IMPROPER CHOOSABILITY OF GRAPHS AND MAXIMUM AVERAGE DEGREE

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Abstract

Improper choosability of planar graphs has been widely studied. In particular, Škrekovski investigated the smallest integer $g_k$ such that every planar graph of girth at least $g_k$ is $k$-improper 2-choosable. He proved [8] that $6 \leq g_1 \leq 9; 5 \leq g_2 \leq 7; 5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. In this paper, we study the greatest real $M(k, l)$ such that every graph of maximum average degree less than $M(k, l)$ is $k$-improper $l$-choosable. We prove that for $l \geq 2$ then $M(k, l) \geq l + \frac{k}{l}$. As a corollary, we deduce that $g_1 \geq 8$ and $g_2 \geq 6$. We also provide an upper bound for $M(k, l)$. This implies that for any fixed $l$, $M(k, l) \xrightarrow{k \to \infty} 2l$.

1 Introduction.

Let $G$ be a graph. We note $V(G)$ its vertex set and $E(G)$ its edge set.

A colouring is an application from the vertex set into a set of colours $S$. If $|S| = l$ we call it $l$-colouring. Let $c$ be a colouring of $G$. The improperty of a vertex $v$ in $G$ under $c$, denoted by $im^c_G(v)$, is the number of neighbours $u$ of $v$ in $G$ such that $c(u) = c(v)$. The improperty of $c$ in $G$ is $im^c_G(c) = \max\{im^c_G(v) - v \in V(G)\}$. A colouring is $k$-improper if its improperty is at most $k$ and a graph is $k$-improper $l$-colourable if it admits a $k$-improper $l$-colouring. The $k$-improper chromatic number of $G$, denoted by $c_k(G)$, is the smallest integer $l$ such that $G$ is $k$-improper $l$-colourable. Note that 0-improper colouring is the usual notion of proper colouring, so the 0-improper chromatic number is exactly the chromatic number usually denoted $\chi(G)$.

One can analogously generalize the notion of choosability. A list-assignment of a graph $G$ is an application $L$ which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v)$. $L$ is an $l$-list-assignment provided each list is of size at least $l$. $G$ is $k$-improper $L$-colourable if there exists a $k$-improper colouring $c$ of $G$ such that $\forall v \in V(G), v \in L(v)$. In this case, $c$ is a $k$-improper $L$-colouring of $G$. $G$ is $k$-improper $l$-choosable if it is $k$-improper $L$-colourable for every $l$-list-assignment $L$.

Colourings of planar graphs have been widely studied. In particular $p_k$ and $p^*_k$, the smallest integers $l$ such that every planar graph is $k$-improper $l$-colourable and $k$-improper $l$-choosable respectively, are known for almost all $k$. Indeed Thomassen showed in [9] that every planar graph is 5-choosable and there are planar graphs which are not 4-choosable [12] so $p^*_0 = 5$. Every planar graph is 4-colourable [1, 2] and there are graphs which are not 1-improper 3-colourable, so $p_0 = p_1 = 4$. But we do not know the exact value of $p^*_1$ which is either 4 or 5. However, it is conjectured that it is 4:

**Conjecture 1 (Eaton and Hull [3], Škrekovski [6])** Every planar graph is 1-improper 4-choosable.
As shown independently by Eaton and Hull [3] and Škrekovski [6], every planar graph is 2-improper 3-choosable and for every $k$, there are planar graphs which are not $k$-improper 2-colourable. Hence $p_k = p_k^* = 3$ for any $k \geq 2$.

Moreover improper colourings of planar graphs have also been studied under some girth restrictions. The girth of graph is the smallest length of a cycle. The well-known theorem of Grötzsch [4, 11] states that every planar graph of girth at least 4 is 3-colourable. Voigt [13] showed a planar graph of girth 4 which is not 3-choosable and Thomassen [10] proved that every planar graph of girth at least 5 is 3-choosable. In [7], Škrekovski showed that every planar graph of girth at least 4 is 1-improper 3-choosable. In [8], Skrekovski investigated $k$-improper 2-choosability of planar graphs in relation with their girth. Denoting by $g_k$ the smallest integer such that every planar graph of girth at least $g_k$ is $k$-improper 2-choosable, he proved that $6 \leq g_1 \leq 9$, $5 \leq g_2 \leq 7$, $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. Hence the only unknown values are $g_1$, $g_2$ and $g_3$.

In this paper, we study the $k$-improper $l$-choosability of graphs in relation with their maximum average degree.

**Definition 1** The maximal average degree of a graph $G$ is:

$$M(G) := \max \{ \frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}, H \text{ subgraph of } G \}.$$  

The girth and the maximum average degree of a planar graph are related to each other:

**Theorem 1** Let $G$ be a planar graph of girth $g$.

$$M(G) < \frac{2g}{g - 2}.$$  

**Proof.** We recall the Euler’s formula for a planar graph $H$: $|V(H)| - |E(H)| + |F(H)| = 2$ with $|F(H)|$ the number of faces of $H$. Note that every subgraph $H$ of $G$ has girth at least $g$, so $g|F(H)| \leq 2|E(H)|$. Thus $2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|$. Hence $\frac{2|E(H)|}{|V(H)|} \leq \frac{2g}{g - 2} - \frac{4}{(g - 2)|V(H)|} < \frac{2g}{g - 2}$ for every subgraph $H$ of $G$. \hfill $\square$

Let $M(k, l)$ be the greatest real such that every graph of maximum average degree less than $M(k, l)$ is $k$-improper $l$-choosable. Obviously, $M(k_1, l) \leq M(k_2, l)$ if $k_1 \leq k_2$.

We have that $M(k, 1) = \frac{2k + 4}{k + 2}$ since a graph is $k$-improper 1-choosable if, and only if, it has maximum degree at most $k$.

In order to introduce our method which uses some discharging process, we first present it in Section 2 for improper 2-choosability: we prove that for every $k \geq 0$,

$$4 - \frac{4}{k + 2} \leq M(k, 2) \leq 4 - \frac{2k + 4}{k^2 + 2k + 2}.$$  

As a corollary, we obtain the following upper bounds for $g_k$ which are better than Škrekovski’s ones: $g_1 \leq 8$, $g_2 \leq 6$, $g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$.

In Section 3 we extend the lower bound of Section 2 to any value of $l$: we prove that for every $l \geq 2$ and $k \geq 0$,

$$l + \frac{lk}{l + k} \leq M(k, l).$$  

Last, we provide for any value of $l$ and $k$ a graph which is not $k$-improper $l$-choosable, and we deduce that $M(k, l) \xrightarrow{k \to \infty} 2l$. 

