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IMPROPER CHOOSABILITY OF GRAPHS AND MAXIMUM AVERAGE DEGREE

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Improper choosability of graphs and maximum average degree

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Abstract

Improper choosability of planar graphs has been widely studied. In particular, Škrekovski investigated the smallest integer g_k such that every planar graph of girth at least g_k is k-improper 2-choosable. He proved [8] that $6 \leq g_1 \leq 9$; $5 \leq g_2 \leq 7$; $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. In this paper, we study the greatest real $M(k,l)$ such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. We prove that for $l \geq 2$ then $M(k,l) \geq l + \frac{lk}{l+k}$. As a corollary, we deduce that $g_1 \leq 8$ and $g_2 \leq 6$. We also provide an upper bound for $M(k,l)$. This implies that for any fixed l, $M(k,l) \xrightarrow[k\to\infty]{} 2l$.

1 Introduction.

Let G be a graph. We note $V(G)$ its vertex set and $E(G)$ its edge set.

A *colouring* is an application from the vertex set into a set of colours S. If $|S| = l$ we call it l-colouring. Let c be a colouring of G. The *improperty* of a vertex v in G under c, denoted by $im^c_G(v)$, is the number of neighbours u of v in G such that $c(u) = c(v)$. The *improperty* of c in G is $im_G(c) = \max\{im^c_G(v) — \ v \in V(G)\}$. A colouring is *$k$-improper* if its improperty is at most k and a graph is *k-improper l-colourable* if it admits a k-improper l-colouring. The *k-improper chromatic number* of G, denoted by $c_k(G)$, is the smallest integer l such that G is k-improper l-colourable. Note that 0-improper colouring is the usual notion of proper colouring, so the 0-improper chromatic number is exactly the chromatic number usually denoted $\chi(G)$.

One can analogously generalize the notion of *choosability.* A *list-assignment* of a graph G is an application L which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v)$. L is an l-list-assignment provided each list is of size at least l. G is k-improper L-colourable if there exists a k-improper colouring c of G such that $\forall v \in V(G), v \in L(v)$. In this case, c is a k-improper L-colouring of G. G is k-improper l-choosable if it is k-improper L-colourable for every l-list-assignment L.

Colourings of planar graphs have been widely studied. In particular p_k and p^*_k, the smallest integers l such that every planar graph is k-improper l-colourable and k-improper l-choosable respectively, are known for almost all k. Indeed Thomassen showed in [9] that every planar graph is 5-choosable and there are planar graphs which are not 4-choosable [12] so $p^*_0 = 5$. Every planar graph is 4-colourable [1, 2] and there are graphs which are not 1-improper 3-colourable, so $p_0 = p_1 = 4$. But we do not know the exact value of p^*_1 which is either 4 or 5. However, it is conjectured that it is 4:

Conjecture 1 (Eaton and Hull [3], Škrekovski [6]) *Every planar graph is 1-improper 4-choosable.*

As shown independently by Eaton and Hull [3] and Škrekovski [6], every planar graph is 2-improper 3-choosable and for every k, there are planar graphs which are not k-improper 2-colourable. Hence $p_k = p_k^* = 3$ for any $k \geq 2$.

Moreover improper colourings of planar graphs have also been studied under some girth restrictions. The *girth* of graph is the smallest length of a cycle. The well-known theorem of Grötzsch [4, 11] states that every planar graph of girth at least 4 is 3-colourable. Voigt [13] showed a planar graph of girth 4 which is not 3-choosable and Thomassen [10] proved that every planar graph of girth at least 5 is 3-choosable. In [7], Škrekovski showed that every planar graph of girth at least 4 is 1-improper 3-choosable. In [8], Škrekovski investigated k-improper 2-choosability of planar graphs in relation with their girth. Denoting by g_k be the smallest integer such that every planar graph of girth at least g_k is k-improper 2-choosable, he proved that $6 \leq g_1 \leq 9$, $5 \leq g_2 \leq 7$, $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. Hence the only unknown values are g_1, g_2 and g_3.

In this paper, we study the k-improper l-choosability of graphs in relation with their maximum average degree.

Definition 1 The maximal average degree of a graph G is:

$$Mad(G) := max\{\frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}, H \text{ subgraph of } G\}.$$

The girth and the maximum average degree of a planar graph are related to each other:

Theorem 1 *Let G be a planar graph of girth g.*

$$Mad(G) < \frac{2g}{g-2}.$$

Proof. We recall the Euler's formula for a planar graph H: $|V(H)| - |E(H)| + |F(H)| = 2$ with $|F(H)|$ the number of faces of H. Note that every subgraph H of G has girth at least g, so $g|F(H)| \leq 2|E(H)|$. Thus $2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|$. Hence $\frac{2|E(H)|}{|V(H)|} \leq \frac{2g}{g-2} - \frac{4g}{(g-2)|V(H)|} < \frac{2g}{g-2}$ for every subgraph H of G. □

Let $M(k,l)$ be the greatest real such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. Obviously, $M(k_1,l) \leq M(k_2,l)$ if $k_1 \leq k_2$.

We have that $M(k,1) = \frac{2k+2}{k+2}$ since a graph is k-improper 1-choosable if, and only if, it has maximum degree at most k.

In order to introduce our method which uses some discharging process, we first present it in Section 2 for improper 2-choosability: we prove that for every $k \geq 0$,

$$4 - \frac{4}{k+2} \leq M(k,2) \leq 4 - \frac{2k+4}{k^2+2k+2}.$$

As a corollary, we obtain the following upper bounds for g_k which are better than Škrekovski's ones: $g_1 \leq 8$, $g_2 \leq 6$, $g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$.

In Section 3 we extend the lower bound of Section 2 to any value of l: we prove that for every $l \geq 2$ and $k \geq 0$,

$$l + \frac{lk}{l+k} \leq M(k,l).$$

Last, we provide for any value of l and k a graph which is not k-improper l-choosable, and we deduce that $M(k,l) \xrightarrow[k\to\infty]{} 2l$.

