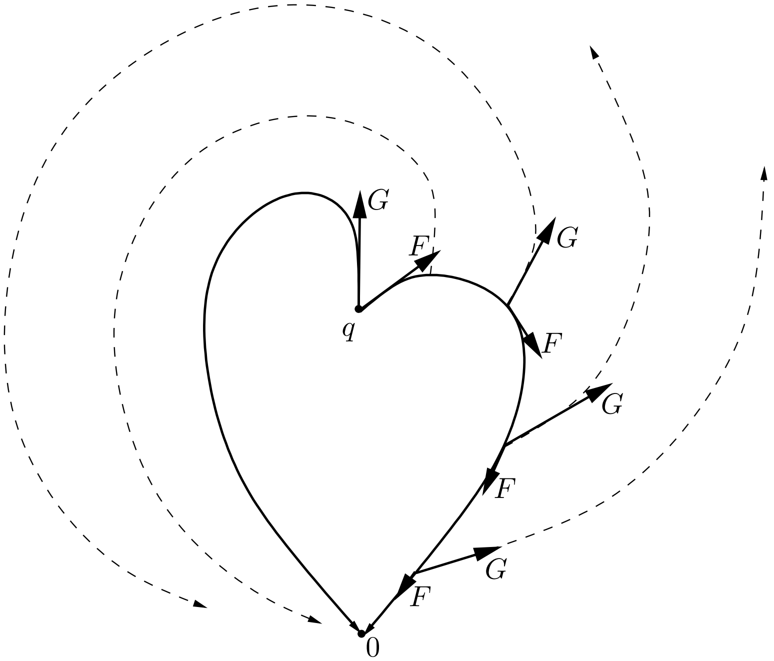

Figure 2: The curves $\gamma^{F,G,t}$.

Let us consider a point q' in $\mathcal{B}(q)$. First, we claim that the index of $\gamma^{F,G,0}$ with respect to this point is +1, the curve describing exactly the boundary of $\mathcal{B}(q)$. Now, by stability of G at 0, it is clear that the index with respect to q' of the curve $\gamma^{F,G,t}$ depends continuously of t, each of these curves converging to 0 when τ goes to

$+\infty$. But when t goes to $+\infty$, then $\gamma_F(q,t)$ goes to 0. Hence, the fact that the curve $\gamma^{F,G,t}$ has index 1 implies that there are points p, close to 0, such that the curve $\gamma_G(p,.)$ turns around q' before converging to 0, that is:

$$\int_0^{+\infty} \frac{d}{dt}(Arg(\gamma_G(p,s)))ds = 2\pi + \mathcal{O}(1/t),$$

where $Arg(p')$ is the argument of p' with respect to q' (defined modulo 2π but not its variation). But this is in contradiction with the fact that G is stable in 0. ■

<u>Second case:</u> Let assume that $\gamma_F(q,.)$ and $\gamma_G(q,.)$ intersect. Then we call t the first positive time such that the point $\gamma_G(q,t)$ is a $\gamma_F(q,\tau)$. We define the closed curve:

$$\begin{array}{ll} \gamma_{F,G}(q,s) = \gamma_F(q,s) & \text{if } s \in [0,\tau], \\ \gamma_{F,G}(q,s) = \gamma_G(q,t+\tau-s) & \text{if } s \in [\tau,\tau+t]. \end{array}$$

It separates the space in two open sets $\mathcal{B}(q)$ and $\mathcal{D}(q)$, $\mathcal{B}(q)$ being bounded.

Let assume that $\frac{1}{2}(F+G)$ strictly points inside $\mathcal{B}(q)$ at q. Hence it is also true at $\gamma_{F,G}(q,s)$ close to q, that is for s close to 0 or close to $\tau+t$. And it will propagate all along the curve $\gamma_{F,G}(q,.)$, except maybe at $\gamma_{F,G}(q,\tau)$. But at $\gamma_{F,G}(q,\tau)$, one can do the same reasoning with $-F$ and $-G$, and hence $\frac{1}{2}(F+G)$ is also pointing strictly inside $\mathcal{B}(q)$ at $\gamma_{F,G}(q,\tau)$. Hence the accessible set is included in $\overline{\mathcal{B}}(q)$, all the admissible $uF+(1-u)G$ pointing inside $\mathcal{B}(q)$ all along $\partial\mathcal{B}(q)$.

