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NONISOTROPIC 3-LEVEL QUANTUM SYSTEMS: COMPLETE SOLUTIONS FOR MINIMUM TIME AND MINIMAL ENERGY

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RÉSUMÉ :

Nous appliquons les techniques de géométrie sous-riemannienne sur les groupes de Lie, et celles pour la synthèse optimale sur les variétés de dimension 2, à l'étude du transfert de population d'un système à trois niveaux d'énergie guidé par deux lasers d'amplitude et de fréquence arbitraires. Dans l'approximation RWA, nous considérons un modèle non isotropique, c'est-à-dire tel que le couplage n'est pas identique pour les deux lasers. Le but est de produire un transfert du premier au troisième niveau en minimisant : 1) le temps de transfert (avec des amplitudes bornées), 2) l'énergie des lasers. Après réduction du problème aux variables réelles, nous commençons à développer, pour 1), une théorie des problèmes "temps-optimal" distributionnels sur les variétés de dimension 2, et nous utilisons, pour 2), les techniques de géométrie sous-riemannienne sur les groupes de Lie de dimension 3.

MOTS CLÉS :

Contrôle Quantique, Contrôle Optimal, Géométrie Sous-Riemannienne, Groupes de Lie, Principe du Maximum de Pontryagin, Commutations

ABSTRACT:

We apply techniques of subriemannian geometry on Lie groups and of optimal synthesis on 2-D manifolds to the population transfer problem in a three-level quantum system driven by two laser pulses, of arbitrary shape and frequency. In the rotating wave approximation, we consider a nonisotropic model i.e. a model in which the two couplings of the lasers are different. The aim is to induce transitions from the first to the third level, minimizing: 1) the time of the transition (with bounded laser amplitudes), 2) the energy of lasers. After reducing the problem to real variables, for the purpose bf 1) we start to develop a theory for time optimal distributional problem on 2-D-manifolds, while for the purpose bf 2) we use techniques of subriemannian geometry on 3-D Lie groups.

KEY WORDS :

Control of Quantum Systems, Optimal Control, Subriemannian Geometry, Lie Groups, Pontryagin Maximum Principle, Bang-Bang

Nonisotropic 3-level Quantum Systems: Complete Solutions for Minimum Time and Minimal Energy

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Abstract We apply techniques of subriemannian geometry on Lie groups and of optimal synthesis on 2-D manifolds to the population transfer problem in a three-level quantum system driven by two laser pulses, of arbitrary shape and frequency. In the rotating wave approximation, we consider a nonisotropic model i.e. a model in which the two couplings of the lasers are different. The aim is to induce transitions from the first to the third level, minimizing **1)** the time of the transition (with bounded laser amplitudes), **2)** the energy of lasers. After reducing the problem to real variables, for the purpose **1)** we start to develop a theory for time optimal distributional problem on 2-D-manifolds, while for the purpose **2)** we use techniques of subriemannian geometry on 3-D Lie groups.

Keywords: Control of Quantum Systems, Optimal Control, Subriemannian Geometry, Lie Groups, Pontryagin Maximum Principle, Bang-Bang.

1 Introduction

The problem of designing an efficient transfer of population between different atomic or molecular levels is crucial in many atomic-physics projects [22, 33, 25, 43, 5]. Often excitation or ionization is accomplished by using a sequence of laser pulses to drive transitions from each state to the next state. The transfer should be as efficient as possible in order to minimize the effects of relaxation or decoherence that are always present. In the recent past years, people started to approach the design of laser pulses by using Geometric Control Techniques (see for instance [2, 21, 40, 31, 32, 41]). Finite dimensional closed quantum systems are in fact left invariant control systems on $SU(n)$, or on the corresponding Hilbert sphere $S^{2n-1} \subset \mathbb{C}^n$, where n is the number of atomic or molecular levels. For these kinds of systems very powerful techniques were developed both for what concerns controllability [3, 23, 29, 30, 42] and optimal control [1, 11, 26].

In this paper we apply techniques of subriemannian geometry on Lie groups and of optimal synthesis on 2-D manifolds to the population transfer problem in a three-level quantum system driven by two external fields (in the rotating wave approximation) of arbitrary shape and frequency. The dynamics is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$):

$$i\frac{d\psi(t)}{dt} = H\psi(t), \qquad (1)$$

where $\psi(.) = (\psi_1(.), \psi_2(.), \psi_3(.)) : \mathbb{R} \to \mathbb{C}^3$, $\sum_{j=1}^{3} |\psi_j(t)|^2 = 1$ (i.e. $\psi(t)$ belong to the sphere $S^5 \subset \mathbb{C}^3$), and:

$$H = \begin{pmatrix} E_1 & \mu_1\mathcal{F}_1 & 0 \\ \mu_1\mathcal{F}_1^* & E_2 & \mu_2\mathcal{F}_2 \\ 0 & \mu_2\mathcal{F}_2^* & E_3 \end{pmatrix}. \qquad (2)$$

Here $(^*)$ indicates the complex conjugation involution.

The controls $\mathcal{F}_1(.), \mathcal{F}_2(.)$, that we assume to be different from zero only in a fixed interval are the external pulsed field, while $\mu_j > 0, \quad (j = 1, 2)$ are the couplings (intrinsic to the quantum system) that we have restricted to couple only levels j and $j + 1$ by pairs.

We call the system (2) isotropic in the ("nongeneric") case in which $\mu_1 = \mu_2$ otherwise we call the system nonisotropic.

This model belongs to a class of systems on which it is possible to eliminate the so called drift term (i.e. the term $diag(E_1, E_2, E_3)$) by a unitary change of coordinates and a change of controls (see below).

The aim is to induce complete population transfer from the state one ($|\psi(0)_1|^2 = 1$) to the state three ($|\psi(0)_3|^2 = 1$) minimizing the costs described in the following.

- **Energy**

$$\int_0^T \left(|\mathcal{F}_1(t)|^2 + |\mathcal{F}_2(t)|^2\right) \, dt, \tag{3}$$

This cost is the energy of the laser pulses. After elimination of the drift and reduction to a real problem (see below), the problem of minimizing this cost becomes a singular-Riemannian problem (or a subriemannian problem when lifted on the group $SO(3)$, see Section 5) and was studied in [13], in the isotropic case. For this cost the final time must be fixed otherwise there are not solutions to the minimization problem. We recall that a minimizer for this cost is parameterized with constant velocity ($|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 = const$). For the costs (3), the controls are assumed to be unbounded. Anyway if the final time T is fixed in such a way the minimizer is parameterized by arc-length ($|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 = 1$), then minimizing the cost (3) is equivalent to minimize time with moduli of controls constrained in the closed set:

$$|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 \leq 1. \tag{4}$$

- **Time with Bounded Controls**

We can also want to minimize time of transfer under the conditions:

$$|\mathcal{F}_1(t)| \leq \nu_1 \text{ and } |\mathcal{F}_2(t)| \leq \nu_2 \tag{5}$$

Here the isotropic case corresponds to $\mu_1\nu_1 = \mu_2\nu_2$. Let notice that if there is no constraint on the laser pulses, then there is no minimizer.

In the isotropic case one can normalize $\mu_1 = \mu_2 = 1$ (this corresponds to make the change of notation $\mu_j\mathcal{F}_j(.) \to \mathcal{F}_i(.)$). While in the nonisotropic case we can normalize $\mu_1 = 1$, $\mu_2 = \alpha > 0$ (this corresponds to make the change of notation $\mu_1\mathcal{F}_1(.) \to \mathcal{F}_1(.)$, $\mu_2 \to \alpha > 0$). Notice that this change of notation modifies the costs only for an (irrelevant) multiplicative constant.

Remark **1** The problem of inducing a transition from the first to the third eigenstate, can be formulated, as usual, at the level of the wave function $\psi(t)$, but also at the level of the time evolution operator (the resolvent), denoted here by $g(t)$:

$$\psi(t) = g(t)\psi(0), \quad g(t) \in U(3), \quad g(0) = id. \tag{6}$$

For $g(t)$ the control system (9) reads:

$$\dot{g}(t) = Hg(t) \tag{7}$$

In the following we will call the optimal control problem for $\psi(t)$ and for $g(t)$ respectively *the "problem downstairs"* and *the "problem upstairs"*. In this paper we take advantages of working both upstairs and downstairs depending on the specific problem. This approach happened to be very useful in some other problem of optimal control on Lie groups, see for instance [16]. Notice that the evolution of the trace part is decoupled from the rest. It follows that we can always assume $E_1 + E_2 + E_3 = 0$ and $g(t) \in SU(3)$.

In [14], it was proved that for a convex cost depending only on the moduli of controls (i.e. amplitudes of the lasers), like those described above, there always exists a minimizer in resonance that connects a source and a target defined by conditions on the moduli of the components of the wave function (e.g. two eigenstates):

$$\mathcal{F}_j(t) = u_j(t)e^{i[(E_{j+1}-E_j)t+\phi_j]}, \quad j = 1, 2, \tag{8}$$

where $\phi_j \in [-\pi, \pi]$ are arbitrary phases and $u_j(.) : \mathbb{R} \to \mathbb{R}$ are the amplitudes of the lasers that should be determined.

Remark **2** In [14] this result was proved for a quite more general n-level system:

$$\begin{cases} i\dfrac{d\psi(t)}{dt} = \mathcal{H}(t)\psi(t) := (D + V(t))\psi \\ \psi(.) := (\psi_1(.), ..., \psi_n(.)) : \mathbb{R} \to \mathbb{C}^n, \quad \sum_i |\psi_i|^2 = 1, \end{cases}$$

where $D = diag(E_1, ..., E_n)$ and $V(t)$ is an Hermitian matrix ($V(t)_{j,k} = V(t)^*_{k,j}$), whose elements are either identically zero or controls.

