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ARC-CHROMATIC NUMBER OF DIGRAPHS IN WHICH EVERY VERTEX HAS BOUNDED OUTDEGREE OR BOUNDED INDEGREE

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Abstract

A *k-digraph* is a digraph in which every vertex has outdegree at most k. A $(k \vee l)$-*digraph* is a digraph in which a vertex has either outdegree at most k or indegree at most l. Motivated by function theory, we study the maximum value $\Phi(k)$ (resp. $\Phi^{\vee}(k,l)$) of the arc-chromatic number over the k-digraphs (resp. $(k \vee l)$-digraphs). El-Sahili [3] showed that $\Phi^{\vee}(k,k) \leq 2k+1$. After giving a simple proof of this result, we show some better bounds. We show $\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k) \leq \theta(2k)$ and $\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^{\vee}(k,l) \leq \theta(2k+2l)$ where θ is the function defined by $\theta(k) = \min\{s : \binom{s}{\lceil \frac{s}{2} \rceil} \geq k\}$. We then study in more details properties of Φ and $\Phi^{\vee}$. Finally, we give the exact values of $\Phi(k)$ and $\Phi^{\vee}(k,l)$ for $l \leq k \leq 3$.

1 Introduction

A *directed graph* or *digraph* D is a pair $(V(D), E(D))$ of disjoint sets (of *vertices* and *arcs*) together with two maps $tail : E(D) \rightarrow V(D)$ and $head : E(D) \rightarrow V(D)$ assigning to every arc e a *tail*, $tail(e)$, and a *head*, $head(e)$. The tail and the head of an arc are its *ends*. An arc with tail u and head v is denoted by uv; we say that u *dominates* v and write $u \rightarrow v$. We also say that u and v are adjacent. The *order* of a digraph is its number of vertices. In this paper, all the digraph we consider are *loopless*, that is that every arc has its tail distinct from its head.

Let D be a digraph. The *line-digraph* of D is the digraph $L(D)$ such that $V(L(D)) = E(D)$ and an arc $a \in E(D)$ dominates an arc $b \in E(D)$ in $L(D)$ if and only if $head(a) = tail(b)$.

A *vertex-colouring* or *colouring* of D is an application c from the vertex-set $V(D)$ into a set of colours S such that for any arc uv, $c(u) \neq c(v)$. The *chromatic number* of D, denoted $\chi(D)$, is the minimum number of colours of a colouring of D.
An *arc-colouring* of D is an application c from the arc-set $E(D)$ into a set of colours S such that if the tail of an arc e is the head of an arc e' then $c(e) \neq c(e')$. Trivially, there is a one-to-one correspondence between arc-colourings of D and colourings of $L(D)$. The *arc-chromatic number* of D, denoted $\chi_a(D)$, is the minimum number of colours of an arc-colouring of D. Clearly $\chi_a(D) = \chi(L(D))$.

A *k-digraph* is a digraph in which every vertex has outdegree at most k. A *$(k \vee l)$-digraph* is a digraph in which a vertex has either outdegree at most k or indegree at most l.

For any digraph D and set of vertices $V' \subset V(D)$, we denote by $D[V']$, the subdigraph of D induced by the vertices of V'. For any subdigraph F of D, we denote by $D - F$ the digraph $D[(V(D) \setminus V(F)]$. For any arc-set $E' \subset E$, we denote by $D - E'$ the digraph $(V(D), E(D) \setminus E')$ and for any vertex $x \in V(D)$, we denote by $D - x$ the digraph induced by $V(D) \setminus \{x\}$.

Let D be a $(k \vee l)$-digraph. We denote by $V^+(D)$, or V^+ if D is clearly understood, the subset of the vertices of D with outdegree at most k, and by $V^-(D)$ or V^- the complementary of $V^+(D)$ in $V(D)$. Also D^+ (resp. D^-) denotes $D[V^+]$ (resp. $D[V^-]$).

In this paper, we study the arc-chromatic number of k-digraphs and $(k \vee l)$-digraphs. This is motivated by the following interpretation in function theory as shown by El-Sahili in [3].

Let f and g be two maps from a finite set A into a set B. Suppose that f and g are *nowhere coinciding*, that is for all $a \in A$, $f(a) \neq g(a)$. A subset A' of A is *(f, g)-independant* if $f(A') \cap g(A') = \emptyset$. We are interested by finding the largest (f, g)-independant subset of A and the minimum number of (f, g)-independant subsets to partition A. As shown by El-Sahili [3], this can be translated into an arc-colouring problem.

Let $D_{f,g}$ and $H_{f,g}$ be the digraphs defined as follows :

- $V(D_{f,g}) = B$ and $(b, b') \in E(D_{f,g})$ if there exists an element a in A such that $g(a) = b$ and $f(a) = b'$. Note that if for all a, $f(a) \neq g(a)$, then $D_{f,g}$ has no loop.
- $V(H_{f,g}) = A$ and $(a, a') \in E(H_{f,g})$ if $f(a) = g(a')$.

We associate to each arc (b, b') in $D_{f,g}$ the vertex a of A such that $g(a) = b$ and $f(a) = b'$. Then (a, a') is an arc in $H_{f,g}$ if, and only if, $head(a) = tail(a')$ (as arcs in $D_{f,g}$). Thus $H_{f,g} = L(D_{f,g})$. Note that for every digraph D, there exists maps f and g such that $D = D_{f,g}$.

It is easy to see that an (f, g)-independant subset of A is an independant set in $H_{f,g}$. In [2] El-Sahili proved the following :

Theorem 1 (El-Sahili [2]) *Let f and g be two nowhere coinciding maps from a finite set A into a set B. Then there exists an (f, g)-independant subset A' of cardinality at least $|A|/4$.*

Let f and g be two nowhere coinciding maps from a finite set A into B. We define $\phi(f, g)$ as the minimum number of (f, g)-independant sets to partition A. Then $\phi(f, g) = \chi(H_{f,g}) = \chi_a(D_{f,g})$.

Let $\Phi(k)$ (resp. $\Phi^\vee(k, l)$) be the maximum value of $\phi(f, g)$ for two nowhere coinciding maps f and g from A into B such that for every z in B, $g^{-1}(z) \leq k$ (resp. either $g^{-1}(z) \leq k$ or

$f^{-1}(z) \leq l$). The condition $f^{-1}(z)$ (resp. $g^{-1}(z)$) has at most k elements means that each vertex has indegree (resp. outdegree) at most k in $D_{f,g}$. Hence $\Phi(k)$ (resp. $\Phi^\vee(k,l)$) is the maximum value of $\chi_a(D)$ for D a k-digraph (resp. $(k \vee l)$-digraph).

Remark 2 Let f and g be two nowhere coinciding maps from A into B. Then A may be partitionned into $\Phi(|A|-1)$ (f,g)-independant sets.