2 Improper 2-choosability

2.1 Lower bound for $M(k, 2)$

In this subsection, we shall prove the following theorem:

Theorem 2 *For all $k \geq 0$, all graphs of maximum average degree less than $\frac{4k+4}{k+2}$ are k-improper 2-choosable.*

Note that if $k = 0$ the result holds trivially. Indeed a graph with maximum average degree less than 2 contains no cycle and so is a forest. Hence it is 2-choosable. Furthermore $M(0, 2) \leq 2$ since an odd cycle is not 2-colourable, so $M(0, 2) = 2$.

For bigger value of k, we will need the following preliminary definitions and results:

Definition 2 If $v \in V(G)$ then $d_G(v)$ denotes the degree of v in the graph G. For all positive integer d, a vertex of degree equals to (resp. at most, resp. at least) d is called a d-vertex (resp. $(\leq d)$-vertex, resp. $(\geq d)$-vertex). For $S \subseteq V(G)$ (resp. $E \subseteq E(G)$) we denote by $G - S$ (resp. $G - E$) the induced subgraph of G obtained by removing the vertices (resp. edges) of S (resp. E) from $V(G)$ (resp. $E(G)$). If $S = \{v\}$ and $E = \{uv\}$, we shall note $G - v = G - S$ and $G - uv = G - E$. The union (resp. intersection) of the graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ (resp. $G = G_1 \cap G_2$) such that $V(G) = V(G_1) \cup V(G_2)$ (resp. $V(G) = V(G_1) \cap V(G_2)$) and $E(G) = E(G_1) \cup E(G_2)$ (resp. $E(G) = E(G_1) \cap E(G_2)$).

A graph is said to be $(k, 2)$-minimal if it is not k-improper 2-choosable but each of its proper subgraphs is.

Lemma 1 (Škrekovski [8]) *Let $k \geq 1$ and let G be a $(k, 2)$-minimal graph. Then*

(i) $\delta \geq 2$.

(ii) Two $(\leq k + 1)$-vertices are not adjacent.

Definition 3 Let D be a digraph. The outdegree (resp. indegree) of a vertex u in D is denoted by $d^+_D(u)$ (resp. $d^-_D(u)$). The *degree* of u is $d_D(u) = d^-_D(u) + d^+_D(u)$; it is the degree of u in the underlying undirected graph.

If u and v are two of its vertices, a (u, v)-dipath is a directed path from u to v.

An *arborescence* is an oriented tree in which every path is directed from a vertex called the *root*. Note that in an arborescence every vertex except the root has indegree 1. The leaves of the arborescence are the vertices of outdegree 0. A vertex which is neither a leaf nor the root is an *internal vertex*. A *quasi-arborescence* is a directed graph obtained from an arborescence by identifying some leaves.

Let u be a vertex of a digraph D. The *outsection* of u in D, denoted $A^+_D(u)$, is the set of vertices v such that there is a (u, v)-dipath in D.

Let G be a $(k, 2)$-minimal graph. We partially orient G using the following process:

1. Orient each edge uv where v is a 2-vertex from u to v.

2. If $k \geq 3$, orient each edge uv where v is a 3-vertex from u to v.

3. While there is an unoriented edge uv where v an i-vertex with $2 + k \leq i < \frac{3k}{2} + 2$ and outdegree $i - 1$, we orient it from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G.

The following proposition, whose proof is left to the reader, follows immediately from the definition of a discharging digraph.

Proposition 1 *Let D be a discharging digraph of a $(k,2)$-minimal graph.*

- *D has no 2-circuit since two $(\leq k+1)$-vertices are not adjacent by Lemma 1 (ii). So it has no circuit at all.*
- *If $k \leq 2$, only vertices of degree 2 or $k+2$ have indegree more than zero. If $k \leq 3$, only vertices of degree 2, 3 or $k+2$ have indegree more than zero.*
- *Every 2-vertex has indegree exactly 2 in D and if $k \geq 3$, every 3-vertex has indegree exactly 3.*
- *For every vertex u, $A^+_D(u)$ is a quasi-arborescence whose leaves have degree 2 (resp. 2 or 3) in G if $k \leq 2$ (resp. $k \geq 3$). In particular, the indegree of the leaves in $A^+_D(u)$ is at most 2 (resp. 3).*

Definition 4 A quasi-arborescence is a $(k,2)$-quasi-arborescence if and only if:

- Every vertex has outdegree at most $\max\{2, 2k-1\}$.
- Every leaf has indegree at most $\min\{k, 3\}$.

Lemma 2 *Let $k \geq 2$. Let Q be a $(k,2)$-quasi-arborescence rooted at u and L a 2-list-assignment of Q. Then any L-colouring of the leaves can be extended in a k-improper L-colouring of D such that u has improperty at most $k-1$.*

Proof. By induction on the number of vertices of Q, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller k-quasi-arborescences. Let $v_1, \dots, v_s$ be the outneighbours of u in Q. Note that $Q - u$ is the union of s $(k,2)$-quasi-arborescences Q_i, $1 \leq i \leq s$ rooted at v_i that are disjoint except possibly on their leaves.

Let c be an L-colouring of the leaves of Q. Then by induction it can be extended in a k-improper L-colouring of each of the Q_i so that $im(v_i) \leq k-1$. Since a leaf of Q has indegree at most $\min\{k,3\}$ and $im_Q(x) = im_{Q_i}(x)$ for every vertex of Q_i which is not a leaf, then the union of these colourings is a k-improper L-colouring of Q such that $im(v_i) \leq k-1$.