2
2 Improper 2-choosability

2.1 Lower bound for \( M(k, 2) \)

In this subsection, we shall prove the following theorem:

**Theorem 2** For all \( k \geq 0 \), all graphs of maximum average degree less than \( \frac{4k+4}{k+2} \) are \( k \)-improper 2-choosable.

Note that if \( k = 0 \) the result holds trivially. Indeed a graph with maximum average degree less than 2 contains no cycle and so is a forest. Hence it is 2-choosable. Furthermore \( M(0, 2) \leq 2 \) since an odd cycle is not 2-colourable, so \( M(0, 2) = 2 \).

For bigger value of \( k \), we will need the following preliminary definitions and results:

**Definition 2** If \( v \in V(G) \) then \( d_G(v) \) denotes the degree of \( v \) in the graph \( G \). For all positive integer \( d \), a vertex of degree equals to (resp. at most, resp. at least) \( d \) is called a \( d \)-vertex (resp. \( \leq d \)-vertex, resp. \( \geq d \)-vertex). For \( S \subseteq V(G) \) (resp. \( E \subseteq E(G) \)) we denote by \( G - S \) (resp. \( G - E \)) the induced subgraph of \( G \) obtained by removing the vertices (resp. edges) of \( S \) (resp. \( E \)) from \( V(G) \) (resp. \( E(G) \)). If \( S = \{v\} \) and \( E = \{vw\} \), we shall note \( G - v = G - S \) and \( G - \{vw\} = G - E \). The union (resp. intersection) of the graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \cup G_2 \) (resp. \( G = G_1 \cap G_2 \)) such that \( V(G) = V(G_1) \cup V(G_2) \) (resp. \( V(G) = V(G_1) \cap V(G_2) \)) and \( E(G) = E(G_1) \cup E(G_2) \) (resp. \( E(G) = E(G_1) \cap E(G_2) \)).

A graph is said to be \((k, 2)\)-minimal if it is not \( k \)-improper 2-choosable but each of its proper subgraphs is.

**Lemma 1** (Škrekovski [8]) Let \( k \geq 1 \) and let \( G \) be a \((k, 2)\)-minimal graph. Then

(i) \( \delta \geq 2 \).

(ii) Two \( (\leq k + 1) \)-vertices are not adjacent.

**Definition 3** Let \( D \) be a digraph. The outdegree (resp. indegree) of a vertex \( u \) in \( D \) is denoted by \( d_D^+(u) \) (resp. \( d_D^-(u) \)). The *degree* of \( u \) is \( d_D(u) = d_D^+(u) + d_D^-(u) \); it is the degree of \( u \) in the underlying undirected graph.

If \( u \) and \( v \) are two of its vertices, a \((u, v)\)-dipath is a directed path from \( u \) to \( v \).

An *arborescence* is an oriented tree in which every path is directed from a vertex called the *root*. Note that in an arborescence every vertex except the root has indegree 1. The leaves of the arborescence are the vertices of outdegree 0. A vertex which is neither a leaf nor the root is an *internal vertex*. A *quasi-arborescence* is a directed graph obtained from an arborescence by identifying some leaves.

Let \( u \) be a vertex of a digraph \( D \). The *outsection* of \( u \) in \( D \), denoted \( A_D^+(u) \), is the set of vertices \( v \) such that there is a \((u, v)\)-dipath in \( D \).

Let \( G \) be a \((k, 2)\)-minimal graph. We partially orient \( G \) using the following process:

1. Orient each edge \( uv \) where \( v \) is a 2-vertex from \( u \) to \( v \).
2. If \( k \geq 3 \), orient each edge \( uv \) where \( v \) is a 3-vertex from \( u \) to \( v \).
3. While there is an unoriented edge \( uv \) where \( v \) an \( i \)-vertex with \( 2 + k \leq i < \frac{3k}{2} + 2 \) and outdegree \( i - 1 \), we orient it from \( u \) to \( v \).
The digraph $D$ induced by the oriented edges is called a *discharging digraph* of $G$.

The following proposition, whose proof is left to the reader, follows immediately from the definition of a discharging digraph.

**Proposition 1** Let $D$ be a discharging digraph of a $(k, 2)$-minimal graph.

- $D$ has no 2-circuit since two $(\leq k + 1)$-vertices are not adjacent by Lemma 1 (ii). So it has no circuit at all.
- If $k \leq 2$, only vertices of degree 2 or $k + 2$ have indegree more than zero. If $k \leq 3$, only vertices of degree 2, 3 or $k + 2$ have indegree more than zero.
- Every 2-vertex has indegree exactly 2 in $D$ and if $k \geq 3$, every 3-vertex has indegree exactly 3.
- For every vertex $u$, $A^+_D(u)$ is a quasi-arborescence whose leaves have degree 2 (resp. 2 or 3) in $G$ if $k \leq 2$ (resp. $k \geq 3$). In particular, the indegree of the leaves in $A^+_D(u)$ is at most 2 (resp. 3).

**Definition 4** A quasi-arborescence is a $(k, 2)$-quasi-arborescence if and only if:

- Every vertex has outdegree at most $\max\{2, 2k - 1\}$.
- Every leaf has indegree at most $\min\{k, 3\}$.

**Lemma 2** Let $k \geq 2$. Let $Q$ be a $(k, 2)$-quasi-arborescence rooted at $u$ and $L$ a 2-list-assignment of $Q$. Then any $L$-colouring of the leaves can be extended in a $k$-improper $L$-colouring of $D$ such that $u$ has improperty at most $k - 1$.

**Proof.** By induction on the number of vertices of $Q$, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller $k$-quasi-arborescences. Let $v_1, \ldots, v_s$ be the outneighbours of $u$ in $Q$. Note that $Q - u$ is the union of $s$ $(k, 2)$-quasi-arborescences $Q_i$, $1 \leq i \leq s$ rooted at $v_i$ that are disjoint except possibly on their leaves.

Let $c$ be an $L$-colouring of the leaves of $Q$. Then by induction it can be extended in a $k$-improper $L$-colouring of each of the $Q_i$ so that $\text{im}(v_i) \leq k - 1$. Since a leaf of $Q$ has indegree at most $\min\{k, 3\}$ and $\text{im}_Q(x) = \text{im}_{Q_i}(x)$ for every vertex of $Q_i$ which is not a leaf, then the union of these colourings is a $k$-improper $L$-colouring of $Q$ such that $\text{im}(v_i) \leq k - 1$.