If $\frac{1}{2}(F+G)$ strictly points outside $\mathcal{B}(q)$ at q, the same reasoning allows to prove that it is pointing outside $\mathcal{B}(q)$ all along $\gamma_{F,G}(q,.)$ and hence that the accessible set is included inside $\overline{\mathcal{D}}(q)$.
Lemma: *$\mathcal{A}(q)$ is included in $\mathcal{B}(q) \cup \gamma_{F,G}(q,[0,\tau+t])$.*

Proof. In order to prove this lemma, we just have to prove that $\frac{1}{2}(F+G)$ cannot point strictly outside $\mathcal{B}(q)$ all along $\partial\mathcal{B}(q)$. In this last case, it would be clear that $0 \in \overset{\circ}{\mathcal{D}}(q)$. But the fact that $\frac{1}{2}(F+G)$ strictly points outside $\mathcal{B}(q)$ all along $\partial\mathcal{B}(q)$ implies that it has a 0 inside $\mathcal{B}(q)$. Which is impossible because $F+G$ is not 0 inside $\mathcal{B}(q)$. ■

Hence, we have proved that in both cases, the accessible set is bounded.

2.1.2 The accessible set is exactly $\overline{\mathcal{B}}(q)$

Lemma: *In fact $\mathcal{A}(q) = \overline{\mathcal{B}}(q)$.*

Proof.

First, we proved before that $\mathcal{A}(q) \subset \overline{\mathcal{B}}(q)$.

Second, $\partial\mathcal{B}(q) \subset \mathcal{A}(q)$ because $\partial\mathcal{B}(q)$ is included in the union of $\gamma_F(q) \cup \gamma_G(q)$.

Now, let us assume that it exists a point $p \neq 0$ in $\mathcal{B}(q)$ which is not in $\mathcal{A}(q)$. Hence, the set $\{\gamma_{-F}(p,t) \mid t \geq 0\}$ is bounded because it cannot cross the boundary of $\mathcal{B}(q)$ which is a subset of $\mathcal{A}(q)$. Actually, if $\gamma_{-F}(p,t) = p'$ was in $\mathcal{A}(q)$ then $p = \gamma_F(p',t)$ would also be in $\mathcal{A}(q)$.

Moreover the set $\{\gamma_{-F}(p,t) \mid t \geq 0\}$ cannot have 0 as accumulation point because of stability. Effectively, for any V_2 neighborhood of 0 it exists V_1 neighborhood of 0 such that if p' is in V_1 then any $\gamma_F(p',t)$ is in V_2. Hence if the set $\{\gamma_{-F}(p,t) \mid t \geq 0\}$ had 0 for accumulation point then there would exist a t such that $\gamma_{-F}(p,t) \in V_1$ hence $p = \gamma_F(\gamma_{-F}(p,t),t)$ would be in V_2, i.e. p would be in any neighborhood of 0 and hence would be 0.

Now it exists V, neighborhood of 0 such that $\{\gamma_{-F}(p,t) \mid t \geq 0\}$ has no intersection with V. Hence for any $t \geq 0$, it exists a point $p' = \gamma_{-F}(p,t)$ in $\mathcal{B}(q)$ such that $\gamma_F(p',t) \notin V$. But this is not possible because $\mathcal{B}(q)$ is bounded and F being globally asymptotically stable, it exists a Lyapunov function for F which gradient push any bounded set in V in finite time.

Hence there is no p in $\overline{\mathcal{B}}(q) - \mathcal{A}(q)$. ■

2.1.3 Proof that the only accumulation point is 0

Let us prove that 0 is the only possible accumulation point. In order to do this, let us consider a point p different from 0. We claim that:

Proposition 2 *There exists a neighborhood V of p such that for any point $p' \in V$ the curves $\gamma_F(p')$ and $\gamma_G(p')$ do not intersect in V and there is a coordinate system in V, centered at p', such that:*

1) the curve $t \mapsto \gamma_F(p', t)$ is the semi-line $t \mapsto (-t, t)$ $(t \geq 0)$,

2) the curve $t \mapsto \gamma_G(p', t)$ is the semi-line $t \mapsto (t, t)$ $(t \geq 0)$,

3) the vector fields F and G are such that their first coordinates F_1 and G_1 are bigger than $\frac{1}{2}$ in V.