Using the so called interaction picture (i.e making the unitary transformation $\psi(t) \to U^{-1}(t)\psi(t)$, where $U(t) = diag(-iE_1, -iE_2, -iE_3)$) and next making the further transformation $\psi(t) \to V^{-1}\psi(t)$, where $V := diag(1, e^{i(-\pi/2-\phi_1)}, e^{i(-\pi-\phi_1-\phi_2)})$ (to kill the phases) the Schrödinger equation becomes the equation:

$$\frac{d\psi(t)}{dt} = H(t)\psi(t), \tag{9}$$

that one can restrict to reals (i.e. to the sphere S^2, i.e. $\psi(.) : \mathbb{R} \to \mathbb{R}^3$, $\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$) by taking a real initial condition. Now the Hamiltonian belongs to $so(3)$:

$$H = \begin{pmatrix} 0 & -u_1 & 0 \\ u_1 & 0 & -\alpha u_2 \\ 0 & \alpha u_2 & 0 \end{pmatrix}. \tag{10}$$

For details on these transformations, see [12, 13, 14]. After elimination of the drift, the costs to which we are interested become:

1. Energy in fixed time: $\displaystyle\int_0^T \left(u_1(t)^2 + u_2(t)^2\right) dt,$
2. Time with bounded controls: $\displaystyle T = \int_0^T 1\, dt,$ under the condition $|u_1(t)| \leq 1$ and $|u_2(t)| \leq 1$.

Downstairs the source and the target become $\psi_1 = \{\pm 1\}$, $\psi_3 = \{\pm 1\}$, while upstairs we have that $g(t) \in SO(3) \subset U(3)$, and the source and the target are respectively the sets (cf. [13]):

$$\begin{aligned} \mathcal{S}^u &:= \left\{ \left(\begin{array}{c|c} Z_2 & 0 \\ \hline 0 & O(2) \end{array} \right) \in SO(3) \right\} = S(Z_2 \times O(2)), \\ \mathcal{T}^u &:= g_0\, \mathcal{S}^u = g_0\, S(Z_2 \times O(2)), \quad \text{where } g_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3), \end{aligned} \tag{11}$$

The most important and powerful tool for the study of optimal trajectories is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [1, 26, 39]). It is a first order necessary condition for optimality and generalizes the Weierstraß conditions of Calculus of Variations to problems with non-holonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle that is a solution to a suitable pseudo–Hamiltonian system.

Anyway, giving a complete solution to an optimization problem (that for us means to give an optimal synthesis, see for instance [6, 11, 19, 20, 38]) remains extremely difficult for several reasons. First, one is faced with the problem of integrating a Hamiltonian system (that generically is not integrable excepted for very special costs). Second, one should manage with "non Hamiltonian solutions" of the PMP, the so called abnormal extremals. Finally, even if one is able to find all the solutions of the PMP it remains the problem of selecting among them the optimal trajectories. For these reasons, usually, one can hope to find a complete solution of an optimal control problem in low dimension only.

This paper is the continuation of a series of papers on optimal control of finite dimensional quantum systems [12, 13, 14].

In [12], the problem of minimizing the energy of the lasers pulses was studied for a two-level system and for an isotropic three-level system. The two-level system (that is a problem on $SU(2)$) was completely solved, while the three-level problem was solved assuming resonance (i.e. controls in the form (8)) as hypothesis.

In [13] the problem of minimizing the isotropic energy in the three-level problem was treated without assuming resonance as hypothesis. In that case, even if the optimal control problem is in a space of quite big dimension ($SU(3)$ has dimension 8) it was possible to get explicit expressions of optimal controls and trajectories thanks to a special Lie-structure of the problem (and to the fact that the cost can be built with the Killing form), that renders the Hamiltonian system associated to the PMP, Liouville integrable. In this paper resonance was obtained as consequence of the minimization process and explicit expressions of amplitudes were exhausted. As explained in Remark 2 and above, in [14] it was proved that it is possible to restricts to resonant trajectories in a quite general n-level system.

This results permits to reduce the problem to real variables (i.e. from $SU(n)$ to $SO(n)$, if working upstairs or from $S^{2n-1} \subset \mathbb{C}^n$ to $S^{n-1} \subset \mathbb{R}^n$ if working downstairs), and therefore to simplify considerably the difficulty of the problem.

This reduction of dimension is crucial in finding complete solutions to the optimal control problem in many cases. For instance for the three-level system (1), (2):

A. the minimum time problem with bounded controls is a problem in dimension 5. In this case, since the dimension of the state space is big, the problem of finding extremals and selecting optimal trajectories can be extremely hard. The fact that one can restrict to minimizers that are in resonance permits to reduce the problem to a bi-dimensional problem, that can be solved with techniques similar to those used in [11, 26, 37, 44]). This is the goal of Section 4;

B the minimum energy problem that is naturally lifted to a left invariant sub-Riemannian problem on the group $SU(3)$. This problem cannot be solved with the techniques used in [13] for the isotropic case, because now the cost is built with a "deformed Killing form". Anyway since we can restrict to resonant minimizers the problem is reduced to a contact sub-Riemannian problem on $SO(3)$, that does not have abnormal extremals (since it is contact) and the corresponding Hamiltonian system is completely integrable (since it leads to a left invariant Hamiltonian system on a Lie group of dimension 3). The complete solutions can be found in terms of Elliptic functions. This is the aim of Section 5.

Remark **3** Thanks to the reduction given by [14], a minimum energy problem for a 4-level system, with levels coupled by pairs, is a singular Riemannian problem on S^3 or can be lifted to a left invariant sub-Riemannian problem on $SO(4)$. Anyway there are no reasons to believe that the Hamiltonian system associated to the PMP is Liouville integrable in this case. See [15] for some numerical solutions to this problem.

In Section 2 we state our main tool i.e. the PMP and we show that there are not abnormal extremals. In Section 3 we states our main results. In Section 4 we treat the minimum time problem. In Subsection 4.1 we start to develop the theory of Time Optimal Syntheses on 2-D manifolds for distributional systems with bounded controls (this section is written in such a way to be as self-consistent as possible) and in the subsequent subsection we build explicitly the time optimal synthesis.

In Section 5, attacking the problem upstairs, we treat the minimum energy problem. We explicitly integrate the Hamiltonian system associated to the PMP and we give almost explicit expressions (in term of elliptic functions) for optimal controls linking the first and the third level.

2 Pontryagin Maximum Principle

Theorem (Pontryagin Maximum Principle) *Consider a control system of the form* $\dot{x} = f(x,u)$ *with a cost of the form* $\int_0^T f^0(x,u)\ dt$*, and initial and final conditions given by* $x(0) \in M_{in}$*,* $x(T) \in M_{fin}$*, where* x *belongs to a manifold* M *and* $u \in U \subset \mathbb{R}^m$*. Assume moreover that* M*,* f*,* f^0 *are smooth and that* M_{in} *and* M_{fin} *are smooth submanifolds of* M*. If the couple* $(u(.), x(.)) : [0,T] \subset \mathbb{R} \to U \times M$ *(with* $u(.)$ *measurable and essentially bounded, and* $x(.)$ *Lipschitz) is optimal, then there exists a never vanishing field of covectors along* $x(.)$*, that is an Lipschitz continuous function* $(\lambda(.), \lambda_0) : t \in [0,T] \mapsto (\lambda(t), \lambda_0) \in T^*_{x(t)}M \times \mathbb{R}$ *(where* $\lambda_0 \leq 0$ *is a constant) such that:*

i) $\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \lambda}(x(t), \lambda(t), u(t))$,

ii) $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), \lambda(t), u(t))$,

where by definition:

$$\mathcal{H}(x, \lambda, u) := < \lambda, f(x,u) > + \lambda_0 f^0(x,u). \tag{12}$$

Moreover:

iii) $\mathcal{H}(x(t), \lambda(t), u(t)) = \mathcal{H}_M(x(t), \lambda(t))$*, for a.e.t* $\in [0,T]$*,*
where $\mathcal{H}_M(x, \lambda) := \max_{v \in U} \mathcal{H}(x, \lambda, v)$.

iiii) $\mathcal{H}_M(x(t), \lambda(t)) = k \geq 0$*, where* k *depends on the final time (if it is fixed) or* $k = 0$ *if it is free.*

v) $< \lambda(0), T_{x(0)}M_{in} > = < \lambda(T), T_{x(T)}M_{fin} > = 0$ *(transversality conditions).*

Definition 1 *The real-valued map on* $T^*M \times U$*, defined in (12) is called* PMP-Hamiltonian*. A trajectory* $x(.)$ *(resp. a couple* $(u(.), \lambda(.))$*) satisfying conditions* **i)**, **ii)**, **iii)** *and* **iiii)** *is called an* extremal *(resp.* extremal pair*). If* $(x(.), \lambda(.))$ *satisfies* **i)**, **ii)**, **iii)** *and* **iiii)** *with* $\lambda_0 = 0$ *(resp.* $\lambda_0 < 0$*), then it is called an* abnormal extremal *(resp. a* normal extremal*). An abnormal extremal is said to be* non trivial *(in the following NTAE) if it does not correspond to controls a.e. vanishing.*

Remark **4** Notice that the definition of abnormal extremal does not depend on the cost but only on the dynamics (in fact if $\lambda_0 = 0$, the cost disappears in (12)). Moreover if $(x(.), \lambda(.))$ is a trivial abnormal extremal on $[c,d]$ then $x(.) = x(c)$ on $[c,d]$.

Proposition 1 *For the control systems downstairs (9), (10), there are no NTAE. The same holds for the control system upstairs (7), (10).*

Proof. In both problems (with or without constraints), downstairs or upstairs, the set of admissible velocities at a point x writes:

$$\{u_1 F_1(x) + u_2 F_2(x) \mid (u_1, u_2) \in U\}$$

where the set of admissible controls U is a subset of $\mathbb{R}^2$ containing 0 in its interior. This implies that, if there were an abnormal extremal, its covector λ should annihilate F_1 and F_2 all along the trajectory, but also, using PMP, their bracket $[F_1, F_2]$. But in all cases, $(F_1, F_2, [F_1, F_2])$ form a generating family of the tangent space which implies that $\lambda \equiv 0$ all along the trajectory. This is forbidden by the PMP. ■

3 Main Results

In this section we present explicitly the optimal controls steering state 1 to state 3 for our two costs. The complete optimal synthesis is presented in Sections 4 and 5.

3.1 Minimum Time

Here we present the time-optimal controls steering state 1 to state 3. For the problem of minimum time there are three different cases, corresponding to the sign of $\alpha - 1$.

If $0 < \alpha < 1$ then the optimal minimizer corresponds to:

- $u_1(t) = u_2(t) = 1$ for $t \in [0, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}[$. The corresponding trajectory steers state 1 to state $(0, \sqrt{1-\alpha^2}, \alpha)$.
- $u_1(t) = 0,\ u_2(t) = 1$ for $t \in [\frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}} + \frac{\arccos \alpha}{\alpha}]$. The corresponding trajectory steers the state $(0, \sqrt{1-\alpha^2}, \alpha)$ to the state 3.