Now, one of the two colours of $L(u)$, say $\alpha$, is assigned to at most $k - 1$ neighbours of $u$ since $s \leq 2k - 1$. Thus setting $c(u) = \alpha$, we obtain the desired colouring. □

Obviously, the above result cannot be extended for $k = 1$ because it is hopeless to extend every $L$-colouring of the leaves in a colouring such that the root has improperty 0. However, one can prove the following weaker result:

**Lemma 3** Let $Q$ be a $(1, 2)$-quasi-arborescence rooted at $u$, $L$ a 2-list-assignment of $Q$ with $L(u) = \{\alpha, \beta\}$ and $c$ an $L$-colouring of $S$ the set of leaves of $Q$ with indegree 1. One the following holds:

(i) $c$ may be extended in a 1-improper $L$-colouring of $Q$ such that $\text{im}(u) = 0$;
(ii) $c$ may be extended in two different 1-improper $L$-colourings of $Q$, one such that $c(u) = \alpha$ and one such that $c(u) = \beta$.

**Proof.** We proceed by induction on the number of vertices of $Q$. Let $v_1$ and $v_2$ be two out-neighbours of $u$ in $Q$. $Q - u$ is the union of two $(1, 2)$-quasi-arborescences $Q_1$ and $Q_2$, rooted at $v_1$ and $v_2$ respectively, that are disjoint except possibly on their leaves. Let $S'$ be the set of leaves in $Q_1 \cap Q_2$ and $L(u) = \{\alpha, \beta\}$. We $L$-colour the leaves of $Q_1$ that have indegree 1 in $Q_1$. By induction, each of the $Q_i$ satisfies (i) or (ii).

If at least one of the $Q_i$ satisfies (ii), then one can extend $c$ to $Q_1 \cup Q_2$ such that $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$. Moreover for any vertex $x$ not in $V(Q_1) \setminus S'$, $im_{Q_1}(x) = im_{Q_2}(x) \leq 1$. If a vertex $s' \in S'$ has improperty 2 then its two neighbours are coloured the same. So recolouring $s'$ with the colour of $L(s') \setminus \{c(s')\}$, we get a 1-improper $L$-colouring of $Q_1 \cup Q_2$. Hence setting $c(u) = \alpha$, we get a 1-improper $L$-colouring of $Q$ such that $im(u) = 0$. Thus $Q$ satisfies (i).

Suppose now $Q_1$ and $Q_2$ both satisfy (i). Then, possibly with recolouring of vertices of $S'$ as before, one can extend $c$ into a 1-improper $L$-colouring of $Q_1 \cup Q_2$ such that $im(v_1) = im(v_2) = 0$. If $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$ then setting $c(v) = \alpha$, we get a 1-improper $L$-colouring of $Q$ such that $im(u) = 0$. Thus $Q$ satisfies (i). If not then assigning to $u$ the colours $\alpha$ and $\beta$, we get the two 1-improper $L$-colourings of $Q$ satisfying (ii). \qed

**Lemma 4** Let $k \geq 3$. Let $D$ be a discharging digraph of a $(k, 2)$-minimal graph $G$.

(i) Every $i$-vertex with $4 \leq i \leq k + 1$ has outdegree zero.

(ii) Every $i$-vertex with $2 + k \leq i \leq 2k + 1$ has outdegree less than $i$.

**Proof.**

(i) Suppose, for a contradiction, that $v$ is a vertex contradicting the assertion and let $u$ be an outneighbour of $v$. Note that $u$ is a $(< \frac{3k}{2} + 2)$-vertex by definition of a discharging digraph.

Let $L$ be a 2-list-assignment of $G$. Let $S$ be the set of leaves of $A^+_D(u)$. By minimality, let $c$ be a $k$-improper $L$-colouring of $G - A^+_D(u)$.

$A^+_D(u)$ is a $(k, 2)$-quasi-arborescence: since it is dominated by $v$ in $D$, $u$ has outdegree less than $\frac{3k}{2} + 1$ and so at most $2k - 1$. Thus by Lemma 2, we can extend $c$ to $G - vu$ so that $im(u) \leq k - 1$. Since the leaves have degree at most $3 \leq k$, the improperty of the leaves is at most $3 \leq k$. So we obtain a $k$-improper $L$-colouring of $G - vu$.

If $c(u) \neq c(v)$ or $im_{G - uv}(v) \leq k - 1$ then $c$ is a $k$-improper $L$-colouring of $G$. Otherwise all the $k + 1$ neighbours of $v$ are coloured the same so recolouring $v$ with its other allowed colour yields a $k$-improper $L$-colouring of $G$.

Hence $G$ is $k$-improper 2-choosable which is a contradiction.

(ii) Suppose, for a contradiction, that $v$ is an $i$-vertex contradicting the assertion.

Let $L$ be 2-list-assignment of $G$ and $c$ a $k$-improper $L$-colouring of $G - v$. There is a colour of $L(v)$, say $\alpha$, that is assigned to at most $k$ neighbours of $v$. Let $v_1, \ldots, v_k$ be these neighbours.
Lemma 5 Let $D$ be a discharging digraph of a $(2,2)$-minimal graph $G$.

(i) The outdegree of a $3$-vertex is zero.

(ii) If $v$ is an $i$-vertex with $i \in \{4;5\}$ then its outdegree is less than $i$.

Lemma 6 Let $D$ be a discharging of a $(1,2)$-minimal graph $G$. There is no $3$-vertex with outdegree $3$ in $D$.

Proof. Suppose, for a contradiction, that $v$ is a $3$-vertex with outdegree $3$. Let $u$ be an outneighbour of $v$. Let $Q_1 = A_D^+(u)$, $Q_2 = A_{D-vu}^+(v)$, $S$ be the set of leaves of $A_D^+(v)$ with indegree $1$ in $A_D^+(v)$ and $S'$ be the set of leaves with indegree $2$ in $A_D^+(v)$.

Let $L$ be a $2$-list-assignment of $G$. By minimality of $G$, let $c$ be a $1$-improper $L$-colouring of $G - A_D^+(v)$. Vertices not in $S$ have no neighbour in $G - A_D^+(v)$ and every vertex of $S$ has exactly one neighbour in $G - A_D^+(v)$. Extend $c$ to $S \cup S'$ by assigning to each vertex of $S$ a colour of its list not assigned to its neighbour in $G - A_D^+(v)$ and any colour of its list to a vertex of $S'$.