The functions $\Phi^\vee$ and Φ are very close to each other:

Proposition 3

$$\Phi(k) \leq \Phi^\vee(k,0) \leq \cdots \leq \Phi^\vee(k,k) \leq \Phi(k)+2$$

Proof. The sole inequality that does not immediatly follow the definitions is $\Phi^\vee(k,k) \leq \Phi(k)+2$. Let us prove it.
Let D be a $(k \vee k)$-digraph. One can colour the arcs in $D^+ \cup D^-$ with $\Phi(k)$ colours. It remains to colour the arcs with tail in V^- and head in V^+ with one new colour and the arcs with tail in V^+ and head in V^- with a second new colour. □

Moreover, we conjecture that $\Phi^\vee(k,k)$ is never equal to $\Phi(k)+2$.

Conjecture 4

$$\Phi^\vee(k,k) \leq \Phi(k)+1$$

In [3], El-Sahili gave the following upper bound on $\Phi^\vee(k,k)$:

Theorem 5 (El-Sahili [3]) $\Phi^\vee(k,k) \leq 2k+1$

In this paper, we first give simple proofs of Theorems 1 and 5. Then, in Section 3, we improve the upper bounds on $\Phi(k)$ and $\Phi^\vee(k,l)$. We show (Theorem 18) that $\Phi(k) \leq \theta(2k)$ if $k \geq 2$, and $\Phi^\vee(k,l) \leq \theta(2k+2l)$ if $k+l \geq 3$, where θ is the function defined by $\theta(k) = \min\{s : \binom{s}{\lceil s/2 \rceil} \geq k\}$. Since $2^s/s \leq \binom{s}{\lceil s/2 \rceil} \leq 2^s/\sqrt{s}$ for $s \geq 2$, once can obtain the following equivalent for θ as $k \to \infty$:

$$\theta(k) = \log(k) + \Theta(\log(\log(k)))$$

Lower bounds for Φ and $\Phi^\vee$ are stated by Corollaries 14 and 15: $\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k)$ and $\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^\vee(k,l)$.
We also establish (Corollary 21) that $\Phi^\vee(k,l) \leq \theta(2k)$ if $\theta(2k) \geq 2l+1$.

In Section 4, we study in more details the relations between $\Phi^\vee(k,l)$ and $\Phi(k)$. We conjecture that if k is very large compared to l then $\Phi^\vee(k,l) = \Phi(k)$. We prove that $\Phi^\vee(k,0) = \Phi(k)$ and conjecture that $\Phi^\vee(k,1) = \Phi(k)$ if $k \geq 1$. We prove that for a fixed k either this latter conjecture holds or Conjecture 4 holds. This implies that $\Phi^\vee(k,1) \leq \Phi(k)+1$.

Finally, in Section 5, we give the exact values of $\Phi(k)$ and $\Phi^\vee(k,l)$ for $l \leq k \leq 3$. They are summarized in the following table :

$\Phi^\vee(0,0)=1$	$\Phi^\vee(1,0)=\Phi(1)=3$	$\Phi^\vee(2,0)=\Phi(2)=4$	$\Phi^\vee(3,0)=\Phi(3)=4$
	$\Phi^\vee(1,1)=3$	$\Phi^\vee(2,1)=4$	$\Phi^\vee(3,1)=4$
		$\Phi^\vee(2,2)=4$	$\Phi^\vee(3,2)=5$
			$\Phi^\vee(3,3)=5$

2 Simple proofs of Theorems 1 and 5

Proof of Theorem 1. Let $D = D_{f,g}$. Let (V_1, V_2) be a partition of $V(D)$ that maximizes the number of arcs a with one end in V_1 and one end in V_2. It is well-known that $a \geq |E(D)|/2$. Now let A_1 be the set of arcs with head in V_1 and tail in V_2 and A_2 be the set of arcs with head in V_2 and tail in V_1. Then A_1 and A_2 corresponds to independant sets of $L(D)$ and $|A_1| + |A_2| = a$. Hence one of the A_i has cardinality at least $a/2 = \frac{|E(D)|}{4}$. □

Before giving a short proof of Theorem 5, we precise few standard definitions.

Definition 6 A *path* is a non-empty digraph P of the form

$$V(P) = \{v_0, v_1, \ldots, v_k\} \qquad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\},$$

where the v_i are all distinct. The vertices v_0 and v_k are respectively called the *origin* and *terminus* of P.

A *circuit* is a non-empty digraph C of the form

$$V(C) = \{v_0, v_1, \ldots, v_k\} \qquad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv_0\},$$

where the v_i are all distinct.

A digraph is *strongly connected* or *strong* if for every two vertices u and v there is a path with origin u and terminus v. A maximal strong subdigraph of a digraph D is called a *strong component* of D. A component I of D is *initial* if there is no arc with tail in $V(D) \setminus V(I)$ and head in $V(I)$. A component I of D is *terminal* if there is no arc with tail in $V(I)$ and head in $V(D) \setminus V(I)$. A digraph is *connected* if its underlying graph is connected.

A digraph D is *l-degenerate* if every subdigraph H has a vertex of degree at most l.

The following lemma corresponding to the greedy colouring algorithm is a piece of folklore.

Lemma 7 *Every l-degenerate digraph is $(l+1)$-colourable.*

Proof of Theorem 5. Let D be a $(k \vee k)$-digraph. Acording to Lemma 7, it suffices to prove that $L(D)$ is $2k$-degenerate.

In every initial strong component C there is a vertex with indegree at most k. Indeed if there is no such vertex then $(k+1)|C| \leq \sum_{v \in C} d^-(v) \leq \sum_{v \in C} d^+(v) \leq k|C|$. Analogously, in every terminal strong component there is a vertex with outdegree at most k.

Now, there is a path originating in a minimal component and terminating in a terminal one. Hence there is a path whose origin has indegree at most k and whose terminus has outdegree at most k. Hence there is an arc e whose tail has indegree at most k and whose head has outdegree at most k. Thus e has degree at most $2k$ in $L(D)$. □

3 Lower and upper bounds for Φ and $\Phi^\vee$

We will now search for bounds on Φ since they also give bounds on $\Phi^\vee$.

Theorem 1 and an easy induction yields $\chi_a(D) \leq \log_{4/3} |D|$. However there exists better upper bounds stated by Poljak and Rödl [4]. For sake of completeness and in order to introduce useful tools, we provide a proof of Theorem 11.

Definition 8 We denote by $\overline{H}_k$ the complementary of the hypercube of dimension k, that is the digraph with vertex-set all the subsets of $\{1,\ldots,k\}$ and with arc-set $\{xy : x \not\subset y\}$.

A *homomorphism* $h : D \to D'$ is a mapping $h : V(D) \to V(D')$ such that for every arc xy of D, $h(x)h(y)$ is an arc of D'.