If $1 < \alpha$ then the optimal minimizer corresponds to:

- $u_1(t) = 1,\ u_2(t) = 0$ for $t \in [0, \arccos(\frac{1}{\alpha})[$. The corresponding trajectory steers the state 1 to the state $(\frac{1}{\alpha}, \sqrt{1 - \frac{1}{\alpha^2}}, 0)$.
- $u_1(t) = u_2(t) = 1$ for $t \in [\arccos(\frac{1}{\alpha}), \arccos(\frac{1}{\alpha}) + \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})]$. The corresponding trajectory steers the state $(\frac{1}{\alpha}, \sqrt{1 - \frac{1}{\alpha^2}}, 0)$ to the state 3.

If $\alpha = 1$ then the optimal minimizer corresponds to $u_1(t) = u_2(t) = 1$ for $t \in [0, \frac{\pi}{\sqrt{2}}]$ and steers, without switching, state 1 to state 3.

3.2 Minimum Energy

Here we present the optimal controls for energy for optimal trajectories starting from state 1. The speed has been normalized to 1.

Any optimal trajectory (ψ_1, ψ_2, ψ_3) for energy, linking state 1 to another state, has control functions u_1 and u_2 such that there exists some real m such that, for any time t:

$$u_1(t) = -\mathrm{cd}\left(\sqrt{\frac{m^2+1-\alpha^{-1/2}}{\alpha^{-1/2}}}t; \sqrt{\frac{1-\alpha^{-1/2}}{m^2+1-\alpha^{-1/2}}}\right), \tag{13}$$

$$u_2(t) = \sqrt{\frac{m^2}{\alpha^{-1/2}(m)^2+1-\alpha^{-1/2})}}\ \mathrm{sd}\left(\sqrt{\frac{m^2+1-\alpha^{-1/2}}{\alpha^{-1/2}}}t; \sqrt{\frac{1-\alpha^{-1/2}}{m^2+1-\alpha^{-1/2}}}\right), \tag{14}$$

where *cd* and *sd* are the two elliptic functions of the second kind defined in [47]. Moreover, it is possible (see 5) to get intricate but explicit expression of (ψ_1, ψ_2, ψ_3) depending only on the real m.

The way of computing the value of m corresponding to the minimizer steering state 1 to state 3 is presented at the end of the section 5.

4 Minimum Time

4.1 Minimum Time for Distributional Systems on 2-D Manifolds

In this section, we start to develop the theory of Time Optimal Syntheses on 2-D manifolds for distributional systems with bounded controls. This section is written to be as self-consistent as possible.

4.1.1 Basic Definitions and PMP

We focus on the following:

Problem (P) *Consider the control system:*

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \quad x \in M, \quad |u_i| \leq 1, \quad i = 1, 2, \tag{15}$$

where:

(H0) *M is a smooth two dimensional manifold. The vector fields $F_1(x)$ and $F_2(x)$ are $\mathcal{C}^\infty$.*

We are interested to the problem of reaching every point of M in minimum time from a fixed point $x_0 \in M$.

The theory developed next will be then applied to our three-level quantum systems:

$$\begin{cases} \dot{x} = u_1 F_1(x) + u_2 F_2(x), \quad x \in S^2, \quad |u_i| \leq 1, \quad i = 1, 2, \\ F_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = -x_2 \dfrac{\partial}{\partial x_1} + x_1 \dfrac{\partial}{\partial x_2}, \\ F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = \alpha \left(-x_3 \dfrac{\partial}{\partial x_2} + x_2 \dfrac{\partial}{\partial x_3} \right). \end{cases} \tag{16}$$

Definition 2 *A control for the system (15) is a measurable function $u(.) = (u_1(t), u_2(t)) : [a, b] \to [-1, 1]^2$. The corresponding trajectory is a Lipschitz continuous map $\gamma : [a_1, a_2] \to M$ such that $\dot{\gamma}(t) = u_1(t) F_1(\gamma(t)) + u_2(t) F_2(\gamma(t))$ for almost every $t \in [a_1, a_2]$. Since the system is autonomous we can always assume that $Dom(\gamma) = [0, a]$.*

For us a solution to the problem **(P)** is an optimal synthesis that is a collection $\{(\gamma_{x_1}, u_{x_1}) : x_1 \in M\}$ of trajectory–control pairs such that $\gamma_{x_1}(0) = x_0$, $\gamma_{x_1}(T) = x_1$, and γ_{x_1} is time optimal.

To the purpose of constructing the optimal synthesis, we use ideas similar to those used by Sussmann, Bressan, Piccoli and the first author in [8, 10, 9, 17, 18, 36, 37, 45, 46] and recently rewritten in [11], for the minimum time stabilization to the origin for the "control affine version" of the same problem ($\dot{x} = F(x) + uG(x)$, $u \in [-1, 1]$, where x belongs to a 2-D-manifold).

Here we use the notation $u = (u_1, u_2)$ and (in a local charts) $x = (x_1, x_2)$, $F_1 = ((F_1)_1, (F_1)_2)$, $F_2 = ((F_2)_1, (F_2)_2)$. Let us introduce some notations to describe different types of controls.

Definition 3 *Let $u(.) = (u_1(.), u_2(.)) : [a, b] \to [-1, 1]^2$ be a control for the control system (15).*

- *$u(.)$ is said to be a bang control if for almost every $t \in [a, b]$, $u(t)$ is constant and belongs to a point of the set $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$.*
- *A switching time of $u(.)$ is a time $t \in [a, b]$ such that for every $\varepsilon > 0$, $u(.)$ is not bang on $(t - \varepsilon, t + \varepsilon) \cap [a, b]$.*
- *If $u_A : [a_1, a_2] \to [-1, 1]^2$ and $u_B : [a_2, a_3] \to [-1, 1]^2$ are controls, their concatenation $u_B * u_A$ is the control:*

  $$(u_B * u_A)(t) := \begin{cases} u_A(t) \text{ for } t \in [a_1, a_2] \\ u_B(t) \text{ for } t \in]a_2, a_3]. \end{cases}$$

  (Notice that in the notation $u_B * u_A$, the control u_A comes first). The control $u(.)$ is called bang-bang if it is a finite concatenation of bang arcs.
- *A u_i-switching time of $u(.) = (u_1(.), u_2(.))$ is a time $t \in [a, b]$ such that for every $\varepsilon > 0$, u_i is neither a.e. equal to 1 nor a.e. equal to -1 on any interval of the form $(t - \varepsilon, t + \varepsilon) \cap [a, b]$.*
- *A trajectory of (15) is a bang trajectory, (resp. bang-bang trajectory), if it corresponds to a bang control, (resp. bang-bang control).*

A key role is played by the following three functions defined on M:

$$\Delta_A(x) := det(F_1(x)), F_2(x)) = (F_1)_1(F_2)_2 - (F_2)_1(F_1)_2, \tag{17}$$
$$\Delta_{B1}(x) := det(F_1(x)), [F_1,F_2](x)) = (F_1)_1([F_1,F_2])_2 - ([F_1,F_2])_1(F_1)_2, \tag{18}$$
$$\Delta_{B2}(x) := det(F_2(x)), [F_1,F_2](x)) = (F_2)_1([F_1,F_2])_2 - ([F_1,F_2])_1(F_2)_2 \tag{19}$$

The set of zeros of $\Delta_A^{-1}(0)$, $\Delta_{B1}^{-1}(0)$, $\Delta_{B2}^{-1}(0)$ of these three functions are respectively the set of points where F_1 and F_2 are parallel, the set of points where F_1 is parallel to $[F_1,F_2]$ and the set of points where F_2 is parallel to $[F_1,F_2]$.

Using PMP one easily understand (see next section) that these loci are fundamental in the construction of the optimal synthesis. In fact assuming that they are smooth embedded one dimensional submanifold of M we have the following:

- in each connected region of $M \setminus (\Delta_A^{-1}(0) \cup \Delta_{B1}^{-1}(0) \cup \Delta_{B2}^{-1}(0))$, every extremal trajectory is bang-bang with at most two switching (one of the control u_1 and one of the control u_2). Moreover the possible switchings are determined. More precisely for every $x \in M \setminus (\Delta_A^{-1}(0) \cup \Delta_{B1}^{-1}(0) \cup \Delta_{B2}^{-1}(0))$ define the functions:

$$f_i(x) := -\frac{\Delta_{B_i}(x)}{\Delta_A(x)}. \tag{20}$$

  If $f_i > 0$ (resp. $f_i < 0$), we have that u_i can only switch from -1 to $+1$ (resp. from $+1$ to -1).

- the support of a u_i-singular trajectories (that are trajectories for which the u_i-switching function identically vanishes and for which u_i can assume values different from ± 1, see Definition 4 below) is always contained in the set $\Delta_{Bi}^{-1}(0)$.

For the problem **(P)**, the PMP says the following:

Theorem (Pontryagin Maximum Principle for the problem (P)) *Consider a control system of the form $\dot{x} = f(x,u)$ with a Define for every $(x,\lambda,u) \in T^*M \times [-1,1] \times [-1,1]$:*

$$\mathcal{H}(x,\lambda,u) = u_1 < \lambda, F_1(x) > +u_2 < \lambda, F_2(x) > +\lambda_0$$

and:

$$\mathcal{H}_M(x,\lambda) = \max\{\mathcal{H}(x,\lambda,u) : u \in [-1,1] \times [-1,1]\}. \tag{21}$$

If $\gamma(.) : [0,a] \to M$ is a (time) optimal trajectory corresponding to a control $u(.) : [0,a] \to [-1,1] \times [-1,1]$, then there exist a nontrivial field of covectors along $\gamma(.)$, that is a Lipschitz continuous function $\lambda(.) : t \in [0,a] \mapsto \lambda(t) \in T^*_{\gamma(t)}M$ never vanishing, and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in Dom(\gamma)$:

i) $\dot{\lambda}(t) = -\lambda(t) \cdot (u_1(t)\nabla F + u_2(t)\nabla G)(\gamma(t))$,

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$,

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = \mathcal{H}_M(\gamma(t), \lambda(t))$.

Remark **5** In this version of PMP, $\lambda(.)$ is always different from zero otherwise the condition **ii)** would imply $\lambda_0 = 0$ (cf. PMP in Section 2).