Now $Q_1$ and $Q_2$ satisfy either (i) or (ii) of Lemma 3. If one of them satisfies (ii), then possibly with recolouring of vertices of $S'$ one can extend $c$ into a $1$-improper $L$-colouring of $G - vu$ such that $c(v) \neq c(u)$. Hence $c$ is a $1$-improper $L$-colouring of $G$.

If $Q_1$ and $Q_2$ satisfy both (i), then possibly with recolouring of vertices of $S'$ one can extend $c$ into a $1$-improper $L$-colouring of $G - vu$ such that $im(v) = im(u) = 0$. Hence $c$ is a $1$-improper $L$-colouring of $G$.

So $G$ is $1$-improper $2$-choosable which is a contradiction. 

Proof of Theorem 2. Let $G$ be a $(k,2)$-minimal graph and $D$ a discharging digraph of $G$. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{k+2}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex $v$:

- If $v$ is a $2$-vertex, it has indegree $2$ so its new charge is $w'(v) = 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$.

- If $v$ is a $3$-vertex and $k \geq 3$, it has indegree $3$ so its new charge is $w'(v) = 3 + 3 \times \frac{k}{k+2} = \frac{6k+6}{k+2} > \frac{4k+4}{k+2}$. If $v$ is a $3$-vertex and $k = 2$ then it has outdegree $0$ by Lemma 5 and indegree $0$ by construction so $w'(v) = 3$.

- If $4 \leq d(v) \leq k + 1$, ($k \geq 3$), then by Lemma 4 (i), $v$ has outdegree zero so its charge is $d(v) \geq 4 > \frac{4k+4}{k+2}$.
• If \( k + 2 \leq d(v) < \frac{3k}{2} + 2 \) then either \( v \) has outdegree at most \( d(v) - 2 \) and so its new charge is at least \( d(v) - (d(v) - 2) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{2k}{k+2} \geq 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2} \), or by Lemmas 4, 5 and 6, it has outdegree \( d(v) - 1 \). In this case, by definition of a discharging digraph, \( v \) has indegree 1 so its new charge is:

\[
d(v) - (d(v) - 1) \times \frac{k}{k+2} + \frac{k}{k+2} = d(v) - (d(v) - 2) \times \frac{k}{k+2} = \frac{4k+4}{k+2}.
\]

• If \( \frac{3k}{2} + 2 \leq d(v) \leq 2k + 1, (k \geq 2) \), then by Lemmas 4 and 5, \( v \) has outdegree at most \( d(v) - 1 \). So \( w'(v) \geq d(v) - (d(v) - 1) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{k}{k+2} \geq \frac{3k+4+k}{k+2} = \frac{4k+4}{k+2} \).

• If \( d(v) \geq 2k + 2 \), then \( w'(v) \geq d(v)(1 - \frac{k}{k+2}) = \frac{2d(v)}{k+2} \geq \frac{4k+4}{k+2} \).

Hence \( Mad(G) \geq \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{|V|} \sum_{v \in V} w'(v) \geq \frac{4k+4}{k+2} \).

Corollary 1 Let \( G \) be a planar graph of girth \( g \).

1. If \( g \geq 8 \) then \( G \) is 1-improper 2-choosable, so \( g_1 \leq 8 \).

2. If \( g \geq 6 \) then \( G \) is 2-improper 2-choosable, so \( g_3 \leq g_2 \leq 6 \).

3. If \( g \geq 5 \) then \( G \) is 4-improper 2-choosable, so \( g_k \leq 4 \) for \( k \geq 5 \).

2.2 Upper bound for \( M(k, 2) \)

Let us fix \( k \geq 1 \). In this section, we shall construct a family of graphs \( (G^k_n)_{n \geq 1} \) such that for all \( n \geq 1 \):

• \( G^k_n \) is not \( k \)-improper 2-colourable.

• \( Mad(G^k_n) = \frac{2n(4k^2 + 6k + 4) + 4k^2 + 6k + 2}{2n(k^2 + 2k + 2) + (k+1)^2} \).

Hence we will deduce:

Theorem 3 For all \( k \geq 1 \), \( M(k, 2) \leq \frac{4k^2 + 6k + 4}{k^2 + 2k + 2} = 4 - \frac{2k + 4}{k^2 + 2k + 2} \).

We denote by \( H_k \) the graph composed of two adjacent vertices \( u \) and \( v \) also connected by \( k + 1 \) disjoint paths of length 2. Take \( k \) copies of \( H_k \) and create the graph \( F_k \) by identifying the vertex \( v \) of each copy. Note that \( F_k \) has one vertex of degree \( k(k + 2) \), \( k \) vertices of degree \( k + 2 \) and \( k(k + 1) \) vertices of degree 2. Now we take \( 2n + 1 \) copies of \( F_k \) and we join the vertices \( v \) of each copy creating a cycle of size \( 2n + 1 \). At last we make a subdivision of all the edges of the cycle but one so as to obtain the graph \( G^k_n \).
Lemma 7 \( G^k_n \) is not \( k \)-improper 2-colourable.

Proof. First remark that in any \( k \)-improper 2 colouring of \( H_k \), \( v \) has impropriety at least 1. Indeed \( v \) is a \((k + 2)\) vertex in \( H_k \), so if it has impropriety zero then its \( k + 2 \) neighbours are coloured the same, but this is impossible since \( u \) is a neighbour of \( v \) adjacent to the \( k + 1 \) remaining neighbours. Hence in any \( k \)-improper colouring of \( F_k \), \( v \) has impropriety \( k \). So in order to colour the whole graph, we must properly colour the subdivided cycle with 2 colours, which is impossible.

Lemma 8 The maximum average degree of \( G^k_n \) is \( M^k_n = \frac{(2n+1)(4k^2+6k+4)-2}{(2n+1)(k^2+2k+2)-1} \).

Proof. As it is easily seen, the maximum average degree of \( G \) is its average degree, which is:

\[
\frac{(2n+1)[(1 \times k(k+2) + 2) + (k \times (k+2)) + (k(k+1) \times 2)] + (2n) \times 2}{(2n+1)(1 + k + k(k+1)) + 2n} = M^k_n.
\]

3 Improper \( l \)-choosability, \( l \geq 2 \)

3.1 Lower bound for \( M(k, l) \)

In this subsection, we shall prove the following theorem:

Theorem 4 For all \( l \geq 2 \) and all \( k \geq 0 \), all graphs of maximum average degree less than \( \frac{l(l+2k)}{l+k} \) are \( k \)-improper \( l \)-choosable.