Now, one of the two colours of $L(u)$, say α, is assigned to at most $k-1$ neighbours of u since $s \leq 2k-1$. Thus setting $c(u) = \alpha$, we obtain the desired colouring. □

Obviously, the above result cannot be extended for $k=1$ because it is hopeless to extend every L-colouring of the leaves in a colouring such that the root has improperty 0. However, one can prove the following weaker result:

Lemma 3 *Let Q be a $(1,2)$-quasi-arborescence rooted at u, L a 2-list-assignment of Q with $L(u) = \{\alpha, \beta\}$ and c an L-colouring of S the set of leaves of Q with indegree 1. One the following holds:*

(i) c may be extended in a 1-improper L-colouring of Q such that $im(u) = 0$;

(ii) c may be extended in two different 1*-improper L-colourings of Q, one such that* $c(u) = \alpha$ *and one such that* $c(u) = \beta$*.*

Proof. We proceed by induction on the number of vertices of Q. Let v_1 an v_2 be two out-neighbours of u in Q. $Q - u$ is the union of two $(1, 2)$-quasi-arborescences Q_1 and Q_2, rooted at v_1 and v_2 respectively, that are disjoint except possibly on their leaves. Let S' be the set of leaves in $Q_1 \cap Q_2$ and $L(u) = \{\alpha, \beta\}$. We L-colour the leaves of Q_i that have indegree 1 in Q_i. By induction, each of the Q_i satisfies (i) or (ii).

If at least one of the Q_i satisfies (ii), then one can extend c to $Q_1 \cup Q_2$ such that $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$. Moreover for any vertex x not in $V(Q_i)\backslash S'$, $im_Q(x) = im_{Q_i}(x) \leq 1$. If a vertex $s' \in S'$ has improperty 2 then its two neighbours are coloured the same. So recolouring s' with the colour of $L(s') \setminus \{c(s')\}$, we get a 1-improper L-colouring of $Q_1 \cup Q_2$. Hence setting $c(u) = \alpha$, we get a 1-improper L-colouring of Q such that $im(u) = 0$. Thus Q satisfies (i).

Suppose now Q_1 and Q_2 both satisfy (i). Then, possibly with recolouring of vertices of S' as before, one can extend c into a 1-improper L-colouring of $Q_1 \cup Q_2$ such that $im(v_1) = im(v_2) = 0$. If $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$ then setting $c(v) = \alpha$, we get a 1-improper L-colouring of Q such that $im(u) = 0$. Thus Q satisfies (i). If not then assigning to u the colours α and β, we get the two 1-improper L-colourings of Q satisfying (ii). □

Lemma 4 *Let* $k \geq 3$*. Let D be a discharging digraph of a* $(k, 2)$*-minimal graph G.*

(i) Every i-vertex with $4 \leq i \leq k + 1$ *has outdegree zero.*

(ii) Every i-vertex with $2 + k \leq i \leq 2k + 1$ *has outdegree less than i.*

Proof.

(i) Suppose, for a contradiction, that v is a vertex contradicting the assertion and let u be an outneighbour of v. Note that u is a $(< \frac{3k}{2} + 2)$-vertex by definition of a discharging digraph.

Let L be a 2-list-assignment of G. Let S be the set of leaves of $A^+_D(u)$. By minimality, let c be a k-improper L-colouring of $G - A^+_D(u)$.

$A^+_D(u)$ is a $(k, 2)$-quasi-arborescence: since it is dominated by v in D, u has outdegree less than $\frac{3k}{2} + 1$ and so at most $2k - 1$. Thus by Lemma 2, we can extend c to $G - vu$ so that $im(u) \leq k - 1$. Since the leaves have degree at most $3 \leq k$, the improperty of the leaves is at most $3 \leq k$. So we obtain a k-improper L-colouring of $G - uv$.

If $c(u) \neq c(v)$ or $im_{G-uv}(v) \leq k - 1$ then c is a k-improper L-colouring of G. Otherwise all the $k + 1$ neighbours of v are coloured the same so recolouring v with its other allowed colour yields a k-improper L-colouring of G.

Hence G is k-improper 2-choosable which is a contradiction.

(ii) Suppose, for a contradiction, that v is an i-vertex contradicting the assertion.

Let L be 2-list-assignment of G and c a k-improper L-colouring of $G - v$. There is a colour of $L(v)$, say α, that is assigned to at most k neighbours of v. Let $v_1, \ldots, v_s$ be these neighbours.

Let $G' = G - \bigcup_{j=1}^{s} A_D^+(v_j)$. And set $c' = c$ for every vertex of G' and every leaf of the $A_D^+(v_j)$. By Lemma 2 applied to each $A_D^+(v_j)$ (which are disjoint except possibly on their leaves), we can extend c' into a k-improper L-colouring of $G - v$ such that $im(v_j) \leq k-1$ for $1 \leq j \leq s$. Now by definition of c', the only neighbours of v that may be assigned α by c' are those of $\{v_1, \ldots, v_s\}$. Hence setting $c'(v) = \alpha$, the L-colouring c' is k-improper.

Hence G is k-improper 2-choosable which is a contradiction.

□

Analogously, one can prove the following lemma when $k = 2$.

Lemma 5 *Let D be a discharging digraph of a $(2,2)$-minimal graph G.*

(i) The outdegree of a 3-vertex is zero.

(ii) If v is an i-vertex with $i \in \{4; 5\}$ then its outdegree is less than i.

Lemma 6 *Let D be a discharging of a $(1,2)$-minimal graph G. There is no 3-vertex with outdegree 3 in D.*

Proof. Suppose, for a contradiction, that v is a 3-vertex with outdegree 3. Let u be an outneighbour of v. Let $Q_1 = A_D^+(u)$, $Q_2 = A_{D-vu}^+(v)$, S be the set of leaves of $A_D^+(v)$ with indegree 1 in $A_D^+(v)$ and S' the set of leaves with indegree 2 in $A_D^+(v)$.

Let L be a 2-list-assignment of G. By minimality of G, let c be a 1-improper L-colouring of $G - A_D^+(v)$. Vertices not in S have no neighbour in $G - A_D^+(v)$ and every vertex of S has exactly one neighbour in $G - A_D^+(v)$. Extend c to $S \cup S'$ by assigning to each vertex of S a colour of its list not assigned to its neighbour in $G - A_D^+(v)$ and any colour of its list to a vertex of S'.