Let c be an arc-colouring of a digraph D into a set of colours S. For any vertex x of D, we denote by $Col_c^+(x)$ or simply $Col^+(x)$ the set of colours assigned to the arcs with tail x. We define $Col^-(x) = S \setminus Col^+(x)$. Note that $Col^-(x)$ contains (but may be bigger than) the set of colours assigned to the arcs with head x. The cardinality of $Col^+(x)$ (resp. $Col^-(x)$) is denoted by $col^+(x)$ (resp. $col^-(x)$).

Theorem 9 *For every digraph D, $\chi_a(D) = \min\{k : D \to \overline{H}_k\}$.*

Proof. Assume that D admits an arc-colouring with $\{1,\ldots,k\}$. It is easy to check that Col^+ is a homomorphism from D to $\overline{H}_k$.
Conversely, suppose that there exists a homomorphism h from D to $\overline{H}_k$. Assign to each arc xy an element of $h(y) \setminus h(x)$, which is not empty. This provides an arc-colouring of D. □

Definition 10 The *complete digraph of order n*, denoted $\vec{K}_n$, is the digraph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and arc-set $\{v_iv_j : i \neq j\}$.

The *transitive tournament of order n*, denoted TT_n, is the digraph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and arc-set $\{v_iv_j : i < j\}$.

The following corollary of Theorem 9 provides bounds on the arc-chromatic number of a digraph according to its chromatic number.

Theorem 11 (Poljak and Rödl [4]) *For every digraph D,*

$$\lceil log(\chi(D)) \rceil \leq \chi_a(D) \leq \theta(\chi(D)).$$

Proof. By definition of the chromatic number, $D \to \vec{K}_{\chi(D)}$. As the subsets of $\{1,\ldots,k\}$ with cardinality $\lceil \frac{k}{2} \rceil$ induce a complete digraph on $\binom{k}{\lceil \frac{k}{2} \rceil}$ vertices in $\overline{H}_k$, we obtain a homomorphism from D to $\overline{H}_{\theta(\chi(D))}$. So $\chi_a(D) \leq \theta(\chi(D))$.

By Theorem 9, we have $D \to \overline{H}_{\chi_a(D)}$. As $\chi(\overline{H}_{\chi_a(D)}) = 2^{\chi_a(D)}$, we obtain $D \to \vec{K}_{2^{\chi_a(D)}}$. □

These bounds are tight since the lower one is achieved by transitive tournaments and the upper one by complete digraphs by Sperner's Lemma (see [5]). However, the lower bound may be increased if the digraph has no sink (vertex with outdegree 0) or/and no source (vertex with indegree 0).

Theorem 12 *Let D be a digraph.*

(i) If D has no sink then $\log(\chi(D) + 1) \leq \chi_a(D)$.

(ii) If D has no source and no sink then $\log(\chi(D) + 2) \leq \chi_a(D)$.

Proof. The proof is identical to the proof of Theorem 11. But if a digraph has no source (resp. no sink) then for every v, $Col^+(v) \neq S$ (resp. $Col^+(v) \neq \emptyset$). □

Again, these two lower bounds are also tight. Let Q_n (resp. W_n) be the tournament of order n obtained from TT_n by reversing the arc $v_1 v_n$ (resp. $v_2 v_n$). One can easily check that $\chi_a(W_n) = \lceil \log(n+1) \rceil = \lceil \log(\chi(W_n)+1) \rceil$ and $\chi_a(Q_n) = \lceil \log(n+2) \rceil = \lceil \log(\chi(Q_n)+2) \rceil$.

Proposition 13 *Every k-digraph is $2k$-degenerate.*

Proof. Every subdigraph of a k-digraph is also a k-digraph. Hence it suffices to prove that every k-digraph has a vertex with degree at most $2k$. Since the sum of outdegrees equals the sum of indegrees, there is a vertex with indegree at most k and thus with degree at most $2k$. □

Corollary 14

$$\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k) \leq \theta(2k+1)$$

Proof. The upper bound follows from Proposition 13, Lemma 7 and Theorem 11. The lower bound comes from a regular tournament on $2k+1$ vertices T_{2k+1} and the complete digraph on $k+1$ vertices $\vec{K}_{k+1}$. Indeed $\chi(T_{2k+1}) = 2k+1$, so $\chi_a(T_{2k+1}) \geq \log(2k+3)$ by Theorem 12 and $\chi_a(\vec{K}_{k+1}) = \theta(k+1)$. □

Corollary 15

$$\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^{\vee}(k,l) \leq \theta(2k+2l+2)$$

Proof. The upper bound follows Proposition 13 and Theorem 11 since every $(k \vee l)$-digraph D is $2k+2l+2$-colourable (D^+ is $2k$-degenerate and so $(2k+1)$-colourable and D^- is $2l$-degnerate and so $(2l+1)$-colourable). The lower bound comes from $\vec{K}_{k+1}$, $\vec{K}_{l+1}$ and a tournament T composed of a regular tournament on $2l+1$ vertices dominating a regular tournament on $2k+1$ vertices. Indeed, $\chi_a(\vec{K}_{k+1}) = \theta(k+1)$, $\chi_a(\vec{K}_{l+1}) = \theta(l+1)$ and T has no source, no sink and chromatic number $2k+2l+2$, so, by Theorem 12, $\chi_a(T) \geq \log(2k+2l+4)$. □

We can obtain a slightly better upper bound on Φ. Bounds on $\Phi^{\vee}$ follow.

Definition 16 For any integer $k \geq 1$, let T_k^+ $(k \geq 1)$ be the complete digraph on $t_1^+, \ldots, t_{2k+1}^+$ minus the arcs $\{t_1^+ t_2^+, t_1^+ t_3^+, \ldots, t_1^+ t_{k+1}^+\}$.

Lemma 17 *Let $k \geq 1$ be an integer. If D is a k-digraph, then there exists a homomorphism h^+ from D to T_k^+ such that if $h^+(x) = t_1^+$ then $d^+(x) = k$.*

Proof. For a fixed k, we prove it by induction on $|V(D)|$.

First, suppose that there exists in D a vertex x with $d^-(x) < k$. Then, $d^+(x) + d^-(x) < 2k$. By induction on $D - x$, there is a homomorphism h from $D - x$ to T_k^+ such that if $h^+(v) = t_1^+$ then $d^+_{D-x}(v) = k$. Hence, $h^+(y) \neq t_1^+$ for every inneighbour y of x, because $d^+_{D-x}(y) < k$. As x has at most $2k-1$ neighbours, we find $i \in \{2, \ldots, 2k+1\}$ such that no neighbour y of x satisfies $h^+(y) = t_i^+$. So, $h^+(x) = t_i^+$ extends h^+ to a homomorphism from D to T_k^+.