4) $\mathcal{A}(p') \cap V = \{(t, t') |\ t \geq 0\ , |t'| \leq t\} \cap V$,

Proof. The fact that $\gamma_F(p')$ and $\gamma_G(p')$ do not intersect in V for V small enough is just a consequence of the continuity with respect to initial conditions of the dynamical systems $\dot{x} = F(x)$ and $\dot{x} = G(x)$ and of their asymptotic stability in 0. Points 1), 2) and 3) are a direct consequence of this fact. Moreover, because of the construction of $\overline{\mathcal{B}}(p')$, it is clear that the only points of the boundary of $\mathcal{A}(p')$ are exactly the points $\gamma_F(p', t)$ and $\gamma_G(p', t)$ for t positive and small. Hence the point 4) is clear. ■

Now, let us choose $p' = \gamma_{-(F+G)}(p, t)$ for t small enough so that p is in a triangle which has vertexes $p' = (0, 0)$, (ϵ, ϵ) and $(\epsilon, -\epsilon)$ for a certain ϵ. We call $T(p', \epsilon)$ this triangle which is a neighborhood of p (see figure 2.1.3).

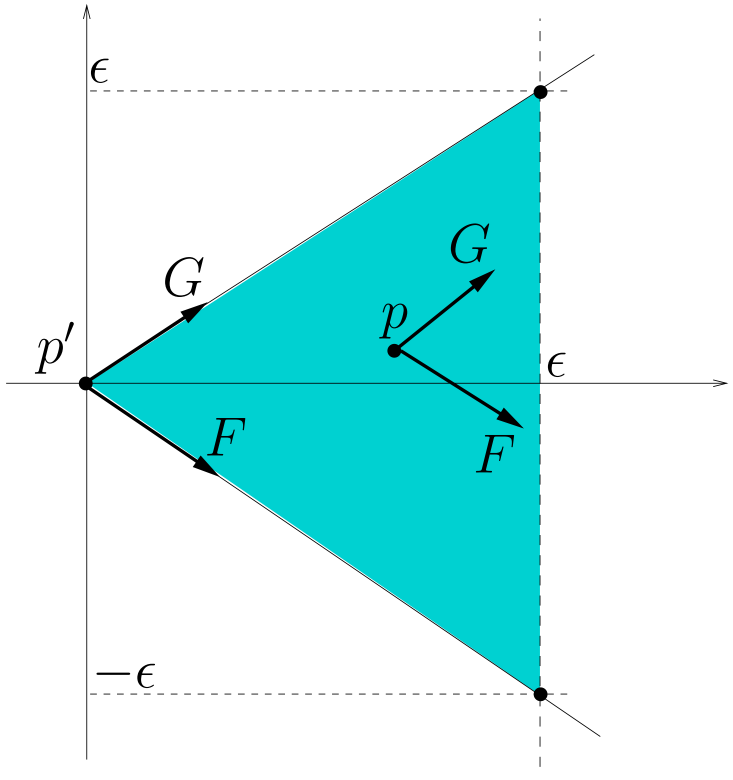

Figure 2: the set $T(p', \epsilon)$

Let us take a point q in $T(p', \epsilon)$. Any admissible curve of the dynamics, starting at q, leaves the triangle $T(p', \epsilon)$ in time less than 2ϵ (because of 3)) by the segment between the two points (ϵ, ϵ) and $(\epsilon, -\epsilon)$. Along this segment the vectors of the dynamics all point strictly outside $T(p', \epsilon)$. Hence the vectors of the dynamics point inside $\mathcal{A}(p') - T(p', \epsilon)$, which is then invariant by the dynamics.

In conclusion, if an admissible trajectory passes in $T(p', \epsilon)$, then it leaves $T(p', \epsilon)$ for $\mathcal{A}(p') - T(p', \epsilon)$ in time less than 2ϵ and never comes back in $T(p', \epsilon)$. Hence, $T(p', \epsilon)$ being a neighborhood of p, p cannot be an accumulation point. ■

Now, the accessible set of a point is bounded and any admissible curve of the control system is without non 0 accumulation point: hence any admissible curve of the dynamics converges to 0. The convergence is proved. ■

2.2 Proof of the stability

Let us take U a neighborhood of 0. Because both F and G are stable at 0, it exists ϵ such that any integral curve of F or G starting in the ball $B(0, \epsilon)$ never leaves U. Hence, for any point q of $B(0, \epsilon)$, $\gamma_F(q, .)$ and $\gamma_G(q, .)$ are included in U. But then also $\mathcal{A}(q)$ is included in U because $\mathcal{A}(q) = \mathcal{B}(q)$ which has $\gamma_F(q, .)$ and $\gamma_G(q, .)$ for boundary.

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