Now Q_1 and Q_2 satisfy either (i) or (ii) of Lemma 3. If one of them satisfies (ii), then possibly with recolouring of vertices of S' one can extend c into a 1-improper L-colouring of $G - vu$ such that $c(v) \neq c(u)$. Hence c is a 1-improper L-colouring of G.
If Q_1 and Q_2 satisfies both (i), then possibly with recolouring of vertices of S' one can extend c into a 1-improper L-colouring of $G - vu$ such that $im(v) = im(u) = 0$. Hence c is a 1-improper L-colouring of G.

So G is 1-improper 2-choosable which is a contradiction. □

Proof of Theorem 2. Let G be a $(k,2)$-minimal graph and D a discharging digraph of G. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{k+2}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex v:

- If v is a 2-vertex, it has indegree 2 so its new charge is $w'(v) = 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$.
- If v is a 3-vertex and $k \geq 3$, it has indegree 3 so its new charge is $w'(v) = 3 + 3 \times \frac{k}{k+2} = \frac{6k+6}{k+2} > \frac{4k+4}{k+2}$. If v is a 3-vertex and $k = 2$ then it has outdegree 0 by Lemma 5 and indegree 0 by construction so $w'(v) = 3$.
- If $4 \leq d(v) \leq k+1$, $(k \geq 3)$, then by Lemma 4 (i), v has outdegree zero so its charge is $d(v) \geq 4 > \frac{4k+4}{k+2}$.

- If $k+2 \le d(v) < \frac{3k}{2}+2$ then either v has outdegree at most $d(v)-2$ and so its new charge is at least $d(v)-(d(v)-2)\times\frac{k}{k+2} = \frac{2d(v)}{k+2}+\frac{2k}{k+2} \ge 2+\frac{2k}{k+2} = \frac{4k+4}{k+2}$, or by Lemmas 4, 5 and 6, it has outdegree $d(v)-1$. In this case, by definition of a discharging digraph, v has indegree 1 so its new charge is:

$$d(v)-(d(v)-1)\times\frac{k}{k+2}+\frac{k}{k+2} = d(v)-(d(v)-2)\times\frac{k}{k+2} \ge \frac{4k+4}{k+2}.$$

- If $\frac{3k}{2}+2 \le d(v) \le 2k+1$, $(k \ge 2)$, then by Lemmas 4 and 5, v has outdegree at most $d(v)-1$. So $w'(v) \ge d(v)-(d(v)-1)\times\frac{k}{k+2} = \frac{2d(v)}{k+2}+\frac{k}{k+2} \ge \frac{3k+4+k}{k+2} = \frac{4k+4}{k+2}$.

- If $d(v) \ge 2k+2$, then $w'(v) \ge d(v)(1-\frac{k}{k+2}) = \frac{2d(v)}{k+2} \ge \frac{4k+4}{k+2}$.

Hence $Mad(G) \ge \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{1}{|V|}\sum_{v\in V} w'(v) \ge \frac{4k+4}{k+2}$. □

Corollary 1 *Let G be a planar graph of girth g.*

1. *If $g \ge 8$ then G is 1-improper 2-choosable, so $g_1 \le 8$.*
2. *If $g \ge 6$ then G is 2-improper 2-choosable, so $g_3 \le g_2 \le 6$.*
3. *If $g \ge 5$ then G is 4-improper 2-choosable, so $g_k \le 4$ for $k \ge 5$.*

2.2 Upper bound for $M(k,2)$

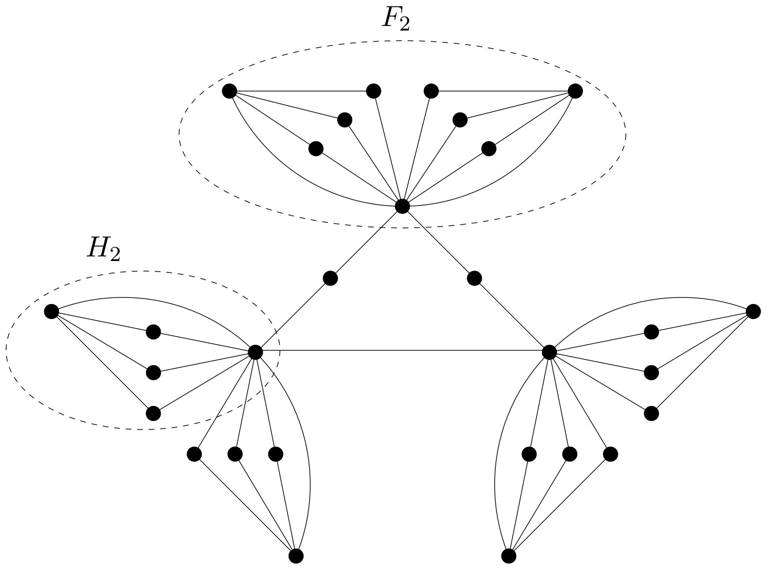
Let us fix $k \ge 1$. In this section, we shall construct a family of graphs $(G_n^k)_{n\ge1}$ such that for all $n \ge 1$:

- G_n^k is not k-improper 2-colourable.
- $Mad(G_n^k) = \dfrac{2n(4k^2+6k+4)+4k^2+6k+2}{2n(k^2+2k+2)+(k+1)^2}$.

Hence we will deduce:

Theorem 3 *For all $k \ge 1$, $M(k,2) \le \dfrac{4k^2+6k+4}{k^2+2k+2} = 4-\dfrac{2k+4}{k^2+2k+2}$.*

We denote by H_k the graph composed of two adjacent vertices u and v also connected by $k+1$ disjoint paths of length 2. Take k copies of H_k and create the graph F_k by identifying the vertex v of each copy. Note that F_k has one vertex of degree $k(k+2)$, k vertices of degree $k+2$ and $k(k+1)$ vertices of degree 2. Now we take $2n+1$ copies of F_k and we join the vertices v of each copy creating a cycle of size $2n+1$. At last we make a subdivision of all the edges of the cycle but one so as to obtain the graph G_n^k.