Suppose now that every vertex v of D has indegree at least k. Since $\sum_{v\in V(D)} d^-(v) = \sum_{v\in V(D)} d^+(v) \leq k|V(D)|$, every vertex has indegree and outdegree k. Hence, by Brooks Theorem (see [1]) either D is $2k$-colourable and $D \to T_k^+[\{t_2^+,\dots,t_{2k+1}^+\}]$, or D is a regular tournament on $2k+1$ vertices. In this latter case, label the vertices of D with $v_1, v_2, \dots, v_{2k+1}$ such that $N^-(v_1) = \{v_2,\dots,v_{k+1}\}$. Then h^+ defined by $h^+(v_i) = t_i^+$ is the desired homomorphism. □

Theorem 18 *Let k and l be two positive integers.*

(i) If $k \geq 2$, then $\Phi(k) \leq \theta(2k)$.

(ii) If $k+l \geq 3$, then $\Phi^\vee(k,l) \leq \theta(2k+2l)$.

Proof. *(i)* If $k=2$, the result follows Corollary 14 since $\theta(4)=\theta(5)=4$. Suppose now that $k \geq 3$. Let D be a k-digraph. By Lemma 17 there is a homomorphism from D to T_k^+. We will provide a homomorphism g from T_k^+ to $\overline{H}_{\theta(2k)}$.
Fix $S_1,\dots,S_{2k}$, $2k$ subsets of $\{1,\dots,\theta(2k)\}$ with cardinality $\lfloor\theta(2k)/2\rfloor$ and S a subset of $\{1,\dots,2k\}$ with cardinality $\lfloor\theta(2k)/2\rfloor - 1$. Without loss of generality, the S_i containing S are $S_1,\dots,S_l$ with $l \leq \lceil\theta(2k)/2\rceil + 1 \leq k$. (One can easily check that $\theta(2k)/2+1 \leq k$ provided that $k \geq 3$.) Now, set $g(t_1^+) = S$ and $g(t_{i+1}^+) = S_i$ for $1 \leq i \leq 2k$. It is straightforward to check that g is a homomorphism.

(ii) Let D be a $(k \vee l)$-digraph. By Lemma 17, there exists a homomorphism h^+ from D^+ to T_k^+ such that if $h^+(x) = t_1^+$ then $d^+(x) = k$ and, by symmetry , a homomorphism h^- from D^- to T_l^-, the complete digraph on $\{t_1^-,\dots,t_{2l+1}^-\}$ minus the arcs $\{t_2^-t_1^-, t_3^-t_1^-, \dots t_{l+1}^-t_1^-\}$, such that if $h^-(x) = t_1^-$ then $d^-(x) = l$. We now provide a homomorphism from D to $\overline{H}_{\theta(2k+2l)}$.
Fix S^+ and S^-, two subsets of $\{1,\dots,\theta(2k+2l)\}$ with cardinality $\lfloor\theta(2k+2l)/2\rfloor - 1$ for S^+ and $\lfloor\theta(2k+2l)/2\rfloor + 1$ for S^- such that $S^+ \not\subset S^-$. (This is possible since $\theta(2k+2l) \geq 4$, because $k+l \geq 2$.) Set $\mathcal{N} = \{U \subset \{1,\dots,\theta(2k+2l)\} : |U| = \lfloor\theta(2k+2l)/2\rfloor\}$. We want a partition of $\mathcal{N}$ into two parts $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| \geq 2k$ and $|\mathcal{B}| \geq 2l$, such that S^+ is included in at most k sets of $\mathcal{A}$ and S^- contains at most l sets of $\mathcal{B}$. Let $\mathcal{N}_{S^+}$ (resp. $\mathcal{N}_{S^-}$) be the set of elements of $\mathcal{N}$ containing S^+ (resp. contained in S^-). We have $|\mathcal{N}_{S^+}| = \lceil\theta(2k+2l)/2\rceil + 1$ and $|\mathcal{N}_{S^-}| = \lfloor\theta(2k+2l)/2\rfloor + 1$; because $k+l \geq 3$, it follows $|\mathcal{N}_{S^+}| \leq k+l$ and $|\mathcal{N}_{S^-}| \leq k+l$. Moreover, the sets $\mathcal{N}_{S^+}$ and $\mathcal{N}_{S^-}$ are disjoint. Let us sort the elements of $\mathcal{N}$ beginning with those of $\mathcal{N}_{S^-}$ and ending with those of $\mathcal{N}_{S^+}$. Let $\mathcal{A}$ be the $2k$ first sets in this sorting and $\mathcal{B}$ what remains ($|\mathcal{B}| \geq 2l$). We claim that $\mathcal{A}$ contains at most k elements of $\mathcal{N}_S$. If not, then $|\mathcal{A}| > \binom{\theta(2k+2l)}{\lfloor\theta(2k+2l)/2\rfloor} - |\mathcal{N}_{S^+}| + k$. We obtain $2k > 2k+2l - |\mathcal{N}_{S^+}| + k$ which contradicts $|\mathcal{N}_{S^+}| \leq k+l$. With same argument, $\mathcal{B}$ contains at most l elements of $\mathcal{N}_{S^-}$.
Finally, set $\mathcal{A} = \{A_1,\dots,A_{2k}\}$ such that none of $A_{k+1},\dots,A_{2k}$ contains S^+ and $\{B_1,\dots,B_{2l}\}$ $2l$ sets of $\mathcal{B}$ such that none of $B_{l+1},\dots,B_{2l}$ is contained in S^-.

Let us define $h : D \to \overline{H}_{2k+2l}$. If $x \in V^+$ and $h^+(x) = t_i^+$ then $h(x) = S^+$ if $i=1$ and $h(x) = A_{i-1}$ otherwise. If $x \in V^-$ and $h^-(x) = t_i^-$ the $h(x) = S^-$ if $i=1$ and $h(x) = B_{i-1}$ otherwise. Let us check that h is a homomorphism. Let xy be an arc of D. T_k^+ is a subdigraph of $\overline{H}_{2k+2l}[\{A_1,\dots,A_{2k},S\}]$ and T_l^- is a subdigraph of $\overline{H}_{2k+2l}[\{B_1,\dots,B_{2l},S^-\}]$. So, $h(x)h(y)$ is an arc of $\overline{H}_{2k+2l}$ if x and y are both in V^+ or both in V^-. Suppose now that $x \in V^+$ and $y \in V^-$, then $h^+(x) \neq t_1^+$ because $d_{D^+}(x) < k$ and $h(x) \neq S^+$. Similarly, $h^-(y) \neq S^-$. Thus $h(x)$ and $h(y)$ are elements of $\mathcal{N}$, so $h(x)h(y) \in E(\overline{H}_{2k+2l})$. Finally, suppose that $x \in V^-$ and

$y \in V^+$. Then $h(x)h(y) \in E(\overline{H}_{2k+2l})$ because no element of $\{B_1, \dots, B_{2l}, S^-\}$ is a subset of an element of $\{A_1, \dots, A_{2k}, S^+\}$. □

Remark 19 Note that the homomorphism provided in *(i)* has for image subsets of $\{1, \dots, \theta(2k)\}$ with cardinality at most $\lfloor \theta(2k)/2 \rfloor$. So, using the method developped in Theorem 9, we provide an arc-colouring of a k-digraph D with $\theta(2k)$ colours which satisfies $col^+(x) \leq \lfloor \theta(2k)/2 \rfloor$, so $col^-(x) \geq \lceil \theta(2k)/2 \rceil$, for every vertex x of D.