4.1.2 Switching Functions, Singular Trajectories and Predicting Switchings

In this section we are interested in determining when the controls switch from $+1$ to -1 (and vice versa) and when they may assume values in $]-1,+1[$. Moreover we explain how to predict which kind of switching can happen, using properties of the vector fields F_1 and F_2. A key role is played by the following:

Definition 4 (Switching Functions) *Let $(x(.), \lambda(.))$ be an extremal pair. The corresponding switching functions are defined as $\phi_i(t) :=< \lambda(t), F_i(x(t)) >$.*

Remark **6** Notice that $\phi_i(.)$ are at least Lipschitz continuous. Moreover using the switching functions **ii)** of PMP reads:

$$\mathcal{H}(x(t), \lambda(t), u(t)) + \lambda_0 = u_1(t)\phi_1(t) + u_2(t)\phi_2(t) + \lambda_0 = 0 \text{ a.e..} \tag{22}$$

The following Lemma, characterizes NTAE.

Lemma 1 *Let $(x(.), \lambda(.))$ (defined on $[a_1, a_2]$) be an extremal pair. Then:*

1. *$(x(.), \lambda(.))$ is a NTAE, if and only if $\phi_1(.) \equiv \phi_2(.) \equiv 0$ on $[a_1, a_2]$;*

2. *if $(x(.), \lambda(.))$ is a NTAE, then $Supp(x(.)) \subset \Delta_A^{-1}(0)$;*

3. *if there are no NTAE, $\phi_1(.)$ and $\phi_2(.)$ never vanish at the same time.*

Proof. Let us prove **1.** The sufficiency is obvious. Let us prove the necessity. Equation (22), with $\lambda_0 = 0$ and **iii)** of PMP, imply that both controls vanish (trivial abnormal extremals) or that $\phi_1(.) = \phi_2(.) \equiv 0$ on $[a_1, a_2]$. Let us prove **2.** From $\phi_1(.) = \phi_2(.) \equiv 0$ we have that $\lambda(.)$ is orthogonal to $F_1(x(.))$ and $F_2(x(.))$. Since $\lambda(.)$ is non trivial it follows that $F_1(x(.))$ is parallel to $F_2(x(.))$. To prove **3,** assume by contradiction that there are no NTAE and that there exists a time $\bar{t}$ for which $\phi_1(\bar{t}) = \phi_2(\bar{t}) = 0$. From (22) there exists a sequence $t_m \nearrow \bar{t}$ such that $u_1(t_m)\phi_1(t_m) + u_2(t_m)\phi_2(t_m) + \lambda_0 = 0$. Since $\phi_i(t_m) \to 0$, $i = 1, 2$, it follows $\lambda_0 = 0$. Contradiction. ■

Remark **7** We recall that for the problem (9), (10) (i.e. (16)), there are no NTAE (cf. Proposition 1). Anyway the presence of NTAE is highly nongeneric also for the system (15) under **(H0)**. See also Remark 10.

The switching functions determine when the controls switch from +1 to -1 and vice versa. In fact from the maximization condition **iii)** one immediately get:

Lemma 2 *Let $(x(.), \lambda(.))$ define on $[0, a]$ be an extremal pair and $\phi_i(.)$ the corresponding switching functions. If $\phi_i(t) \neq 0$ for some $t \in]0, a[$, then there exists $\varepsilon > 0$ such that $x(.)$ corresponds to a constant control $u_i = sgn(\phi_i)$ on $]t - \varepsilon, t + \varepsilon[$. Moreover if $\phi_i(.)$ has a zero at t, $\dot{\phi}_i(t)$ exists and it is strictly greater than zero (resp. smaller than zero) then there exists $\varepsilon > 0$ such that $x(.)$ corresponds to constant control $u_i = +1$ on $]t - \varepsilon, t[$ and to constant control $u_i = -1$ on $]t, t + \varepsilon[$ (resp. to constant control $u_i = -1$ on $]t - \varepsilon, t[$ and to constant control $u_i = +1$ on $]t, t + \varepsilon[$).*

Notice that on any interval where $\phi_i(.)$ has no zero (respectively finitely many zeroes) the corresponding control u_i is bang (respectively bang-bang). We are then interested in differentiating ϕ_i. One immediately get:

Lemma 3 *Let $(x(.), \lambda(.))$, defined on $[0, a]$ be an extremal pair and $\phi_i(.)$ the corresponding switching functions. Then a.e. it holds:*

$$\dot{\phi}_1(t) = u_2(t) < \lambda(t), [F_2, F_1](x(t)) >, \tag{23}$$
$$\dot{\phi}_2(t) = u_1(t) < \lambda(t), [F_1, F_2](x(t)) > . \tag{24}$$

From Lemma 2 it follows that u_i can assume values different from ± 1 on some interval $[a_1, a_2]$ only if the corresponding switching function vanishes identically there.

Remark **8** Lemma 1 asserts that, if there are no NTAE (as for the system (16)), then u_1 and u_2 never switch at the same time. In this case, from Lemma 3 it follows that in a neighborhood of a u_1-switching, $\phi_1(t)$ is a $\mathcal{C}^1$ function. A similar statement holds for $\phi_2(t)$.

Definition 5 *An extremal trajectory $x(.)$ defined on $[a_1, a_2]$ is said to be u_i-singular if the corresponding switching function $\phi_i(.)$ vanishes identically on $[a_1, a_2]$.*

Remark **9** From Lemma 1 it follows that if $(x(.), \lambda(.))$ is a NTAE, if and only if it is a u_1-u_2-singular trajectory.

We are now going to show that out from the set $\Delta_A^{-1}(0) \cup \Delta_{B1}^{-1}(0) \cup \Delta_{B2}^{-1}(0)$, only one switching for each control is permitted and that u_i singular trajectories must run on the set $\Delta_{Bi}^{-1}(0)$.

Definition 6 *A point $x \in M$ is called an ordinary point if $x \notin \Delta_A^{-1}(0) \cup \Delta_{B1}^{-1}(0) \cup \Delta_{B2}^{-1}(0)$. If x is an ordinary point, then $F_1(x), F_2(x)$ form a basis of T_xM and we define the scalar functions $f_1(x), f_2(x)$ to be the coefficients of the linear combination: $[F_1, F_2](x) = f_2(x)F_1(x) - f_1(x)F_2(x)$.*

The function f_1 and f_2 are crucial in studying which kind of switchings can happen near ordinary points. But first we need a relation between f_1, f_2, and the functions Δ_A, Δ_{B_1}, Δ_{B_2}.

Lemma 4 *Let x an ordinary point then:*

$$f_i(x) = -\frac{\Delta_{B_i}(x)}{\Delta_A(x)}, \qquad i = 1, 2. \tag{25}$$

Proof. We have:

$$\begin{aligned} \Delta_{B_1}(x) &= Det\big(F_1(x), [F_1, F_2](x)\big) = Det(F_1(x), f_2(x)F_1(x) - f_1(x)F_2(x)) \\ &= -f_1(x)Det\big(F_1(x), F_2(x)\big) = -f_1(x)\Delta_A(x), \end{aligned}$$

and similarly for Δ_{B_2}. ■

On a set of ordinary points the structure of optimal trajectories is particularly simple:

Proposition 2 *Let $\Omega \in M$ be an open set such that every $x \in \Omega$ is an ordinary point. Then all extremal trajectories $x(.)$ of $\Sigma|_\Omega$ are bang-bang with at most a u_1-switching and a u_2-switching. Moreover if $f_i > 0$ (resp. $f_i < 0$) throughout Ω then γ corresponds to control u_i equal to $+1$, -1 or has a $-1 \to +1$ switching (resp. has a $+1 \to -1$ switching).*

Proof. Let $x(.) :]a_1, a_2[\to \Omega$ be an extremal trajectory and $\phi_i(.)$, $i = 1, 2$ the corresponding switching functions. If $\phi_i(.)$, $i = 1, 2$ has no zeros, then $x(.)$ is bang and the conclusion follows.

Let t_1 be a zero of $\phi_1(.)$. The time t_1 cannot be a zero of $\phi_2(.)$ otherwise $x(t_1)$ could not be an ordinary point (we would have $\Delta_A(x(t_1) = 0)$. It follows that $\phi_1(.)$ is $\mathcal{C}^1$ in a neighborhood of t_1. Also t_1 cannot be a zero of $\dot{\phi}_1(.)$ otherwise $x(t_1)$ could not be an ordinary point (we would have $\Delta_{B_1}(x(t_1) = 0)$. Since u_2, in a neighborhood of t_1, is a.e. constantly equal to $\bar{u} \in \{\pm 1\}$ we can set $u_2(t_1) = \bar{u}$. We have:

$$\begin{aligned} \dot{\phi}_1(t_1) &= u_2(t_1) < \lambda(t_1), [F_2, F_1](x(t_1)) >= u_2(t_1) < \lambda(t_1), (-f_2F_1 + f_1F_2)x(t_1)) > \\ &= u_2(t_1)f_1(x(t_1)) \ < \lambda(t_1), F_2(x(t_1)) >= u_2(t_1)f_1(x(t_1)) \ \phi_2(t_1). \end{aligned} \tag{26}$$

Now from Lemma 2 we have that $sgn(u_2(t_1)) = sgn(\phi_2(t_1))$, and being $\phi_2(t_1) \neq 0$ it follows $sgn(\dot{\phi}_1(t_1)) = sgn(f_1(x(t_1)))$. Using again Lemma 2, it follows that if $f_1 > 0$ (resp. $f_1 < 0$) then we can have only a $+1 \to -1$ switching (resp. a $-1 \to +1$ switching). A similar proof can be done for a zero of ϕ_2. ■

We are now interested in analyzing properties of u_i-singular trajectories.

Lemma 5 *Let $(x(.), \lambda(.))$ be u_i-singular trajectory on $[a_1, a_2]$, then $Supp(x(.)|_{[a_1,a_2]}) \subset \Delta_{Bi}^{-1}(0)$.*

Proof. . To simplify the notation assume $i = 1$, the case $i = 2$ being similar. From $\phi_1(.) =< \lambda(.), F_1(x(.)) >\equiv 0$ on $[a_1, a_2]$ we have that $\dot{\phi}_1(.) = u_2(.) < \lambda(.), [F_1, F_2](x(.)) >\equiv 0$ a.e. in $[a_1, a_2]$. Since $(x(.), \lambda(.))$ is not a NTAE, it follows that u_2 is a.e. equal to $+1$ or -1 in $[a_1, a_2]$. Therefore $\lambda(.)$ is orthogonal both to $F_1(x(.))$ and $[F_1, F_2](x(.))$. Being $\lambda(.)$ non trivial it follows that $F_1(x(.))$ is parallel to $[F_1, F_2](x(.))$. It follows the conclusion. ■

Remark **10** From Lemma 1 and 5 it follows that if $(x(.), \lambda(.))$ is a NTAE, then $Supp(x(.)) \subset \Delta_A^{-1}(0) \cap \Delta_{B1}^{-1}(0) \cap \Delta_{B2}^{-1}(0)$. This is clearly a highly non generic situation.