The result of the theorem is trivial if \( k = 0 \) since a graph of maximum average degree less than \( l \) is \((l-1)\)-degenerate (i.e. each of its subgraph has a vertex of degree at most \( l-1 \)). Hence it is \( l \)-choosable. For bigger values of \( k \), we will need some preliminary results.
Definition 5 A graph is said to be \((k, l)\)-minimal if it is not \(k\)-improper \(l\)-choosable but every of its proper subgraph is.

Lemma 9 Let \(G\) be a graph, \(L\) a list-assignment and \(c\) an \(L\)-colouring. If a vertex \(v\) has improperty at least \(d(v) - |L(v)| + 2\) under \(c\), then there exists an \(L\)-colouring \(c'\) of \(G\) such that \(c'(u) = c(u)\) if \(u \neq v\) and \(im_{c'}(v) = 0\).

Proof. Let \(c(v) = \alpha\). Then \(v\) has at most \(d(v) - (d(v) - |L(v)| + 2) = |L(v)| - 2\) neighbours that are not coloured with \(\alpha\). Hence there exists a colour \(\beta \in L(v)\) that does not colour any neighbour of \(v\). So setting \(c(v) = \beta\) we obtain the desired colouring. \(\square\)

We now prove a generalization of Lemma 1.

Lemma 10 Let \(k \geq 1\) and let \(G\) be a \((k, l)\)-minimal graph. Then:

(i) \(\delta \geq l\).

(ii) Two \((\leq l + k - 1)\)-vertices are not adjacent.

Proof.

(i) Let \(L\) be an \(l\)-list-assignment and suppose \(v\) is a \((\leq l - 1)\)-vertex. By minimality let \(c\) be a \(k\)-improper \(L\)-colouring of \(G - v\). As \(v\) has at most \(l - 1\) neighbours in \(G\), there exists a colour, say \(\alpha\), that is not assigned to any neighbour of \(v\). Hence colouring \(v\) with \(\alpha\) yields a \(k\)-improper \(L\)-colouring of \(G\).

Hence \(G\) is \(k\)-improper \(l\)-choosable, a contradiction.

(ii) Let \(L\) be an \(l\)-list-assignment and suppose, for a contradiction, that \(u\) and \(v\) are two neighbours of degree at most \(l + k - 1\). By minimality, let \(c\) be a \(k\)-improper \(L\)-colouring of \(G - \{uv\}\). Then \(c\) is an \(L\)-colouring of \(G\) such that each vertex has improperty at most \(k\), except possibly \(u\) and \(v\) which may have improperty \(k + 1\). But in this case we use Lemma 9 to recolour these vertices and obtain a \(k\)-improper \(L\)-colouring of \(G\).

Hence \(G\) is \(k\)-improper \(l\)-choosable, a contradiction. \(\square\)

Definition 6 Let \(G\) be a \((k, l)\)-minimal graph. We partially orient \(G\) using the following process:

1. Orient each edge \(uv\) where \(v\) is a \((\leq l + k - 1)\)-vertex from \(u\) to \(v\).

2. While there is an \(i\)-vertex \(v\) with \(l + k \leq i < l + k + \frac{k}{2}\) having outdegree exactly \(i - l + 1\) and indegree 0, we orient one of its unoriented incident edges \(uv\) from \(u\) to \(v\).

The digraph \(D\) induced by the oriented edges is called a discharging digraph of \(G\).

The following remark follows from the definition of a discharging digraph.

Remark 1
• Only vertices of degree less than \( l + k + \frac{1}{k} \) can have indegree more than zero.

• For \( i \leq l + k - 1 \), every \( i \)-vertex has indegree exactly \( i \) in \( D \).

**Definition 7** A quasi-arborescence rooted at \( u \) is a \((k, l)\)-quasi-arborescence if and only if:

- Every vertex has outdegree at most \( \max\{2, 2k - 1\} \).

- Every leaf has indegree at most \( l + k - 1 \)

Now we generalize Lemmas 2 and 3.

**Lemma 11** Let \( k \geq 2 \) and let \( Q \) be a \((k, l)\)-quasi-arborescence rooted at \( u \). Let \( L \) be a list-assignment of \( Q \) such that \( |L(v)| \geq \max\{1, dQ(v) - k + 1\} \) if \( v \) is a leaf and \( |L(v)| \geq 2 \) otherwise. We denote by \( S \) the set of leaves that have indegree at least \( k + 1 \) in \( Q \) (and hence a colour-list of size at least 2). Any \( L \)-colouring of the leaves extends in an \( L \)-colouring of \( Q \) such that:

- \( \forall v \in S, \text{im}(v) \leq k \).

- \( \forall v \notin S, \text{im}(v) \leq k \).

Furthermore, possibly by recolouring some vertices of \( S \), this \( L \)-colouring of \( G \) can be made \( k \)-improper.

**Proof.** By induction on the number of vertices of \( Q \), the result being trivially true if \( |V(Q)| = 1 \). Suppose now that \( |V(Q)| > 1 \) and the result holds for smaller \((k, l)\)-quasi-arborescences.

Let \( v_1, \ldots, v_s \) be the outneighbours of \( u \) in \( Q \). Note that \( Q - u \) is the union of \( s \) \((k, l)\)-quasi-arborescences \( Q_i \) rooted at \( v_i \), \( 1 \leq i \leq s \), that are disjoint except possibly on their leaves. We start by \( L \)-colouring all the leaves of \( Q \).

By induction we extend this colouring to an \( L \)-colouring of each of the \( Q_i \) such that \( \text{im}(v_i) \leq k - 1 \). Note that \( \text{im}_Q(x) = \text{im}_{Q_i}(x) \leq k \) for every vertex of \( Q_i \) which is not a leaf and \( \text{im}_Q(x) \leq k \) for each leaf not in \( S \). One of the two colours of \( L(u) \), say \( \alpha \), is assigned to at most \( k - 1 \) neighbours of \( u \) since \( \deg(u) \leq 2k - 1 \). Hence setting \( c(u) = \alpha \), we obtain the first desired colouring.

Now, we can recolour each leaf \( f \) of \( S \) with impropriety at least \( k + 1 \) using Lemma 9 since \( d_Q(f) - |L(f)| + 2 \leq d_Q(f) - d_Q(f) + k - 1 + 2 = k + 1 \). This concludes the proof.