We will now improve the bound *(ii)* of Theorem 18 when l is very small compared to k.

Lemma 20 *Let D be a $(k \vee l)$-digraph and D^1 the subdigraph of D induced by the arcs with tail in V^+. If there exists an arc-colouring of D^1 with $m \geq 2l+1$ colours such that for every x in V^+, $col^-(x) \geq l+1$ then $\chi_a(D) = m$.*

Proof. We will extend the colouring as stated into an arc-colouring of D.

First, we extend this colouring to the arcs of D^-. Since $\sum_{v \in V^-} d^+_{D^-}(v) = \sum_{v \in V^-} d^-_{D^-}(v) \leq l|V^-|$, there is a vertex $v_1 \in V^-$ such that $d^+_{D^-}(v) \leq l$. And so on, by induction, there is an ordering $(v_1, v_2, \dots, v_p)$ of the vertices of D^- such that, for every $1 \leq i \leq p$, v_i dominates at most l vertices in $\{v_j : j > i\}$. Let us colour the arcs of D^- in decreasing order of their head; that is first colour the arcs with head v_p then those with head v_{p-1}, and so on. This is possible since at each stage i, an arc uv_i has at most $2l < m$ forbidden colours (l ingoing u and l outgoing v_i to a vertex in $\{v_j : j > i\}$).

It remains to colour the arcs with tail in V^- and head in V^+. Let v^-v^+ be such an arc. Since $col^-(v^+) \geq l+1$ and $d^-(v^-) \leq l$, there is a colour α in $Col^-(v^+)$ that is assigned to no arc ingoing v^-. Hence, assigning α to v^-v^+, we extend the arc-colouring to v^-v^+. □

Corollary 21 *If $\theta(2k) \geq 2l+1$, then $\phi^\vee(k,l) \leq \theta(2k)$.*

Proof. The digraph D^1, as defined in Lemma 20, is a k-digraph. The result follows directly from Remark 19 and Lemma 20. □

4 Relations between $\Phi(k)$ and $\Phi^\vee(k,l)$

Conjecture 22 *Let l be a positive integer. There exists an integer k_l such that if $k \geq k_l$ then $\Phi^\vee(k,l) = \Phi(k)$.*

We now prove Conjecture 22, for $l = 0$, showing that $k_0 = 1$.

Theorem 23 *If $k \geq 1$,*

$$\Phi^\vee(k,0) = \Phi(k).$$

Proof. Let $D = (V, A)$ be a $(k \vee 0)$ digraph. Let V_0 be the set of vertices with indegree 0. Let D' be the digraph obtained from D by splitting each vertex v of V_0 into $d^+(v)$ vertices with outdegree 1. Formally, for each vertex $v \in V_0$ incident to the arcs $vw_1, \dots, vw_{d^+(v)}$, replace v

by $\{v_1, v_2, \ldots, v_{d^+(v)}\}$ and vw_i by v_iw_i, $1 \leq i \leq d^+(v)$. By construction, D' is a k-digraph and $L(D) = L(D')$. So $\chi_a(D) = \chi_a(D') \leq \Phi(k)$. $\square$

We conjecture that if $l = 1$, Conjecture 22 holds with $k_1 = 1$.

Conjecture 24 *If $k \geq 1$,*

$$\Phi^\vee(k, 1) = \Phi(k)$$

Theorem 25 *If $\Phi(k) = \Phi(k-1)$ or $\Phi(k) = \Phi(k+1)$ then $\Phi^\vee(k, 1) = \Phi(k)$.*

Proof. By Lemma 20, it suffices to prove that if $\Phi(k) = \Phi(k-1)$ or $\Phi(k) = \Phi(k+1)$ every k-digraph admits an arc-colouring with $\Phi(k)$ colours such that for every vertex x, $col^-(x) \geq 2$.

Suppose that $\Phi(k) = \Phi(k-1)$. Let D be a k-digraph and D' be a $(k-1)$-digraph such that $\chi_a(D') = \Phi(k)$. Let C be the digraph constructed as follows: for every vertex $x \in V(D)$ add a copy $D'(x)$ of D' such that every vertex of $D'(x)$ dominates x. Then C is a k-digraph, so it admits an arc-colouring c with $\Phi(k)$ colours. Note that c is also an arc-colouring of D which is a subdigraph of C. Let us prove that for every vertex $x \in V(D)$, $col^-(x) \geq 2$.

Suppose, reductio ad absurdum, that there is a vertex $x \in V(D)$ such that $col^-(x) \leq 1$. Since there are arcs ingoing x in C (those from $V(D'(x))$), then $Col^-(x)$ is a singleton $\{\alpha\}$. Now every arc vx with $v \in D'(x)$ is coloured α so any arc $uv \in E(D'(x))$ is not coloured α. Hence c is an arc-colouring with $\Phi(k) - 1$ colours which is a contradiction.

The proof is analogous if $\Phi(k) = \Phi(k+1)$ with D' a k-digraph such that $\chi_a(D') = \Phi(k)$. Then C is a $(k+1)$-digraph and we get the result in the same way. $\square$

The next theorem shows that for a fixed integer k, one of the Conjectures 24 and 4 holds.

Theorem 26 *Let k be an integer. Then $\Phi^\vee(k, 1) = \Phi(k)$ or $\Phi^\vee(k, k) \leq \Phi(k) + 1$.*

Proof. Suppose that $\Phi^\vee(k, 1) \neq \Phi(k)$. Let C be a $(k \vee 1)$-digraph such that $\chi_a(C) = \Phi(k) + 1$ and C^1 the subdigraph of C induced by the arcs with tail in $V^+(C)$. By Lemma 20, for every arc-colouring of C^1 with $\Phi(k)$ colours there exists a vertex x of C^+ with $col^-(x) \leq 1$.