Lemma 6 *Let $(x(.), \lambda(.))$ be u_1-singular trajectory (resp. u_2-singular trajectory) on $[a_1, a_2]$. Then:*

- $<\nabla\Delta_{B_1}(x(t)), F_1(x(t))>\neq 0$ *(resp.* $(<\nabla\Delta_{B_2}(x(t)), F_2(x(t))>\neq 0$*).*
- *Define respectively the functions:*

$$\varphi_1(t) \quad = \quad -u_2(t)\frac{<\nabla\Delta_{B_1}(x(t)), F_2(x(t))>}{<\nabla\Delta_{B_1}(x(t)), F_1(x(t))>}, \tag{27}$$

$$\varphi_2(t) \quad = \quad -u_1(t)\frac{<\nabla\Delta_{B_2}(x(t)), F_1(x(t))>}{<\nabla\Delta_{B_2}(x(t)), F_2(x(t))>}, \tag{28}$$

Then $\varphi_1(.)$ *(resp.* $\varphi_2(.)$*) satisfies* $|\varphi_1(t)|\leq 1$ *(resp.* $|\varphi_2(t)|\leq 1$*) a.e. in* $[a_1,a_2]$*. Moreover* $x(.)$ *corresponds to the control* $\varphi_1(.)$ *(resp.* $\varphi_2(.)$*) in* $[a_1,a_2]$*.*

Proof. Assume $i=1$, the case $i=2$ being similar.

The first conclusion follows by the fact that if F_1 is tangent to $\Delta_{B1}^{-1}(0)$ then the condition $x([a_1,a_2]))\subset\Delta_{B1}^{-1}(0)$ (implied by Lemma 5) cannot be satisfied.

Now let $u_1(.)$, $u_2(.)$ be the controls corresponding to $x(.)$, that is $\dot{x}(t)=u_1(t)F_1(x(t))+u_2(t)F_2(x(t))$, for almost every t. From $\Delta_{B_1}(x(t))=0$, we have for a.e. t:

$$0=\frac{d}{dt}\Delta_{B_1}(x(t))=\nabla\Delta_{B_1}(x(t))\cdot(u_1(t)F_1(x(t))+u_2(t)F_2(x(t))).$$

This means that $u_1(t)=\varphi_1(t)$. The condition $|\varphi_1(t)|\leq 1$ is simply a consequence of the fact that $x(.)$ is by assumption an admissible trajectory. ■

4.1.3 More on Singular Trajectories

In general if the set $\Delta_{B1}^{-1}(0)$ and $\Delta_{B2}^{-1}(0)$ are not empty, this does not means that a singular trajectory is running them.

The descriptions of singular trajectories is in very intricate in general (see for instance [11] for the control affine case). Here we describes what happen in the simplest case (that is enough for what follows) i.e. when we restrict to a subset S_1 (resp. S_2) of $\Delta_{B1}^{-1}(0)$ (resp. $\Delta_{B2}^{-1}(0)$) satisfying the following conditions:

(C1) S_1 is a smooth one-dimensional connected embedded submanifold of M,

(C2) S_1 does not intersect $\Delta_A^{-1}(0)$ and $\Delta_{B2}^{-1}(0)$ (resp. $\Delta_A^{-1}(0)$ and $\Delta_{B1}^{-1}(0)$),

(C3) S_1 is not tangent to that the vectors F_1+F_2 and $-F_1+F_2$ (resp. not tangent to the vectors F_1+F_2 and F_1-F_2 are not tangent to S_2),

(C4) f_1 (resp. f_2) changes sign on S_1 (resp. S_2).

See Figure 1 A.

Let us treat the case of S_1, being the case of S_2 similar. The first observation is that since there are no NTAE, u_2 cannot switches on S_1. Let for instance assume $u_2=1$.

Due to **(C3)**, F_1+F_2 and $-F_1+F_2$ point either on opposite, or on the same sides of S_1 (see Figure 1 B, cases 1 and 2). Since the admissible velocities belongs to the segment joining F_1+F_2 and $-F_1+F_2$ it follows that only in case 1 of Figure 1 A it is possible to run on S_1.

Moreover using a similar argument to that of Lemma 12 p.47 of [11], one can prove that if $f_1<0$ on the side where F_1+F_2 points, then a trajectory running on S_1 cannot be optimal.

These facts are collected in the following:

Lemma 7 *Let* S_1 *(resp.* S_2*) be a set satisfying conditions* **(C1)**, **(C2)**, **(C3)**, **(C4)** *above. Then* S_1 *(resp* S_2*) can not contain an optimal* u_1*-singular trajectory (resp. an optimal* u_2*-singular trajectory) if one of the two condition are satisfied.*

OO1 F_1+F_2 *and* $-F_1+F_2$ *(resp.* F_1+F_2 *and* F_1-F_2*) point on the same side of* S_1 *(resp.* S_2*);*

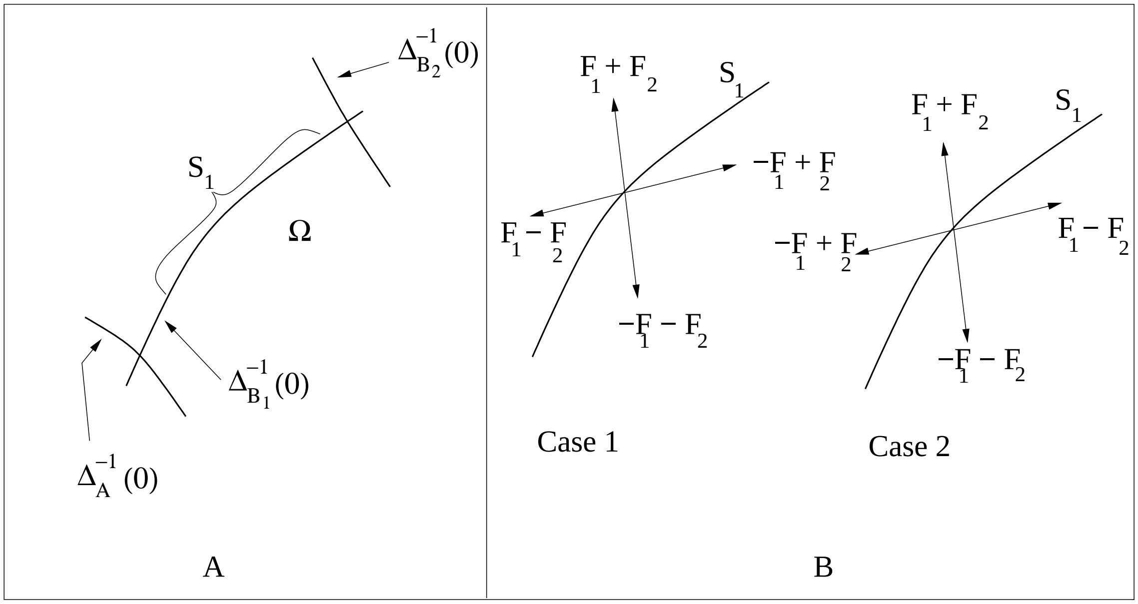


Figure 1:

OO2 *$F_1 + F_2$ and $-F_1 + F_2$ point on opposite sides of S_1 (resp. $F_1 + F_2$ and $F_1 - F_2$ point on opposite sides of S_2), and in a neighborhood of S_1 (resp. S_2) we have the following. On the side where points $F_1 + F_2$, we have $f_1 < 0$ (resp. $f_2 < 0$).*

If on S_1 (resp. on S_2) **OO1** and **OO2** are not satisfied (that means that S_1 is a candidate optimal trajectory) then following [11] we call S_1 a $\underline{u_1\text{-turnpike}}$ (resp. a $\underline{u_2\text{-turnpike}}$).

An extremal trajectory $x(.)$, corresponding to constant control +1 or -1 crossing a u_1-turnpike at a point $\bar{x} = x(\bar{t})$, can enter it only if the corresponding switching function is vanishing at $\bar{t}$, otherwise, after $\bar{t}$, it will correspond to the same control.

Moreover, using an argument similar to that of Lemma 11 p.46 of [11] one can prove that if an extremal trajectory enters a turnpike, then it can exit it with control +1 or −1 in every successive time.

These facts are stated more precisely in the following Lemma, and illustrated in Figure 2.

Lemma 8 *Let $(x(.), \lambda(.)) : [0, a] \to M$ be an extremal pair that verifies $x(a) = \bar{x}$, $\bar{x} \in S_i$ where S_i is a u_i-turnpike, and assume that $\phi_i(a) =< \lambda(a), G(x(a)) >= 0$. Moreover let b, c two real numbers such that $a \leq b < c$, and let $x'(.) : [0, c] \to M$ be a trajectory such that:*

- $x(.)'|_{[0,a]} = x(.)$,
- $x(.)'([a, b]) \subset S_i$.
- $x(.)'|_{]b,c]}$ *correspond to constant control* $+1$ *or to constant control* -1.

Then $x(.)'$ is extremal. Moreover if $\phi_i'(.)$ is the switching function corresponding to $x'(.)$ then $\phi'|_{[a,b]} \equiv 0$.

4.2 Time optimal synthesis

In this subsection, we apply the theory develop in Subsection (4.1) to the problem (9),(10) where we want to minimize the time of transfer under the condition $|u_1| \leq 1$, $|u_2| \leq 1$.