The graph G_1^2

Lemma 7 *G_n^k is not k-improper 2-colourable.*

Proof. First remark that in any k-improper 2 colouring of H_k, v has improperty at least 1. Indeed v is a $(k+2)$ vertex in H_k, so if it has improperty zero then its $k+2$ neighbours are coloured the same, but this is impossible since u is a neighbour of v adjacent to the $k+1$ remaining neighbours. Hence in any k-improper colouring of F_k, v has improperty k. So in order to colour the whole graph, we must properly colour the subdivided cycle with 2 colours, which is impossible. □

Lemma 8 *The maximum average degree of G_n^k is $M_n^k = \frac{(2n+1)(4k^2+6k+4)-2}{(2n+1)(k^2+2k+2)-1}$.*

Proof. As it is easily seen, the maximum average degree of G is its average degree, which is:

$$\frac{(2n+1)[(1 \times k(k+2)+2) + (k \times (k+2)) + (k(k+1) \times 2)] + (2n) \times 2}{(2n+1)(1+k+k(k+1)) + 2n} = M_n^k.$$

□

3 Improper l-choosability, $l \geq 2$

3.1 Lower bound for $M(k,l)$

In this subsection, we shall prove the following theorem:

Theorem 4 *For all $l \geq 2$ and all $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.*

The result of the theorem is trivial if $k = 0$ since a graph of maximum average degree less than l is $(l-1)$-*degenerate* (i.e. each of its subgraph has a vertex of degree at most $l-1$). Hence it is l-choosable. For bigger values of k, we will need some preliminary results.

Definition 5 A graph is said to be (k,l)-minimal if it is not k-improper l-choosable but every of its proper subgraph is.

Lemma 9 *Let G be a graph, L a list-assignment and c an L-colouring. If a vertex v has improperty at least $d(v) - |L(v)| + 2$ under c, then there exists an L-colouring c' of G such that $c'(u) = c(u)$ if $u \neq v$ and $im_{c'}(v) = 0$.*

Proof. Let $c(v) = \alpha$. Then v has at most $d(v) - (d(v) - |L(v)| + 2) = |L(v)| - 2$ neighbours that are not coloured with α. Hence there exists a colour $\beta \in L(v)$ that does not colour any neighbour of v. So setting $c(v) = \beta$ we obtain the desired colouring. □

We now prove a generalization of Lemma 1.

Lemma 10 *Let $k \geq 1$ and let G be a (k,l)-minimal graph. Then:*

(i) $\delta \geq l$.

(ii) Two $(\leq l + k - 1)$-vertices are not adjacent.

Proof.

(i) Let L be an l-list-assignment and suppose v is a $(\leq l-1)$-vertex. By minimality let c be a k-improper L-colouring of $G - v$. As v has at most $l - 1$ neighbours in G, there exists a colour, say α, that is not assigned to any neighbour of v. Hence colouring v with α yields a k-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

(ii) Let L be an l-list-assignment and suppose, for a contradiction, that u and v are two neighbours of degree at most $l + k - 1$. By minimality, let c be a k-improper L-colouring of $G - \{uv\}$. Then c is an L-colouring of G such that each vertex has improperty at most k, except possibly u and v which may have improperty $k + 1$. But in this case we use Lemma 9 to recolour these vertices and obtain a k-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

□

Definition 6 Let G be a (k,l)-minimal graph. We partially orient G using the following process:

1. Orient each edge uv where v is a $(\leq l + k - 1)$-vertex from u to v.
2. While there is an i-vertex v with $l + k \leq i < l + k + \frac{k}{l}$ having outdegree exactly $i - l + 1$ and indegree 0, we orient one of its unoriented incident edges uv from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G.

The following remark follows from the definition of a discharging digraph.

Remark 1

- *Only vertices of degree less than $l+k+\frac{k}{l}$ can have indegree more than zero.*
- *For $i \leq l+k-1$, every i-vertex has indegree exactly i in D.*

Definition 7 A quasi-arborescence rooted at u is a (k,l)-*quasi-arborescence* if and only if:

- Every vertex has outdegree at most $\max\{2, 2k-1\}$.
- Every leaf has indegree at most $l+k-1$

Now we generalize Lemmas 2 and 3.

Lemma 11 *Let $k \geq 2$ and let Q be a (k,l)-quasi-arborescence rooted at u. Let L be a list-assignment of Q such that $|L(v)| \geq \max\{1, d_Q(v)-k+1\}$ if v is a leaf and $|L(v)| \geq 2$ otherwise. We denote by S the set of leaves that have indegree at least $k+1$ in Q (and hence a colour-list of size at least 2). Any L-colouring of the leaves extends in an L-colouring of Q such that:*

- *$im(u) \leq k-1$.*
- *$\forall v \notin S, im(v) \leq k$.*

Furthermore, possibly by recolouring some vertices of S, this L-colouring of G can be made k-improper.

Proof. By induction on the number of vertices of Q, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller (k,l)-quasi-arborescences. Let $v_1, \dots, v_s$ be the outneighbours of u in Q. Note that $Q-u$ is the union of s (k,l)-quasi-arborescences Q_i rooted at v_i, $1 \leq i \leq s$, that are disjoint except possibly on their leaves. We start by L-colouring all the leaves of Q.

By induction we extend this colouring to an L-colouring of each of the Q_i such that $im(v_i) \leq k-1$. Note that $im_Q(x) = im_{Q_i}(x) \leq k$ for every vertex of Q_i which is not a leaf and $im_Q(x) \leq k$ for each leaf not in S. One of the two colours of $L(u)$, say α, is assigned to at most $k-1$ neighbours of u since $deg(u) \leq 2k-1$. Hence setting $c(u) = \alpha$, we obtain the first desired colouring.