Let D be a $(k \vee k)$-digraph. Let D^1 (resp. D^2) be the subdigraph of D induced by the arcs with tail in $V^+(D)$ (resp. head in $V^-(D)$). Let E' be the set of arcs of D with tail in V^- and head in V^+. Let F^1 be the digraph constructed from C^1 as follows : for every vertex $x \in V^+(C)$, add a copy $D^+(x)$ of D^+ and the arcs $\{u(x)x : uv \in E(D), u \in V^+(D),$ and $v \in V^-(D)\}$. Then F^1 is a k-digraph so it admits an arc-colouring c_1 with $\{1, \ldots, \Phi(k)\}$. Now there is a vertex $x \in V^+(C)$ such that $col^-(x) \leq 1$. So all the arcs from $D^+(x)$ to x are coloured the same. Free to permute the labels, we may assume they are coloured 1. Since $F^1[V(D^+(x)) \cup x]$ has the same line-digraph than D^1, the arc-colourings of $F^1[V(D^+(x)) \cup x]$ is in one-to-one correspondence with the arc-colourings of D^1. So D^1 admits an arc-colouring c^1 with $\{1, \ldots, \Phi(k)\}$ such that every arc with head in V^- is coloured 1.

Analogously, D^2 admits an arc-colouring c^2 with $\{1, \ldots, \Phi(k)\}$ such that every arc with head in V^- is coloured 1. The union of c_1 and c_2 is an arc-colouring of $D - E'$ with $\{1, \ldots, \Phi(k)\}$. Hence assigning $\Phi(k) + 1$ to every arc of E', we obtain an arc-colouring of D with $\Phi(k) + 1$ colours. $\square$

Corollary 27 $\Phi^\vee(k, 1) \leq \Phi(k) + 1$.

Note that since $\Phi(k)$ is bounded by $\theta(2k)$, the condition $\Phi^{\vee}(k,1) = \Phi(k)$ or $\Phi^{\vee}(k,k) \leq \Phi(k)+1$ is very often true. Indeed, we conjecture that it it always true and that Φ behaves "smoothly".

Conjecture 28 *(i) If $k \geq 1$, $\Phi(k+1) \leq \Phi(k)+1$.*

(ii) If $k \geq 1$, $\Phi(k+2) \leq \Phi(k)+1$.

(iii) $\Phi(k_1k_2) \leq \Phi(k_1)+\Phi(k_2)$.

Note that *(ii)* implies *(i)* and Conjecture 24.

The arc-set of a (k_1+k_2)-digraph D may trivially be partionned into two sets E_1 and E_2 such that $(V(D), E_1)$ is a k_1-digraph and $(V(D), E_2)$ is a k_2-digraph. So $\Phi(k_1+k_2) \leq \Phi(k_1)+\Phi(k_2)$. In particular, $\Phi(k+1) \leq \Phi(k)+\Phi(1) = \Phi(k)+3$. Despite we were not able to prove Conjecture 28-*(i)*, we now improve the above trivial result.

Theorem 29 *If $k \geq 1$ then, $\Phi(k+1) \leq \Phi(k)+2$.*

Proof. Let D be a $(k+1)$-digraph. Free to add arcs, we may assume that $d^+(v) = k+1$ for every $v \in V(D)$. Let $T_1, \dots, T_p$ be the terminal components of D. Each T_i contains a circuit C_i which has a chord. Indeed consider a maximal path P in T_i and two arcs with tail its terminus and head in P, by maximality. One can extend $\bigcup C_i$ into a subdigraph F spanning D such that $d^+_F(v) \geq 1$ for every $v \in V(D)$ and the sole circuits are the C_i, $1 \leq i \leq p$. In fact, F is the union of p connected components $F_1, ..., F_p$, each F_i being the union of C_i and inarborescences $A_i^1, \dots, A_i^{q_i}$ with roots $r_i^1, \dots, r_i^{q_i}$ in C_i such that $(V(C_i), V(A_i^1) \setminus \{r_i^1\}, \dots, V(A_i^p) \setminus \{r_i^p\})$ is a partition of $V(D)$.

Now $D-F$ is a k-digraph. So we colour the arcs of $D-F$ with $\Phi(k)$ colours. Let α and β be two new colours. Let us colour the arcs of F. Let $1 \leq i \leq p$. If C_i is an even circuit then F_i is bipartite and its arcs may be coloured by α and β. If C_i is an odd circuit, consider its chord xy in $E(D-F)$. In the colouring of $D-F$, $Col^+(x) \not\subset Col^+(y)$ thus there is an arc $x'y'$ of $E(C_i)$ such that $Col^+(x') \not\subset Col^+(y')$. Hence we may assign to $x'y'$ a colour of $Col^+(x') \setminus Col^+(y')$. Now $F_i - x'y'$ is bipartite and its arcs may be coloured by α and β. □

5 Φ and $\Phi^{\vee}$ for small value of k or l.

5.1 $\Phi(1)$, $\Phi^{\vee}(1,0)$ and $\Phi^{\vee}(1,1)$.

Theorem 30

$$\Phi^{\vee}(1,1) = \Phi^{\vee}(1,0) = \Phi(1) = 3$$

Proof. By Theorem 5, $\Phi^{\vee}(1,1) \leq 3$. The 3-circuit is its own line-digraph and is not 2-colourable. □

5.2 $\Phi(2)$ and $\Phi^\vee(2,l)$, for $l \leq 2$.

The aim of this subsection is to prove Theorem 35, that is $\Phi(2) = \Phi^\vee(2,0) = \Phi^\vee(2,1) = \Phi^\vee(2,2) = 4$. Therefore, we first exhibit a 2-digraph which is not 3-arc-colourable. Then we show that $\Phi^\vee(2,2) \leq 4$.

Definition 31 For any integer $k \geq 1$, the *rotative tournament* on $2k+1$ vertices, denoted R_{2k+1}, is the tournament with vertex-set $\{v_1,\dots,v_{2k+1}\}$ and arc-set $\{v_iv_j : \ j-i \text{ (modulo } 2k+1) \in \{1,\dots,k\}\}$.

Proposition 32 *The tournament R_5 is not 3-arc-colourable. So $\Phi(2) \geq 4$.*

Proof. Suppose that R_5 admits a 3-arc-colouring c in $\{1,2,3\}$. Then, for any two vertices x and y, $Col^+(x) \neq Col^+(y)$ and $1 \leq col^+(x) \leq 2$. Hence there is a vertex, say v_1, such that $col^+(v_1) = 1$, say $Col^+(v_1) = \{1\}$. Then $Col^+(v_2)$ and $Col^+(v_3)$ are subsets of $\{2,3\}$ and $Col^+(v_2) \not\subset Col^+v_3$. It follows that $col^+(v_3) = 1$. Repeating the argument for v_3, we obtain $col^+(v_5) = 1$ and then $col^+(v_i) = 1$, for every $1 \leq i \leq 5$, which is a contradiction. □

In order to prove that $\Phi^\vee(2,2) \leq 4$, we need to show that every $(2 \vee 2)$-digraph admits homomorphism h into $\overline{H}_4$. In order to exhibit such a homorphism, we first show that there is a homomorphism h^+ from D^+ into a subdigraph S_2^+ of $\overline{H}_4$ and a homomorphism h^- from D^- into a subdigraph S_2^- of $\overline{H}_4$ with specific properties allowing us to extend h^+ and h^- into a homomorphism h from D into $\overline{H}_4$.