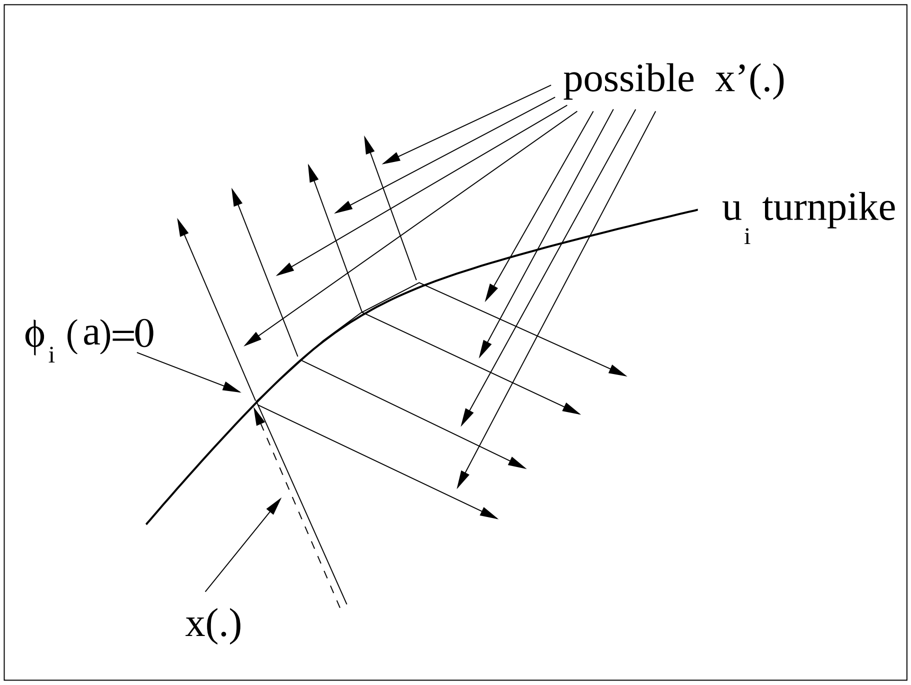


Figure 2:

Let us recall that in paper [14], we proved that there always exist minimizers in resonance. But more precisely, we also proved that there always exist minimizers with positive coordinates. And we described how one can reconstruct all other resonant minimizers from this last one. Hence, in the following, we construct the optimal synthesis in the part of the sphere with positive coordinates. We start from $(1,0,0)$ and we want to construct the optimal synthesis from this point, and in particular the minimizer reaching the point $(0,0,1)$, staying in the positive part of the sphere S^2, that we denote:

$$S^{2+} = \{x \in S^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}.$$

4.2.1 Preliminary computations

If we denote (x_1, x_2, x_3) the coordinates in $\mathbb{R}^3$, then the system (9),(10) writes:

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x),$$

where:

$$F_1(x) = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \text{ and } F_2(x) = \alpha \left(-x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} \right).$$

One can compute that:

$$[F_1, F_2] = \alpha \left(-x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3} \right).$$

and that:

$$\Delta_A = \alpha x_2, \qquad (29)$$

$$\Delta_{B1} = -\alpha x_1, \qquad (30)$$

$$\Delta_{B2} = -\alpha^2 x_3. \qquad (31)$$

Hence:

$$\Delta_A^{-1}(0) = \{x \in S^2 \mid x_2 = 0\}, \tag{32}$$
$$\Delta_{B1}^{-1}(0) = \{x \in S^2 \mid x_1 = 0\}, \tag{33}$$
$$\Delta_{B2}^{-1}(0) = \{x \in S^2 \mid x_3 = 0\}. \tag{34}$$

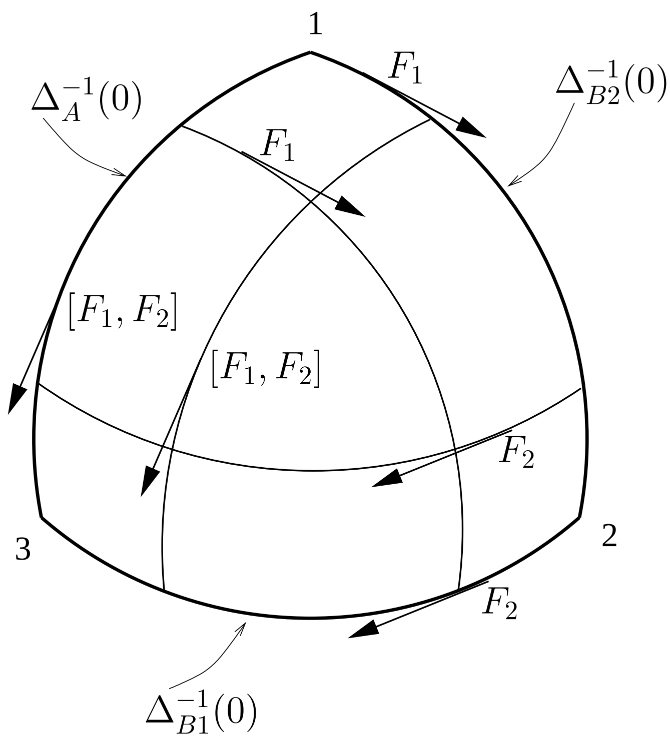
They form the boundary of S^{2+}. Hence, outside $\Delta_A^{-1}(0)$, one can check that:

$$f_1 = -\frac{x_1}{x_2} \tag{35}$$
$$f_2 = \alpha\frac{x_3}{x_2} \tag{36}$$

Now, in the interior of S^{2+}, $f_1 < 0$ and $f_2 > 0$. Hence the only possible switchings in the interior of S^{2+} are:

- $u_1 : +1 \longrightarrow -1$,
- $u_2 : -1 \longrightarrow +1$.

Finally, we want to compute functions φ_1 and φ_2 defined in the previous subsection. Here they are identically equal to 0 where they are defined. Indeed, we have that $\Delta_{B1} = -\alpha x_1$ hence F_2 is orthogonal to its gradient hence $\varphi_1 = 0$ when it is defined. Moreover, φ_1 is not defined when F_1 is also orthogonal to the gradient of Δ_{B1}, hence when F_1 and F_2 are parallel that is along $\Delta_A^{-1}(0)$. The same computations work for φ_2.



4.2.2 The Synthesis

Before describing the time-optimal synthesis let us first make some remarks:

<u>*Remark*</u> **11** 1. In order to build the time-optimal controls steering state 1 to state 3, we need to compute all the synthesis because we are dealing with a global problem. Local considerations around a special trajectory cannot be enough.

2. The problem of constructing the optimal trajectory steering state 1 to state 3 is almost symmetric with respect to the transformation $(\alpha \to \frac{1}{\alpha})$ and if we exchange the roles of state 1 and state 3.
3. The optimal synthesis cannot present chattering. We do not give details about this fact but, roughly speaking, the fact that only certain types of switchings are possible forbid the chattering.

Let us now detail some facts:

- As explained before, it is enough to describe the optimal synthesis in S^{2+}.
- Because there is no chattering and $\varphi_1 = \varphi_2 = 0$, any optimal control is a piecewise constant function taking values $+1$, -1, or 0.
- To enter in the interior of S^{2+}, the control u_2 should take the value $+1$. Hence, after entering S^{2+}, because $f_2 > 0$ in $S^{2+} - \Delta_{B2}^{-1}(0)$, we have $u_2 = 1$ until the end of the transfer.
- To live in S^{2+}, a trajectory should start with $u_1 = 1$ and $u_2 = 0$ or $u_2 = 1$.
- If an optimal trajectory starts following $S^{2+} \cap \Delta_{B2}^{-1}(0)$ (corresponding to controls $u_1 = 1,\ u_2 = 0$), then the only admissible switch of u_2 is $0 \longrightarrow +1$.
- In the interior of S^{2+}, the only admissible switch is for u_1 from value $+1$ to value -1, and the value of u_2 is $+1$.
- At a point of $S^{2+} \cap \Delta_{B1}^{-1}(0)$, the control u_1 can also switch from value $+1$ to value 0 and from value 0 to value -1.
- For all the values of α, the curve $t \mapsto (\cos(t), \sin(t), 0)$ corresponding to controls $(u_1 = 1, u_2 = 0)$ is an optimal trajectory.

Now, let us complete the optimal synthesis.

One technical point is to compute the switching locus, that is the points of S^{2+} where an optimal curve switches, and also the switching time along the optimal curves. In fact, in the special case of this system, the computations are not too intricate. Effectively, on an interval of time where the controls are constant, one can easily compute the evolution of coordinates of the curve and of the corresponding covector. Hence it is possible to compute the switching functions ϕ_1 and ϕ_2 along an admissible trajectory and find their roots.

First case: $\alpha < 1$**.** Let us describe the a priori admissible curves:

- The curve $t \mapsto (\cos(t), \sin(t), 0)$ corresponding to controls $(u_1 = 1, u_2 = 0)$ is optimal. The switching function $\phi_2 = 0$ along this curve hence a priori u_2 can switch to $+1$ at any time. ϕ_1 is constant and positive.
- The curves with controls $(u_1 = 1, u_2 = 0)$ at the beginning and switching to controls $(u_1 = 1, u_2 = 1)$ before state 2. They join $\Delta_{B1}^{-1}(0)$. The computations show that the switching functions do not change of sign along these trajectories.
- The curve with controls $(u_1 = 1, u_2 = 1)$ that joins $\Delta_{B1}^{-1}(0)$. Computations show that, depending on the initial condition on the covector, the switching function ϕ_1 can go to 0 at any instant until $\Delta_{B1}^{-1}(0)$, hence control u_1 can switch.
- Hence are also admissible the curves with controls $(u_1 = 1, u_2 = 1)$ at the beginning and switching to $(u_1 = -1, u_2 = 1)$. After this switching, no other switch is possible as proved before.
- Are also admissible the curves with controls $(u_1 = 1, u_2 = 1)$ until $\Delta_{B1}^{-1}(0)$, switching to controls $(u_1 = 0, u_2 = 1)$ following $\Delta_{B1}^{-1}(0)$, and switching again before state 3 to controls $(u_1 = -1, u_2 = 1)$.
- Finally is also admissible the curve with controls $(u_1 = 1, u_2 = 1)$ until $\Delta_{B1}^{-1}(0)$ and switching to controls $(u_1 = 0, u_2 = 1)$ following $\Delta_{B1}^{-1}(0)$ until state 3.

No other curve is admissible with respect to all the facts we give before. In fact, each one is optimal. Among this family, each one is the only one passing by its final point, except the one corresponding to controls $(u_1 = 1, u_2 = 1)$. But it is the first part of other optimal curves, hence it is optimal.

Second case: $\alpha = 1$**.** This case is the limit case of the previous one. The only optimal curves of the previous case are the 4 first ones.

Third case: $\alpha > 1$**.** The a priori admissible curves are:

- The curve $t \mapsto (\cos(t), \sin(t), 0)$ corresponding to controls $(u_1 = 1, u_2 = 0)$ is optimal. The switching function $\phi_2 = 0$ along this curve hence a priori u_2 can switch to $+1$ at any time. ϕ_1 is constant and positive.