The above result cannot be extended for \( k = 1 \). However one can prove the following:

**Lemma 12** Let \( Q \) be a \((1, l)\)-quasi-arborescence rooted at \( u \), \( L \) be a list-assignment of \( Q \) such that \( |L(v)| \geq 2 \) if \( v \) is not a leaf, and \( |L(v)| \geq d_Q(v) \) otherwise. We denote by \( S \) the set of leaves with indegree at least 2. Let \( c \) be an \( L \)-colouring of the leaves. One of the followings holds:

(i) \( c \) can be extended in an \( L \)-colouring of \( Q \) such that \( \text{im}(u) = 0 \) and \( \text{im}(v) \leq 1 \) if \( v \notin S \);

(ii) \( c \) can be extended in two different \( L \)-colourings of \( Q \) \( c_1 \) and \( c_2 \) such that \( c_1(v) = c_2(v) \) if \( v \neq u \) and \( \text{im}^{c_i}(v) \leq 1 \) if \( v \notin S \).

Furthermore, possibly by recolouring vertices of \( S \), all these \( L \)-colourings can be made \( 1 \)-improper. Moreover, if \( |L(u)| \geq 3 \) then (i) holds.
Proof. By induction on the number of vertices, the result being obvious if \( |V(Q)| = 1 \).

\( Q - u \) is the union of two \((1, l)\)-quasi-arborescences \( Q_1 \) and \( Q_2 \) rooted at \( v_1 \) and \( v_2 \) respectively. They are disjoint except possibly on their leaves. Let \( c \) be an \( L \)-colouring of the leaves of \( Q \). By induction we extend \( c \) to \( Q_1 \) and \( Q_2 \). Note that for each vertex \( v \) of \( Q - S \) \( \text{im}_Q(v) = \text{im}_Q(v) \leq 1 \).

If at least one of the \( Q_i \) satisfies (ii), or if \( |L(u)| \geq 3 \), we can suppose that \( \{c(v_1), c(v_2)\} \neq L(u) \) and hence we extend \( c \) into an \( L \)-colouring of \( Q \) fulfilling (i).

If both \( Q_1 \) and \( Q_2 \) satisfy (i), then either \( c(v_1) = c(v_2) \) and hence setting \( c(u) \in L(u) \{c(v_1)\} \) yields an \( L \)-colouring of \( Q \) that satisfies (i); or colouring \( u \) with two colours of its list gives the two desired colourings of (ii).

Now we can recolour with impropriety zero each leaf \( f \in S \) that has impropriety at least 2 in \( Q \) using Lemma 9, since \( d_Q(f) - |L(f)| + 2 \leq 2 \). This concludes the proof. \( \square \)

Using these results, we can say more about the structure of a discharging digraph. The following lemma generalizes Proposition 1.

**Lemma 13** Let \( D \) be a discharging digraph of a \((k, l)\)-minimal graph \( G \).

(i) Every vertex \( u \) with \( l + k \leq d(u) \leq l + 2k - 1 \) has outdegree at most \( d(u) - l + 1 \). In particular, \( D \) is acyclic.

(ii) For every vertex \( u \), \( A_D^+(u) \) is a \((k, l)\)-quasi-arborescence. In particular, the indegree of the leaves in \( A_D^+(u) \) is at most \( l + k - 1 \).

**Proof.** (ii) follows easily from (i). So, let us prove (i).

Let \( L \) be an \( l \)-list-assignment of \( G \). First, \( D \) has no 2-circuit since two \((\leq l + k - 1)\)-vertices are not adjacent by Lemma 10. Note also that in order to create a circuit in \( D \), it is necessary to create a vertex \( u \) of outdegree at least \( d(u) - l + 2 \). Now suppose, for a contradiction, that \( D \) contains a vertex \( u \) of outdegree at least \( d(u) - l + 2 \) and let \( D' \) be the digraph obtained just after having created the first such vertex \( u \). Let \( u \rightarrow v \) be the last edge that is oriented in \( D' \). \( u \) has \( d(u) - l + 2 \) outneighbours (including \( v \)) while \( v \) has \( d(v) - l + 1 \) outneighbours. We distinguish two cases depending whether the orientation of \( uv \) creates a circuit (which is necessary the first), or not.

**First Case:** the orientation of \( uv \) creates a circuit \( C \). Let \( w \) be the inneighbour of \( u \) in \( C \). We define \( Q_1 = A_{D' - uw}^+(v) \), \( Q_2 = A_{D' - uw}^+(u) \) and \( Q = Q_1 \cup Q_2 \). Note that \( Q_1 \) and \( Q_2 \) are \((k, l)\)-quasi-arborescences which are disjoint, except possibly on some leaves. In particular the outdegree in \( D' \) of every internal vertex \( x \) of \( Q \) is at most \( d_G(x) - l + 1 \). More precisely every internal vertex \( x \neq w \) satisfies \( d_{D'}^+(x) = d_G(x) - l + 1 \) while \( d_{D'}^+(w) = d_G(w) - l \) and for all every vertex \( v \) \( d_{D'}^+(x) = 1 \). Recall that \( d_G(w) = d^+(w) + d^-(w) \). Let \( F \) be the set of leaves in \( Q \), \( S \) the set of leaves that have indegree at least \( k + 1 \) in \( Q \) and \( \bar{S} = F \setminus S \). We define \( \bar{Q} = Q - \bar{S} \). By minimality, let \( c \) be a \( k \)-improper \( L \)-colouring of \( G' = G - \bar{Q} \). Let \( f \in S \): if \( f \) has impropriety at least \( k - d_G(f) + 1 \), then using Lemma 9 we recolour it with impropriety 0 since \( d_G(f) - |L(f)| + 2 = d_G(f) - d_G(f) - l + 2 \leq l + k - 1 - d_G(f) - l + 2 = k - d_G(f) + 1 \).

Now, let \( L_1 \) be the following list-assignment of \( Q_1 \):

\[ L_1(x) = L(x) \setminus \{z \mid \exists z \in N_{G - Q_1}(x), c(z) = \alpha \} \text{ if } x \notin S, \text{ and } L_1(x) = \{c(x)\} \text{ otherwise.} \]

Note that if \( x \neq w \) is an internal vertex then:

\[ |L_1(x)| \geq l - (d_G(x) - d_Q(x)) = l - d_G(x) + d_G(x) - l + 1 + 1 = 2 \]
and since $d^+(u) = d_G(w) - l$ but $u$ is yet uncoloured:

$$|L_1(w)| \geq l - (d_G(w) - d_{Q_1}(w)) + 1 = l - d_G(w) + d_G(w) - l + 1 + 1 = 2.$$  

For the root $v$, $d^-(v) = 0$ but $u$ is uncoloured yet so:

$$|L_1(v)| \geq l - (d_G(v) - d_{Q_1}(v)) + 1 = l - d_G(v) + d_G(v) - l + 1 + 1 = 2,$$

and for a leaf $f \in S$:

$$|L_1(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$  

Thus we may apply Lemmas 11 and 12. To do so, we $L_1$-colour all the leaves in $Q$.