Definition 33 Let S_2^+ be the digraph with vertex-set $\{s_1^+,\dots,s_6^+\}$ with arc-set $\{s_i^+s_j^+ : i \neq j\} \setminus \{s_2^+s_1^+, s_4^+s_3^+, s_6^+s_5^+\}$.

Lemma 34 *Let D be a 2-digraph. There exists a homomorphism h^+ from D to S_2^+ such that the vertices x with $h^+(x) \in \{s_2^+, s_4^+, s_6^+\}$ have outdegree 2.*

Proof. Let us prove it by induction on $|V(D)|$. If $d^+(x) \leq 1$ for every vertex x of D then D is 3-colourable and $D \rightarrow S_2^+[\{s_1^+, s_3^+, s_5^+\}]$. So, we assume that there exists a vertex x with outdegree 2. By induction hypothesis, there is a homomorphism $h^+ : D - x \rightarrow S_2^+$ with the required condition. Note that every inneighbour of x has outdegree at most 1 in $D - x$ and thus can not have image s_2^+, s_4^+ or s_6^+ by h^+. Denote by y and z the outneighbours of x. The set $\{h^+(y), h^+(z)\}$ does not intersect one set of $\{s_1^+, s_2^+\}$, $\{s_3^+, s_4^+\}$ and $\{s_5^+, s_6^+\}$, say $\{s_1^+, s_2^+\}$. Then, setting $h^+(x) = s_2^+$, we extend h^+ into a homomorphism from D to S_2^+ with the required condition. □

Theorem 35

$$\Phi(2) = \Phi^\vee(2,0) = \Phi^\vee(2,1) = \Phi^\vee(2,2) = 4$$

Proof.

By Proposition 32, $4 \leq \Phi(2) \leq \Phi^\vee(2,0) \leq \Phi^\vee(2,1) \leq \Phi^\vee(2,2)$.

Let us prove that $\Phi^\vee(2,2) \leq 4$. Let D be a $(2\vee 2)$-digraph. We will provide a homomorphism from D to $\overline{H}_4$.

Let S_2^- be the dual of S_2^+, that is the digraph on $\{s_1^-, \dots, s_6^-\}$ with arc-set $\{s_i^- s_j^- : i \neq j\} \setminus \{s_1^- s_2^-, s_3^- s_4^-, s_5^- s_6^-\}$. By Lemma 34, there is a homomorphism $h^+ : D^+ \to S_2^+$ such that if $h^+(x) \in \{s_2^+, s_4^+, s_6^+\}$ then $d_{D^+}^+(x) = 2$. Symmetrically, there exists a homomorphism $h^- : D^- \to S_2^-$ such that if $h^-(x) \in \{s_2^-, s_4^-, s_6^-\}$ then $d_{D^-}^-(x) = 2$.

Let S_2 be the digraph obtained from the disjoint union of S_2^+ and S_2^- by adding the arcs of $\{s_i^- s_j^+ : 1 \leq i \leq 6, 1 \leq j \leq 6\} \cup \{s_i^+ s_j^- : i = 1,3,5,\ j = 1,3,5\}$. The mapping $h : D \to S_2$ defined by $h(x) = h^+(x)$ if $x \in V^+$ and $h(x) = h^-(x)$ if $x \in V^-$ is a homomorphism. Indeed if xy is an arc of D with $x \in V^+$ and $y \in V^-$, conditions on h^+ and h^- imply that $h(x) = h^+(x) \in \{s_1^+, s_3^+, s_5^+\}$ and $h(y) = h^-(y) \in \{s_1^-, s_3^-, s_5^-\}$. To conclude, Figure 1 provides a homomorphism g from S_2 to $\overline{H}_4$. The non-oriented arcs on the figure corresponds to circuits of length 2 and all the arcs from S_2^- to S_2^+ are not represented. □

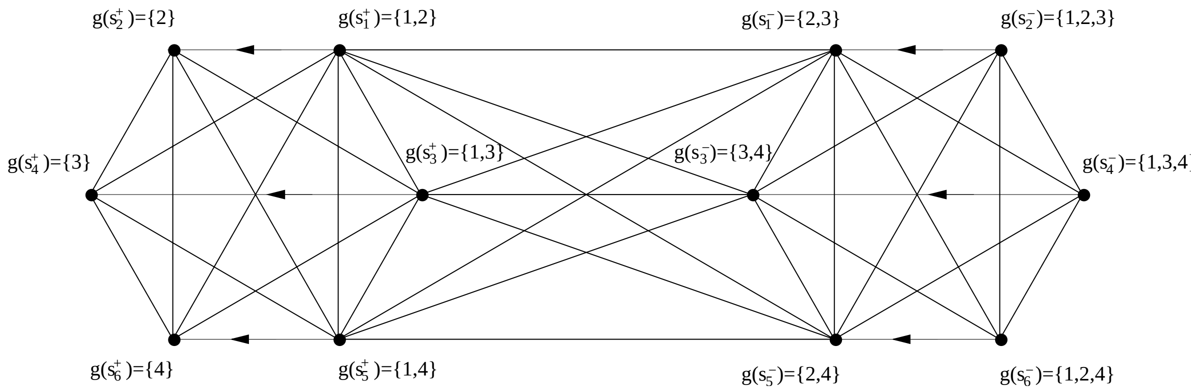

Figure 1: The homomorphism g from S_2 to $\overline{H}_4$.

5.3 $\Phi(3)$ and $\Phi^\vee(3, l)$, for $l \leq 3$.

Theorem 36

$$\Phi(3) = \Phi^\vee(3,0) = \Phi^\vee(3,1) = 4$$

Proof. $4 \leq \Phi(2) \leq \Phi(3) \leq \Phi^\vee(3,0) \leq \Phi^\vee(3,1) \leq \theta(6) = 4$ by Corollary 21. □

In the remaining of this subsection, we shall prove Theorem 42, that is $\Phi^\vee(3,2) = \Phi^\vee(3,3) = 5$. Therefore, we first exhibit a $(3 \vee 2)$-digraph which is not 4-arc-colourable. Then we show that $\Phi^\vee(3,3) \leq 5$.