- The curves with controls $(u_1 = 1, u_2 = 0)$ at the beginning, switching to controls $(u_1 = 1, u_2 = 1)$ after time $\arccos(\frac{1}{\alpha})$ but before state 2 . They join $\Delta_{B1}^{-1}(0)$. The computations show that the switching functions do not change of sign along these trajectories.

- The curve with controls $(u_1 = 1, u_2 = 0)$ at the beginning, switching to controls $(u_1 = 1, u_2 = 1)$ at $t = \arccos(\frac{1}{\alpha})$ and joining state 3. The computations show that the switching function ϕ_1 goes to 0 exactly at state 3.

- The curves with controls $(u_1 = 1, u_2 = 0)$ at the beginning, switching to controls $(u_1 = 1, u_2 = 1)$ before $t = \arccos(\frac{1}{\alpha})$. Computations show that they should switch a second time, to controls $(u_1 = -1, u_2 = 1)$, and that the time between the two switches is constant equal to $\frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})$.

- The curves with controls $(u_1 = 1, u_2 = 1)$ at the beginning, that switch before time $t = \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})$ to controls $(u_1 = -1, u_2 = 1)$.

No other curve is admissible and each one of these curves is optimal, being the only one passing by its end point.

To complete the picture of the synthesis in this third case, one should build the switching curve inside S^{2+}. It is the union of two arcs of circle C_1 and C_2: C_1 starts from state 1 following the control $(u_1 = 1, u_2 = 1)$ during time $t = \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})$ and arriving at the point $(1 - \frac{1}{\alpha^2}, \frac{\sqrt{\alpha^2-1}}{\alpha^2}, \frac{1}{\alpha})$. C_2 is the great arc of circle between the point $(1 - \frac{1}{\alpha^2}, \frac{\sqrt{\alpha^2-1}}{\alpha^2}, \frac{1}{\alpha})$ and state 3. C_2 is the image of the great arc of circle between state 1 and point $(\frac{1}{\alpha}, \sqrt{1 - \frac{1}{\alpha^2}}, 0)$ by the rotation corresponding to the dynamics $(u_1 = 1, u_2 = 1)$ during time $\frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})$.

Figure 3 shows the time-optimal synthesis in the first and third cases.

5 Minimal Energy

In this section, we give a complete and explicit integration of the PMP for the three-level anisotropic sub-Riemannian quantum problem. This provides almost explicit expressions (in term of elliptic functions) for optimal controls linking the first and the third level.

In the sequel, for convenience of computations, we put the constant α in the cost instead of in the dynamics. This gives rise to an equivalent minimization problem. Moreover we set $\beta = 1/\alpha^2$ and we get the presentation below.

5.1 Statement of the Problem

At the level of the group $SO(3)$, the dynamic is governed by the following right invariant system :

$$\dot{g} = \begin{pmatrix} 0 & u_1 & 0 \\ -u_1 & 0 & u_2 \\ 0 & -u_2 & 0 \end{pmatrix} g = dR_g \begin{pmatrix} 0 & u_1 & 0 \\ -u_1 & 0 & u_2 \\ 0 & -u_2 & 0 \end{pmatrix}. \tag{37}$$

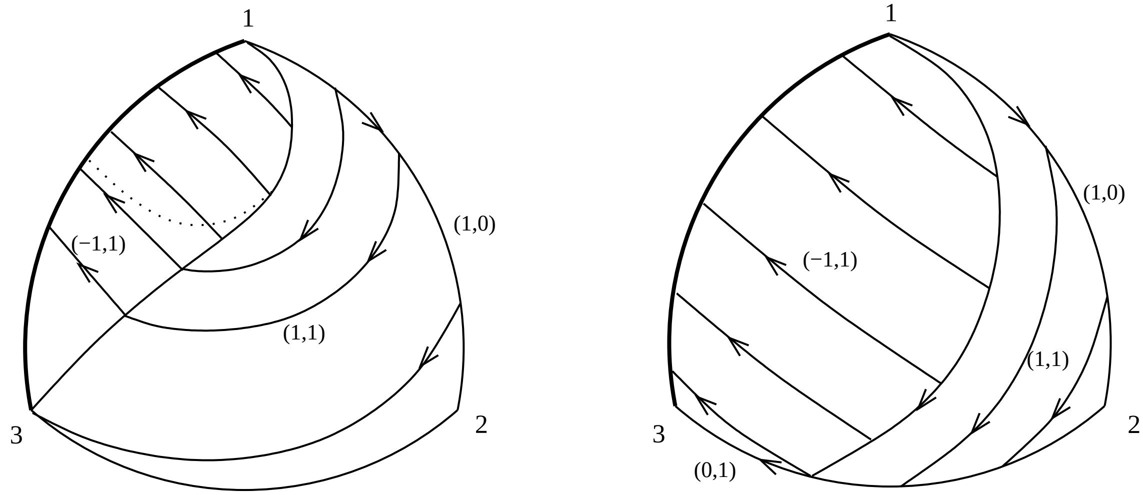


Figure 3: Image of the synthesis for $\alpha > 1$ and $\alpha < 1$

where $u_k : [0 \dots T] \to \mathbb{R}$, for $k = 1, 2$ is a measurable bounded function. The (right invariant) cost $J(u_1, u_2)$ is given by :

$$J(u_1, u_2) = \int_0^T (u_1^2 + \beta u_2^2)\, dt, \tag{38}$$

where β is a fixed strictly positive real number. We will denote by $(x_1(t), x_2(t), x_3(t))$ the first column of the square matrix g. This vector $(x_1(t), x_2(t), x_3(t))$ lies in the unit Euclidean sphere S^2 of $\mathbb{R}^3$. The problem is:

Problem: Find a control $u = (u_1, u_2)$ linking first and third level, and minimizing cost $J(u_1, u_2)$.

The source $\mathcal{S}$ is the smooth submanifold of $SO(3)$ of all matrices that leave invariant the direction of the first vector of the canonical basis of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \left(\begin{array}{c|c} Z_2 & 0 \\ \hline 0 & O(2) \end{array} \right) \in SO(3) \right\} = S\left(Z_2 \times O(2)\right). \tag{39}$$

The target $\mathcal{T}$ is the left translate of $\mathcal{S}$ by matrix:

$$g_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{40}$$

That is, $\mathcal{T}$ is the smooth submanifold of $SO(3)$ of all matrices which send the direction of the third vector of the canonical basis of $\mathbb{R}^3$ to the direction of the first one.

The case where $\beta = 1$ is treated in [13]. We shall assume $\beta \neq 1$ in the following. Moreover, up to a renormalization $J' = 1/\beta J$ of the cost, we may assume that $0 < \beta < 1$.

5.2 Integration of the Hamiltonian System Given by the PMP

5.2.1 Hamiltonian of the System

For each $\lambda_0 \in \mathbb{R}$, and each control $u = (u_1, u_2) \in L^\infty([0 \dots T], \mathbb{R})^2$, the Hamiltonian of the system (37) with cost (38) is:

$$\begin{array}{rcl} H_u : \mathfrak{so}(3)^* & \to & \mathbb{R} \\ P & \mapsto & H_u(P) = P\left[\begin{pmatrix} 0 & u_1 & 0 \\ -u_1 & 0 & u_2 \\ 0 & -u_2 & 0 \end{pmatrix}\right] + \lambda_0(u_1^2 + \beta u_2^2) \end{array} . \tag{41}$$

Using this notation, the Pontryagin Maximum Principle writes:

Theorem (Pontryagin Maximum Principle for Right Invariant Problems) *Consider $(u(.), g(.))$ a measurable control and the associated absolutely continuous trajectory of the system (41) belonging to $L^\infty([0 \dots T], \mathbb{R})^2 \times Ac([0 \dots T], SO(3))$. If it minimizes the cost (38), then there exists a constant $\lambda_0 \leq 0$ and an absolutely continuous function $P \in Ac([0 \dots T], \mathfrak{so}(3)^*)$ such that the couple (P, λ_0) never vanishes and for almost all t in $[0 \dots T]$:*

$$\left\{ \begin{array}{rcl} \frac{dg(t)}{dt} & = & dH_{u(t)}(P(t))g(t) \\ \frac{dP(t)}{dt} & = & -ad^*_{dH_{u(t)}(P(t))}P(t). \end{array} \right. \tag{42}$$

Moreover:

$$H_{u(t)}(P(t)) = \max_{(v_1,v_2)\in\mathbb{R}^2} H_{(v_1,v_2)}(P(t)) \text{ (Maximality condition)}, \tag{43}$$

and:

$$P(t)(T_{g(t)}g(t)\mathcal{S}g^{-1}(t)) = 0 \text{(Transversality condition)}. \tag{44}$$

In this case, since there are no NTAE, (see Proposition (1)) we can normalize $\lambda_0 = -1/2$.

The group $SO(3)$ of special orthogonal matrices of order 3 being semi-simple and compact, the canonical Killing form $Kill(x, y) = tr(xy)$ is negative definite on the Lie algebra $\mathfrak{so}(3)$. Hence $Kill(\cdot, \cdot)$ provides an isomorphism between $\mathfrak{so}(3)$ and its dual space $\mathfrak{so}(3)^*$. Using this identification, to each linear form $p : \mathfrak{so}(3) \to \mathbb{R}$ in $\mathfrak{so}(3)^*$, we can associate a skew symmetric matrix M_p of $\mathfrak{so}(3)$. For each matrix N in $\mathfrak{so}(3)$, $Kill(M_p, N) = p(N)$ by definition of M_p. Equation (42) for P can be rewritten in the famous Lax-Poincaré form (see [27] for a proof and further discussion) :

$$\frac{dM_P(t)}{dt} = [dH_u(P(t)), M_P(t)], \tag{45}$$

where we have identified $\mathfrak{so}(3)^{**} = \mathfrak{so}(3)$.