Suppose first $k \geq 2$. By Lemma 11, we obtain an $L_1$-colouring $c_1$ of $Q_1$ such that $im_{Q_1}(v) \leq k - 1$. Note that $c_1$ extends $c$ into an $L$-colouring of $G - Q_2$ such that each vertex has improperty at most $k$ except possibly some vertices of $S$. Furthermore, $im_{G - Q_2}(v) \leq k - 1$. We define a list-assignment $L_2$ of $Q_2$ by $L_2(u) = L(u) \setminus \{\alpha - 3z \neq v \in N_{G - Q_2}(u), c(z) = \alpha\}$, $L_2(x) = \{c(x)\}$ if $x$ is a leaf and $L_2(x) = L(x) \setminus \{\alpha - 3z \in N_{G - Q_2}(x), c(z) = \alpha\}$ otherwise. Note that we have $|L_2(u)| \geq 2$. We now apply Lemma 11 so as to get an $L_2$-colouring of $Q_2$ and hence an $L$-colouring of $G$. Every vertex not in $S \cup \{u, v\}$ has improperty at most $k$. If $x \in \{u, v\}$ then: $im_G(x) \leq im_{G - Q_2}(x) + 1 \leq k - 1 + 1 = k$ since there cannot be in $L_2(u)$ the colour of a neighbour of $u$ in $G - (Q_2 - v)$. If $f \in S$ has improperty at least $k + 1$, then we recolour it with improperty 0 using Lemma 9 since $d_{Q_2}(f) - |L(f)| + 2 \leq l + k - 1 - 2 = k + 1$. Thus we obtain a $k$-improper $L$-colouring of $G$.

Suppose now $k = 1$. Applying Lemma 12, we obtain an $L_1$-colouring of $G - Q_2$ such that every vertex not in $S$ has improperty at most 1, and either $v$ has improperty 0 ($i$), or it has improperty 1 and we can indifferently colour it with two colours of its list ($ii$). Note that if $v$ has one neighbour distinct from $u$ which is an internal vertex in $Q_2$ then $|L_1(v)| \geq 3$ so we may suppose that $v$ fulfils ($i$). Defining $L_2$ as before, we can apply Lemma 12 to $Q_2$ so as to obtain an $L_2$-colouring of $Q_2$ and hence an $L$-colouring of $G$ such that $u$ fulfils ($i$) or ($ii$). Now, every vertex not in $S \cup \{u, v\}$ has improperty at most 1. If $v$ satisfies ($i$), then either $u$ also satisfies ($i$) or $u$ satisfies ($ii$) but in this case we may suppose $u$ and $v$ are coloured differently so in all cases they have improperty at most 1 in $G$. If $v$ satisfies ($ii$), then the only neighbour of $v$ in $Q_2$ is $u$. Hence we may safely suppose that $u$ and $v$ are coloured differently, so they have improperty at most 1 in $G$.

Finally, we can recolour each leaf of $S$ that has improperty at least 2 by using Lemma 9 and thus we obtain a 1-improper $L$-colouring of $G$.

Hence $G$ is $k$-improper $l$-choosable, a contradiction.

**Second Case:** there is no circuit in $D'$. Then $Q = A^+_D(u)$ is a quasi-arborescence. Moreover each internal vertex $v$ has outdegree at most (and hence exactly) $d(v) - l + 1$. Let $v_1, \ldots, v_s$ be the outneighbours of $u$, we define $Q_j = A^+_D(v_j)$, $1 \leq j \leq s$. The $Q_i$ are $(k, l)$-quasi-arborescences that are disjoint except possibly on their leaves. Let $F$ be the set of leaves in $Q$, $S$ the set of leaves with indegree at least $k + 1$ in $Q$ and $\hat{S} = F \setminus S$. We define $Q = Q - \hat{S}$. Let $L$ be an $l$-list-assignment of $G$. By minimality, let $c$ be a $k$-improper $L$-colouring of $G' = G - Q$. Let $f$ be a leaf in $\hat{S}$. If $f$ has improperty at least $k - d_Q(f) + 1$, we recolour it with improperty 0 using Lemma 9 since: $d_G(f) - |L(f)| + 2 \leq d_G(f) - d_Q(f) - l + 2 \leq l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1$.  

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For each vertex \( v \in Q \), we define \( L'(v) = L(v) \setminus \{ \alpha \} \) if \( v \notin S \) and \( L'(v) = \{ c(v) \} \) otherwise. Note that for an internal vertex \( v \):

\[
|L'(v)| \geq l - (d_G(v) - d_Q(v)) = l - d_G(v) + d_G(v) - l + 1 + 1 = 2.
\]

For a leaf \( f \in S \):

\[
|L'(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.
\]

Suppose first \( k \geq 2 \). We \( L' \)-colour all the leaves, use Lemma 11 so as to extend it into an \( L' \)-colouring of each of the \( Q_i \), and possibly with recolouring some leaves in \( S \) we get a \( k \)-improper \( L \)-colouring of \( G - u \) such that \( \text{im}(v_j) \leq k - 1 \), \( 1 \leq j \leq s \).

Now \( |L'(u)| \geq |L(u)| - d(u) + d'_D(u) = l - d(u) + d(u) - l + 2 \geq 2 \). And \( u \) has \( d^+(u) = d(u) - l + 2 \leq 2k + 1 \) out-neighbours in \( D' \). Thus there is a colour of \( L'(u) \), say \( \alpha \), that is assigned to at most \( k \) out-neighbours of \( u \). Thus setting \( c(u) = \alpha \) yields a \( k \)-improper \( L \)-colouring of \( G \) by definition of \( L' \).

Suppose now \( k = 1 \). We \( L' \)-colour all the leaves, use Lemma 12 so as to extend it into an \( L' \)-colouring of each of the \( Q_i \), and possibly with recolouring some leaves in \( S \) we get a 1-improper \( L \)-colouring of \( G - u \) such that for each \( v_j \) either \( \text{im}(v_j) = 0 \) or \( v_j \) can safely be recoloured with another colour of \( L'(v_j) \).