Now, we can recolour each leaf f of S with improperty at least $k+1$ using Lemma 9 since $d_Q(f) - |L(f)| + 2 \leq d_Q(f) - d_Q(f) + k - 1 + 2 = k+1$. This concludes the proof. □

The above result cannot be extended for $k = 1$. However one can prove the following:

Lemma 12 *Let Q be a $(1,l)$-quasi-arborescence rooted at u, L be a list-assignment of Q such that $|L(v)| \geq 2$ if v is not a leaf, and $|L(v)| \geq d_Q(v)$ otherwise. We denote by S the set of leaves with indegree at least 2. Let c be an L-colouring of the leaves. One of the followings holds:*

(i) c can be extended in an L-colouring of Q such that $im(u) = 0$ and $im(v) \leq 1$ if $v \notin S$;

(ii) c can be extended in two different L-colourings of Q c_1 and c_2 such that $c_1(v) = c_2(v)$ if $v \neq u$ and $im^{c_i}(v) \leq 1$ if $v \notin S$.

Furthermore, possibly by recolouring vertices of S, all these L-colourings can be made 1-improper. Moreover, if $|L(u)| \geq 3$ then (i) holds.

Proof. By induction on the number of vertices, the result being obvious if $|V(Q)| = 1$.

$Q-u$ is the union of two $(1,l)$-quasi-arborescences Q_1 and Q_2 rooted at v_1 and v_2 respectively. They are disjoint except possibly on their leaves. Let c be an L-colouring of the leaves of Q. By induction we extend c to Q_1 and Q_2. Note that for each vertex v of $Q-S$ $im_Q(v) = im_{Q_i}(v) \leq 1$.

If at least one of the Q_i satisfies (ii), or if $|L(u)| \geq 3$, we can suppose that $\{c(v_1), c(v_2)\} \neq L(u)$ and hence we extend c into an L-colouring of Q fulfilling (i).

If both Q_1 and Q_2 satisfy (i), then either $c(v_1) = c(v_2)$ and hence setting $c(u) \in L(u) \setminus \{c(v_1)\}$ yields an L-colouring of Q that satisfies (i); or colouring u with two colours of its list gives the two desired colourings of (ii).

Now we can recolour with improperty zero each leaf $f \in S$ that has improperty at least 2 in Q using Lemma 9, since $d_Q(f) - |L(f)| + 2 \leq 2$. This concludes the proof. □

Using these results, we can say more about the structure of a discharging digraph. The following lemma generalizes Proposition 1.

Lemma 13 *Let D be a discharging digraph of a (k,l)-minimal graph G.*

(i) Every vertex u with $l + k \leq d(u) \leq l + 2k - 1$ has outdegree at most $d(u) - l + 1$. In particular, D is acyclic.

(ii) For every vertex u, $A^+_D(u)$ is a (k,l)-quasi-arborescence. In particular, the indegree of the leaves in $A^+_D(u)$ is at most $l + k - 1$.

Proof. (ii) follows easily from (i). So, let us prove (i).

Let L be an l-list-assignment of G. First, D has no 2-circuit since two $(\leq l+k-1)$-vertices are not adjacent by Lemma 10. Note also that in order to create a circuit in D, it is necessary to create a vertex u of outdegree at least $d(u) - l + 2$. Now suppose, for a contradiction, that D contains a vertex u of outdegree at least $d(u) - l + 2$ and let D' be the digraph obtained just after having created the first such vertex u. Let $u \to v$ be the last edge that is oriented in D'. u has $d(u) - l + 2$ outneighbours (including v) while v has $d(v) - l + 1$ outneighbours. We distinguish two cases depending whether the orientation of uv creates a circuit (which is necessary the first), or not.

First Case: the orientation of uv creates a circuit C. Let w be the inneighbour of u in C. We define $Q_1 = A^+_{D'-wu}(v)$, $Q_2 = A^+_{D'-uv}(u)$ and $Q = Q_1 \cup Q_2$. Note that Q_1 and Q_2 are (k,l)-quasi-arborescences which are disjoint, except possibly on some leaves. In particular the outdegree in D' of every internal vertex x of Q is at most $d_G(x) - l + 1$. More precisely every internal vertex $x \neq w$ satisfies $d^+_{D'}(x) = d_G(x) - l + 1$ while $d^+_{D'}(w) = d_G(w) - l$ and for all every vertex v $d^-_{D'}(x) = 1$. Recall that $d_Q(w) = d^+(w) + d^-(w)$. Let F be the set of leaves in Q, S the set of leaves that have indegree at least $k+1$ in Q and $\bar{S} = F \setminus S$. We define $\mathring{Q} = Q - \bar{S}$. By minimality, let c be a k-improper L-colouring of $G' = G - \mathring{Q}$. Let $f \in S$: if f has improperty at least $k - d_Q(f) + 1$, then using Lemma 9 we recolour it with improperty 0 since $d_{G'}(f) - |L(f)| + 2 = d_G(f) - d_Q(f) - l + 2 \leq l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1$. Now, let L_1 be the following list-assignment of Q_1:
$L_1(x) = L(x) \setminus \{\alpha —\ \exists z \in N_{G-Q_1}(x), c(z) = \alpha\}$ if $x \notin \bar{S}$, and $L_1(x) = \{c(x)\}$ otherwise. Note that if $x \neq w$ is an internal vertex then:

$$|L_1(x)| \geq l - (d_G(x) - d_{Q_1}(x)) = l - d_G(x) + d_G(x) - l + 1 + 1 = 2$$

and since $d^+(w) = d_G(w) - l$ but u is yet uncoloured:

$$|L_1(w)| \geq l - (d_G(w) - d_{Q_1}(w)) + 1 = l - d_G(w) + d_G(w) - l + 1 + 1 = 2.$$

For the root v, $d^-(v) = 0$ but u is uncoloured yet so:

$$|L_1(v)| \geq l - (d_G(v) - d_{Q_1}(v)) + 1 = l - d_G(v) + d_G(v) - l + 1 + 1 = 2,$$

and for a leaf $f \in S$:

$$|L_1(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$

Thus we may apply Lemmas 11 and 12. To do so, we L_1-colour all the leaves in Q.