5.2.2 Expressions of the Controls

Let us define m_1, m_2, m_3 three absolutely continuous functions from $[0 \dots T]$ to $\mathbb{R}$ by:

$$M_P(t) = \begin{pmatrix} 0 & m_1(t) & m_3(t) \\ -m_1(t) & 0 & m_2(t) \\ -m_3(t) & -m_2(t) & 0 \end{pmatrix}, \tag{46}$$

for any time t in $[0 \dots T]$. From maximality condition (43), we deduce that for any t in $[0 \dots T]$, $m_1(t) = u_1(t)$ and $m_2(t) = \beta u_2(t)$. From the transversality condition (44), we deduce $u_2(0) = 0$. Equation (45) writes:

$$\left\{\begin{array}{rclc} \frac{du_1(t)}{dt} & = & & m_3(t)u_2(t) \\ \frac{du_2(t)}{dt} & = & -\frac{1}{\beta} & m_3(t)u_1(t) \\ \frac{dm_3(t)}{dt} & = & (\beta-1) & u_1(t)u_2(t) \end{array}\right. \tag{47}$$

It is easy to check that the two functions K_1 and K_2 defined by:

$$\begin{array}{rccl} K_1: & [0\dots T] & \rightarrow & \mathbb{R} \\ & t & \mapsto & u_1(t)^2+\beta u_2(t)^2, \end{array} \tag{48}$$

and

$$\begin{array}{rccl} K_2: & [0\dots T] & \rightarrow & \mathbb{R} \\ & t & \mapsto & u_1(t)^2+\frac{1}{1-\beta}m_3(t)^2, \end{array} \tag{49}$$

have zero derivative and hence are constant on the time interval $[0\dots T]$. Due to cauchy-Schwarz inequality, we can assume that K_1 is equal to 1: it just implies that the corresponding trajectories are parametrized by arclength, as in Riemannian geometry. Moreover, thanks to the symmetries of the system, we may assume that $u_1(0)$ and $m_3(0)$ are negative and that u_2 is positive on a neighborhood of 0. Using now the classical theory of elliptic functions (see [47] chapter XXII for example), we can express u_1, u_2 and m_3 as elliptic functions of order 2. Indeed, writing, for all t in $[0\dots T]$:

$$u_2(t)=\sqrt{\frac{1-u_1(t)^2}{\beta}}, \tag{50}$$

and:

$$m_3(t)=-\sqrt{(1-\beta)+m_3(0)^2-(1-\beta)u_1(t)^2}, \tag{51}$$

we can express u_1 as the solution of:

$$\frac{du_1(t)}{dt}=-\sqrt{\frac{1}{\beta}-\frac{1}{\beta}u_1(t)^2}\sqrt{m_3(0)^2+(1-\beta)-(1-\beta)u_1(t)^2}, \tag{52}$$

such that:

$$u_1(0)=-1. \tag{53}$$

The solution of (52) with initial condition (53) is known (see [47], chapter XXII) to be:

$$u_1(t)=-\mathrm{cd}\left(\sqrt{\frac{m_3(0)^2+1-\beta}{\beta}}t,\sqrt{\frac{1-\beta}{m_3(0)^2+1-\beta}}\right), \tag{54}$$

for all t in $[0\dots T]$. We shall define:

$$k=\sqrt{\frac{1-\beta}{m_3(0)^2+1-\beta}}, \tag{55}$$

the modulus of the elliptic function of the second order *cd* as defined in [47]. Substituting now (54) in constant functions (48) and (49), we get expressions for u_2 and m_3:

$$u_2(t) = \sqrt{\frac{m_3(0)^2}{\beta(m_3(0)^2+1-\beta)}}\mathrm{sd}\left(\sqrt{\frac{m_3(0)^2+1-\beta}{\beta}}t,k\right), \tag{56}$$

$$m_3(t) = m_3(0)\mathrm{dn}\left(\sqrt{\frac{m_3(0)^2+1-\beta}{\beta}}t,k\right). \tag{57}$$

From the maximality condition (43), we can write that, for any time t:

$$\begin{aligned} p_1(t).x_2(t) - p_2(t).x_1(t) &= u_1(t), && (58) \\ p_2(t).x_3(t) - p_3(t).x_2(t) &= \beta u_2(t), && (59) \end{aligned}$$

where:

$$\begin{aligned} p_1(t) &= u_1(t)x_2(t) + m_3(t)x_3(t), && (60) \\ p_2(t) &= -u_1(t) * x_1(t) + \beta u_2(t) * x_3(t), && (61) \\ p_3(t) &= -m_3(t)x_1(t) - \beta u_2(t)x_2(t). && (62) \end{aligned}$$

Substituting equations (60), (61) and (62) in (58) and (59), we get a system of two equations of second degree with unknown (x_1, x_2, x_3):

$$u1(t)x_2(t)^2 + m_3(t)x_2(t)x_3(t) + u_1(t)x_1(t)^2 - \beta u_2(t)x_1(t)x_3(t) = u_1(t) \quad (63)$$

$$-u_1(t)x_1(t)x_3(t) + \beta u_2(t)x_3(t)^2 + x_2(t)m_3(t)x_1(t) + \beta u_2(t)x_2(t)^2 = \beta u_2(t) \quad (64)$$

By derivation of this two equations, we get two more equations of second degree with unknown (x_1, x_2, x_3):

$$\begin{aligned} &\dot{u_1}(t)x_2(t)^2 + 2x_2(t)u_1(t)u_2(t)x_3(t) + x_2(t)\dot{m_3}(t)x_3(t) - m_3(t)u_2(t)x_2(t)^2 \\ &-m_3(t)x_3(t)u_1(t)x_1(t) + m_3(t)x_3(t)^2u_2(t) + \dot{u_1}(t)x_1(t)^2 - \beta \dot{u_2}(t)x_1(t)x_3(t) \\ &+\beta u_2(t)^2x_1(t)x_2(t) - \beta u_1(t)u_2(t)x_2(t)x_3(t) = \dot{u_1}(t), \quad (65) \end{aligned}$$

and

$$\begin{aligned} &-x_3(t)\dot{u_1}(t)x_1(t) - u_1^2(t)x_3(t)x_2(t) + \beta\dot{u_2}(t)x_3(t)^2 + u_2(t)u_1(t)x_1(t)x_2(t) \\ &+\dot{m_3}(t)x_1(t)x_2(t) + u_1(t)m_3(t)x_2(t)^2 + \beta\dot{u_2}(t)x_2(t)^2 - 2\beta u_1(t)u_2(t)x_1(t)x_2(t) \\ &-m_3(t)u_1(t)x_1(t)^2 + m_3(t)u_2(t)x_1(t)x_3(t) = \beta\dot{u_2}(t). \quad (66) \end{aligned}$$

Since $(x_1(t), x_2(t), x_3(t))$ lies in the sphere S^2 for any t in $\mathbb{R}$, we also get that

$$x_1(t)^2 + x_2(t)^2 + x_3(t)^2 = 1. \quad (67)$$

The system (63), (64), (65), (66), (67) is linear in the 6 unknowns $X_{i,j} = x_i\ x_j, i, j = 1..3$. It is possible (but not very pleasant) to solve this system and then to deduce an expression of $x_1(t)$, $x_2(t)$ and $x_3(t)$ depending only on u_1, u_2, m_3 and their first derivatives. Replacing now u_1, u_2, m_3, $\dot{u_1}$, $\dot{u_2}$ and $\dot{m_3}$ by their expression given by equations (54), (56) and (57), we get fully explicit expressions for $x_1(t)$, $x_2(t)$ and $x_3(t)$ depending only on the initial value $m_3(0)$ of m_3. In other words, any choice of the value of the constant $m_3(0)$ will fully determined the trajectory. What remains to do now is to find the right $m_3(0)$ such that the corresponding trajectory reaches the target in the shortest possible time.

5.2.3 Determination of the Initial Covector

To determine for which values of $m_3(0)$ the trajectory associated to the control $u = (u_1, u_2)$ we have determined reaches the target, we will use a classical isospectral property for invariant Hamiltonian systems on Lie groups (see [26], pp 384-390 for a proof and further discussion):

Property 1 *Denoting again by (g, P) a solution of the Hamiltonian system (42), at any time t of $[0 \ldots T]$:*

$$M_P(t) = g(t)^{-1}M_P(0)g(t). \quad (68)$$

In our case, at time T, trajectory g reaches the target $\mathcal{T}$. So $g(T)$ belongs to $\mathcal{T}$. In other words, there exists a real number θ such that:

$$g(T) = \begin{pmatrix} 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \\ 1 & 0 & 0 \end{pmatrix}. \quad (69)$$

Hence, at time T, matrix $M_p(T)$ writes:

$$\begin{pmatrix} 0 & 0 & -\cos(\theta)+m_3(0)\sin(\theta) \\ 0 & 0 & -\sin(\theta)-m_3(0)\cos(\theta) \\ \cos(\theta)-m_3(0)\sin(\theta) & \sin(\theta)+m_3(0)\cos(\theta) & 0 \end{pmatrix} \quad (70)$$

In particular, we get that:

$$u_1(T) = 0 \quad (71)$$

$$u_2(T) = -\frac{1}{\beta}\left(\sin(\theta)+m_3(0)\cos(\theta)\right) \quad (72)$$

$$m_3(T) = -\cos(\theta)+m_3(0)\sin(\theta). \quad (73)$$

From (71), we deduce that there exists some positive integer n such that:

$$T = n.r', \quad (74)$$

where r' is the first strictly positive root of elliptic function dn. Equations (72) and (73) are equivalent to the system:

$$\left\{ \begin{array}{rcrcl} -\cos(\theta) & + & m_3(0)\sin(\theta) & = & \sqrt{m_3(0)^2+(1-\beta)} \\ m_3(0)\cos(\theta) & + & \sin(\theta) & = & \sqrt{\beta}. \end{array} \right. \quad (75)$$

Equations appearing in (75) can be seen as a system of linear equations in $\cos(\theta)$ and $\sin(\theta)$ with coefficients depending on β and $m_3(0)$. An easy computation leads to:

$$\sin(\theta) = \frac{m_3(0)\sqrt{m_3(0)^2+1-\beta}+\sqrt{\frac{1}{\beta}}}{m_3(0)^2+1}, \quad (76)$$

$$\cos(\theta) = -\frac{\sqrt{m_3(0)^2+1-\beta}-m_3(0)\sqrt{\frac{1}{\beta}}}{m_3(0)^2+1}. \quad (77)$$

The trajectory associated to a certain $m_3(0)$ reaches the target at time T given by equation (74) if and only if:

$$x_3(T) = \pm 1, \quad (78)$$

where x_3 is given by the system (60,.. 67). By Fillipov theorem (see [1] for example), we know that there exists a trajectory linking the source and the target and realizing the infimum of all costs. That is to say that the set of the solutions of equation (78) admits a minimum.

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