The same calculation as above shows there exists a colour of \( L'(u) \), say \( \alpha \), that is assigned to at most 1 neighbour of \( u \), say \( v_1 \). We set \( c(u) = \alpha \). If \( v_1 \) satisfies the first condition, we have a 1-improper \( L \)-colouring of \( G \). If \( v_1 \) satisfies the second condition then we may suppose that \( c(u) \neq c(v) \) and thus we also have a 1-improper \( L \)-colouring of \( G \).

Hence \( G \) is \( k \)-improper \( L \)-choosable, a contradiction.

\[ \square \]

**Proof of Theorem 4.** Let \( G \) be a \((k, l)\)-minimal graph and \( D \) a discharging digraph of \( G \). We start with a charge \( w(v) = d(v) \) on each vertex and we apply the following discharging rule: every vertex gives \( \frac{d(v)k}{l+k} \) to each of its out-neighbours.

Let us examine the new charge \( w'(v) \) of a vertex \( v \):

- If \( d(v) \leq l + k - 1 \) it has indegree \( d(v) \) so its new charge is \( w'(v) = d(v) + \frac{d(v)k}{l+k} \geq l + \frac{lk}{l+k} \).

- If \( l + k \leq d(v) < l + k + \frac{1}{k} \) then either \( v \) has outdegree at most \( d(v) - l \) and its new charge is at least \( d(v) - (d(v) - l) \times \frac{k}{l+k} = \frac{d(v)}{l+k} + \frac{lk}{l+k} \geq l + \frac{lk}{l+k} \), or by Lemma 13, it has outdegree \( d(v) - l + 1 \). In this case, by definition of a discharging digraph, \( v \) has indegree 1 so its new charge is:

\[
w'(v) = d(v) + (d(v) - l + 1) \times \frac{k}{l+k} = \frac{d(v)}{l+k} + \frac{k}{l+k} \geq l + \frac{k}{l+k}.
\]

- If \( l + k + \frac{1}{k} \leq d(v) \leq l + 2k - 1 \), then by Lemma 13, \( v \) has outdegree at most \( d(v) - l + 1 \). So \( w'(v) \geq d(v) - (d(v) - l + 1) \times \frac{k}{l+k} = \frac{d(v)}{l+k} + \frac{k-1}{l+k} = l + \frac{kl}{l+k} \).

- If \( d(v) \geq l + 2k \), then \( w'(v) \geq d(v)(1 - \frac{k}{l+k}) = \frac{d(v)}{l+k} \geq \frac{l^2 + 2kl}{l+k} = l + \frac{kl}{l+k} \).
Hence $\text{Mad}(G) \geq \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{|V|} \sum_{v \in V} w'(v) \geq l + \frac{kl}{l+k}$.

3.2 Upper bound for $M(k,l)$

In this section we shall construct for all $l \geq 2$ and all $k \geq 1$, a graph $G^k_l$ which is not $k$-improper $l$-colourable. So its maximum average degree will give an upper bound for $M(k,l)$. To construct $G^k_l$, take $k+1$ copies of $H_k$ (defined in Subsection 2.2) and identify their vertex $v$. We define $G^k_l$, $l \geq 3$, inductively. First we create the graph $M^k_l$ by taking $k$ copies of $G^k_l$ and adding a vertex $w$ which we join to every other vertices. Then we take $l-1$ copies $M^1, \ldots, M^{l-1}$ of $M^k_l$ and we join all the vertices $w_1, \ldots, w_{l-1}$ (so that they form a complete graph of size $l-1$). Now, we add $k+2$ vertices $z_0, z_1, \ldots, z_{k+1}$ each joined to each of the $w_i$, $1 \leq i \leq l-1$. Last we add the edges $z_0z_i$ for $1 \leq i \leq k+1$.

![Diagram of $G^k_l$](image)

**Lemma 14** For all $l \geq 2$ and all $k \geq 1$, the graph $G^k_l$ is not $k$-improper $l$-colourable.

**Proof.** The result is clear for $G^k_2$. Suppose the result is true for $l-1 \geq 2$ and let us prove it for $G^k_l$. First note that in any $k$-improper $l$-colouring of $M^i$, the vertex $w_i$ has improperty $k$. Indeed, $w_i$ has a neighbour of its colour in each copy of $G^k_{l-1}$ since otherwise $G^k_{l-1}$ would be $k$-improper $(l-1)$-colourable. Hence each of the $w_i$, $1 \leq i \leq l-1$, cannot have any neighbour of its colour in $G^k_l - M^i$. In particular, as the subgraph induced by $w_1, \ldots, w_{l-1}$ is complete, all the $z_i$, $0 \leq i \leq k+1$, must be coloured the same. But then $w_0$ must have improperty $k+1$. □

**Lemma 15** The number of vertices of $G^k_l$ is:

$$n^k_l = 2l + (l+1)k + \sum_{i=2}^l \frac{(l-1)!}{(l-i)!} k^i.$$  

In particular, it is a polynomial in $k$ of degree $l$ and dominant coefficient $(l-1)!$.

**Proof.** $n^k_l$ satisfies: $n^k_2 = k^2 + 3k + 3$ and $\forall l \geq 3, n^k_l = (k \times n^k_{l-1} + 1) \times (l-1) + k + 2$. □

Let $s^k_l$ denotes the sum of the degrees of the vertices in $G^k_l$. We have the following result:
Lemma 16 $s^k_l$ is a polynomial in $k$ of degree $l$ whose dominant coefficient is $2l!$.

Proof. $s^k_l$ satisfies: $s^k_2 = 4k^2 + 10k + 6$ and $s^k_l = (l-1)(k \times s^k_{l-1} + 2k \times n^k_{l-1} + l + k) + (l+1)k + 2l$ if $l \geq 3$. Hence it is a polynomial in $k$ of degree $l$. Furthermore, denoting by $c^k_l$ its dominant coefficient, we have: $c^k_2 = 4$ and $\forall l \geq 3$, $c^k_l = (l-1) \times c^k_{l-1} + 2k \times (l-1)!$. Thus $c^k_l = 2l!$. \hfill $\square$

Proposition 2 $\lim_{k \to \infty} Mad(G^k_l) = 2l$.

Proof. It is clear that the maximum average degree of $G^k_l$ is its average degree. Then by Lemmas 15 and 16, we have:

$$\lim_{k \to \infty} Mad(G^k_l) = 2 \frac{l!}{(l-1)!} = 2l.$$ \hfill $\square$

Corollary 2 For any fixed $l$, $\lim_{k \to +\infty} M(k,l) = 2l$.

Proof. It follows from Theorem 4 and Proposition 2. \hfill $\square$

References