Suppose first $k \geq 2$. By Lemma 11, we obtain an L_1-colouring c_1 of Q_1 such that $im_{Q_1}^{c_1}(v) \leq k-1$. Note that c_1 extends c into an L-colouring of $G - Q_2$ such that each vertex has improperty at most k except possibly some vertices of S. Furthermore, $im_{G-Q_2}(v) \leq k-1$. We define a list-assignment L_2 of Q_2 by $L_2(u) = L(u) \setminus \{\alpha$ — $\exists z \neq v \in N_{G-Q_2}(u), c(z) = \alpha)\}$, $L_2(x) = \{c(x)\}$ if x is a leaf and $L_2(x) = L(x) \setminus \{\alpha$ — $\exists z \in N_{G-Q_2}(x), c(z) = \alpha\}$ otherwise. Note that we have $|L_2(u)| \geq 2$. We now apply Lemma 11 so as to get an L_2-colouring of Q_2 and hence an L-colouring of G. Every vertex not in $S \cup \{u, v\}$ has improperty at most k. If $x \in \{u, v\}$ then: $im_G(x) \leq im_{G-Q_i}(x)+1 \leq k-1+1 = k$ since there cannot be in $L_2(u)$ the colour of a neighbour of u in $G - (Q_2 - v)$. If $f \in S$ has improperty at least $k+1$, then we recolour it with improperty 0 using Lemma 9 since $d_G(f) - |L(f)| + 2 \leq l + k - 1 - l + 2 = k + 1$. Thus we obtain a k-improper L-colouring of G.

Suppose now $k = 1$. Applying Lemma 12, we obtain an L_1-colouring of $G - Q_2$ such that every vertex not in S has improperty at most 1, and either v has improperty 0 (*i*), or it has improperty 1 and we can indifferently colour it with two colours of its list (*ii*). Note that if v has one neighbour distinct from u which is an internal vertex in Q_2 then $|L_1(v)| \geq 3$ so we may suppose that v fulfils (*i*). Defining L_2 as before, we can apply Lemma 12 to Q_2 so as to obtain an L_2-colouring of Q_2 and hence an L-colouring of G such that u fulfils (*i*) or (*ii*). Now, every vertex not in $S \cup \{u, v\}$ has improperty at most 1. If v satisfies (*i*), then either u also satisfies (*i*) or u satisfies (*ii*) but in this case we may suppose u and v are coloured differently so in all cases they have improperty at most 1 in G. If v satisfies (*ii*), then the only neighbour of v in Q_2 is u. Hence we may safely suppose that u and v are coloured differently, so they have improperty at most 1 in G.

Finally, we can recolour each leaf of S that has improperty at least 2 by using Lemma 9 and thus we obtain a 1-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

Second Case: there is no circuit in D'. Then $Q = A^+_{D'}(u)$ is a quasi-arborescence. Moreover each internal vertex v has outdegree at most (and hence exactly) $d(v) - l + 1$. Let $v_1, \ldots, v_s$ be the outneighbours of u, we define $Q_j = A^+_{D'}(v_j)$, $1 \leq j \leq s$. The Q_i are (k, l)-quasi-arborescences that are disjoint except possibly on their leaves. Let F be the set of leaves in Q, S the set of leaves with indegree at least $k+1$ in Q and $\bar{S} = F \setminus S$. We define $\mathring{Q} = Q - \bar{S}$. Let L be an l-list-assignment of G. By minimality, let c be a k-improper L-colouring of $G' = G - \mathring{Q}$. Let f be a leaf in $\bar{S}$. If f has improperty at least $k - d_Q(f) + 1$, we recolour it with improperty 0 using Lemma 9 since: $d_{G'}(f) - |L(f)| + 2 \leq d_G(f) - d_Q(f) - l + 2 \leq l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1.$

For each vertex $v \in Q$, we define $L'(v) = L(v) \setminus \{\alpha — \ \exists w \in N_G(v), c(w) = \alpha\}$ if $v \notin \bar{S}$ and $L'(v) = \{c(v)\}$ otherwise. Note that for an internal vertex v:

$$|L'(v)| \geq l - (d_G(v) - d_Q(v)) = l - d_G(v) + d_G(v) - l + 1 + 1 = 2.$$

For a leaf $f \in S$:

$$|L'(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$

Suppose first $k \geq 2$. We L'-colour all the leaves, use Lemma 11 so as to extend it into an L'-colouring of each of the Q_i, and possibly with recolouring some leaves in S we get a k-improper L-colouring of $G - u$ such that $im(v_j) \leq k - 1$, $1 \leq j \leq s$.

Now $|L'(u)| \geq |L(u)| - d(u) + d^+_{D'}(u) = l - d(u) + d(u) - l + 2 \geq 2$. And u has $d^+(u) = d(u) - l + 2 \leq 2k + 1$ outneighbours in D'. Thus there is a colour of $L'(u)$, say α, that is assigned to at most k outneighbours of u. Thus setting $c(u) = \alpha$ yields a k-improper L-colouring of G by definition of L'.

Suppose now $k = 1$. We L'-colour all the leaves, use Lemma 12 so as to extend it in an L'-colouring of each of the Q_i, and possibly with recolouring some leaves in S we get a 1-improper L-colouring of $G - u$ such that for each v_j either $im(v_j) = 0$ or v_j can safely be recoloured with another colour of $L'(v_j)$.

The same calculation as above shows there exists a colour of $L'(u)$, say α, that is assigned to at most 1 neighbour of u, say v_i. We set $c(u) = \alpha$. If v_i satisfies the first condition, we have a 1-improper L-colouring of G. If v_i satisfies the second condition then we may suppose that $c(u) \neq c(v)$ and thus we also have a 1-improper L-colouring of G.

Hence G is k-improper L-choosable, a contradiction

□

Proof of Theorem 4. Let G be a (k,l)-minimal graph and D a discharging digraph of G. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{l+k}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex v:

- If $d(v) \leq l + k - 1$ it has indegree $d(v)$ so its new charge is $w'(v) = d(v) + \frac{d(v)k}{l+k} \geq l + \frac{lk}{l+k}$.
- If $l + k \leq d(v) < l + k + \frac{k}{l}$ then either v has outdegree at most $d(v) - l$ and so its new charge is at least $d(v) - (d(v) - l) \times \frac{k}{l+k} = \frac{ld(v)}{l+k} + \frac{lk}{l+k} \geq l + \frac{lk}{l+k}$, or by Lemma 13, it has outdegree $d(v) - l + 1$. In this case, by definition of a discharging digraph, v has indegree 1 so its new charge is:

$$w'(v) = d(v) - (d(v) - l + 1) \times \frac{k}{l+k} + \frac{k}{l+k} = d(v) - (d(v) - l) \times \frac{k}{l+k} \geq l + \frac{k}{l+k}.$$

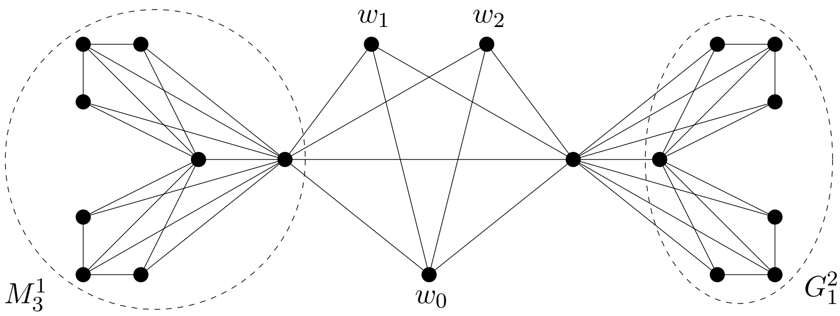
- If $l + k + \frac{k}{l} \leq d(v) \leq l + 2k - 1$, then by Lemma 13, v has outdegree at most $d(v) - l + 1$. So $w'(v) \geq d(v) - (d(v) - l + 1) \times \frac{k}{l+k} = \frac{ld(v)}{l+k} + \frac{kl-k}{l+k} \geq \frac{l^2+2kl}{l+k} = l + \frac{kl}{l+k}$.
- If $d(v) \geq l + 2k$, then $w'(v) \geq d(v)(1 - \frac{k}{l+k}) = \frac{ld(v)}{l+k} \geq \frac{l^2+2kl}{l+k} = l + \frac{kl}{l+k}$.

Hence $Mad(G) \geq \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{1}{|V|}\sum_{v\in V} w'(v) \geq l + \frac{kl}{l+k}$.

□

3.2 Upper bound for $M(k,l)$

In this section we shall construct for all $l \geq 2$ and all $k \geq 1$, a graph G_l^k which is not k-improper l-colourable. So its maximum average degree will give an upper bound for $M(k,l)$. To construct G_2^k, take $k+1$ copies of H_k (defined in Subsection 2.2) and identify their vertex v. We define G_l^k, $l \geq 3$, inductively. First we create the graph M_l^k by taking k copies of G_{l-1}^k and adding a vertex w which we join to every other vertices. Then we take $l-1$ copies $M^1, \dots, M^{l-1}$ of M_l^k and we join all the vertices $w_1, \dots, w_{l-1}$ (so that they form a complete graph of size $l-1$). Now, we add $k+2$ vertices $z_0, z_1, \dots, z_{k+1}$ each joined to each of the w_i, $1 \leq i \leq l-1$. Last we add the edges $z_0 z_i$ for $1 \leq i \leq k+1$.



The graph G_3^1

Lemma 14 *For all $l \geq 2$ and all $k \geq 1$, the graph G_l^k is not k-improper l-colourable.*

Proof. The result is clear for G_2^k. Suppose the result is true for $l-1 \geq 2$ and let us prove it for G_l^k. First note that in any k-improper l-colouring of M^i, the vertex w_i has improperty k. Indeed, w_i has a neighbour of its colour in each copy of G_{l-1}^k since otherwise G_{l-1}^k would be k-improper $(l-1)$-colourable. Hence each of the w_i, $1 \leq i \leq l-1$, cannot have any neighbour of its colour in $G_l^k - M^i$. In particular, as the subgraph induced by $w_1, \dots w_{l-1}$ is complete, all the z_i, $0 \leq i \leq k+1$, must be coloured the same. But then w_0 must have improperty $k+1$. □

Lemma 15 *The number of vertices of G_l^k is:*

$$n_l^k = 2l + (l+1)k + \sum_{i=2}^{l} \frac{(l-1)!}{(l-i)!} k^i.$$

In particular, it is a polynom in k of degree l and dominant coefficient $(l-1)!$.

Proof. n_l^k satisfies: $n_2^k = k^2 + 3k + 3$ and $\forall l \geq 3, n_l^k = (k \times n_{l-1}^k + 1) \times (l-1) + k + 2$. □

Let s_l^k denotes the sum of the degrees of the vertices in G_l^k. We have the following result:

Lemma 16 *s_l^k is a polynom in k of degree l whose dominant coefficient is $2l!$.*

Proof. s_l^k satisfies: $s_2^k = 4k^2+10k+6$ and $s_l^k = (l-1)(k \times s_{l-1}^k + 2k \times n_{l-1}^k + l + k) + (l+1)k + 2l$ if $l \geq 3$. Hence it is a polynom in k of degree l. Furthermore, denoting by c_l^k its dominant coefficient, we have: $c_2^k = 4$ and $\forall l \geq 3, c_l^k = (l-1) \times c_{l-1}^k + 2k \times (l-1)!$. Thus $c_l^k = 2l!$. □

Proposition 2 $\lim_{k \to \infty} Mad(G_l^k) = 2l$.

Proof. It is clear that the maximum average degree of G_l^k is its average degree. Then by Lemmas 15 and 16, we have:

$$\lim_{k \to \infty} Mad(G_l^k) = 2\frac{l!}{(l-1)!} = 2l.$$

□

Corollary 2 *For any fixed l, $\lim_{k \to +\infty} M(k,l) = 2l$.*

Proof. It follows from Theorem 4 and Proposition 2. □

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