Definition 37 Let G^- be the digraph obtained from the rotative tournament on five vertices R_5, with vertex set $\{v_1^-, \dots, v_5^-\}$ and arc-set $\{v_i^- v_j^- : j - i \text{ (modulo } 5) \in \{1,2\}\}$ and five copies of the 3-circuits $R_3^1, \dots, R_3^5$ by adding, for $1 \leq i \leq 5$, the arcs vv_i^-, for $v \in R_3^i$.

Let G^+ be the digraph obtained from the rotative tournament on seven vertices R_7, with vertex set $\{v_1^+, \dots, v_7^+\}$ and arc-set $\{v_i^+ v_j^+ : j - i \text{ (modulo } 7) \in \{1,2,3\}\}$ and seven copies of the rotative tournament of, $R_5^1, \dots, R_5^7$ by adding, for $1 \leq i \leq 7$, the arcs vv_i^+, for $v \in R_5^i$.

Finally, let G be the $(3 \vee 2)$-digraph obtained from the disjoint union of G^- and G^+ by adding all the arcs of the form $v^- v^+$ with $v^- \in V(G^-)$ and $v^+ \in V(G^+)$. See Figure 2.

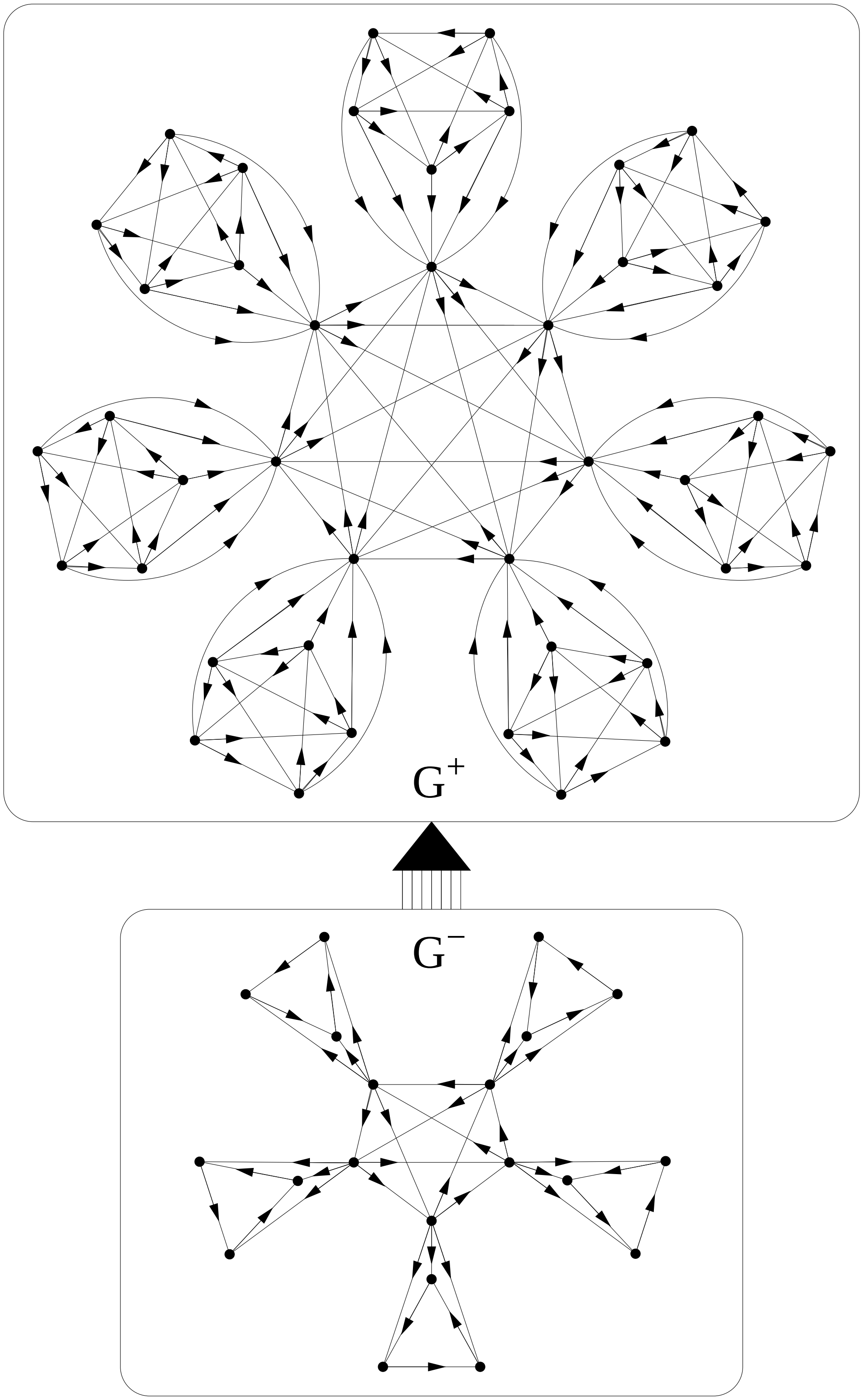

Figure 2: The non 4-arc-colourable $(3 \vee 2)$-digraph G.

Proposition 38 *The digraph G is not 4-arc-colourable. So $\Phi(3,2) \geq 5$.*

Proof. Suppose for a contradiction that G admits an arc-colouring c in $\{1,2,3,4\}$.

Let v^+ be a vertex of G^+ and v^- a vertex of G^-. Then since v^-v^+ is an arc, $Col^+(v^+) \neq Col^+(v^-)$. We will show:

(1) there are at least two 2-subsets S of $\{1,2,3,4\}$ such that a vertex $v^- \in G^-$ satisfies $Col^+(v^-) = S$;

(2) there are at least five 2-subsets S of $\{1,2,3,4\}$ such that a vertex $v^+ \in G^+$ satisfies $Col^+(v^+) = S$.

This gives a contradiction since there are only six 2-subsets in $\{1,2,3,4\}$.

Let us first show (1). Every vertex of G^- satisfies $col^+ \geq 2$ otherwise all the arcs of R_7 in G^+ must be coloured with three colours, a contradiction to Theorem 12. Hence, since in R_5 all the Col^+ are distinct and not $\{1,2,3,4\}$, a vertex of R_5, say v_1^-, has $col^+ = 2$, say $Col^+(v_1^-) = \{1,2\}$. Consider now the vertices of R_3^1. None of them has $Col^+ = \{1,2,3\}$ nor $Col^+ = \{1,2,4\}$ since they are dominated by v_1^-. Moreover they all have different Col^+ since R_3^1 is a tournament. Hence one of them, say v, satisfies $col^+(v) = 2$. Now $Col^+(v) \neq Col^+(v_1^-)$ since $v_1^- \rightarrow v$.

Let us now prove (2). Let $\mathcal{S} = \{$2-subsets S such that $\exists\ v^+ \in G^+, Col^+(v^+) = S\}$ and suppose that $|\mathcal{S}| \leq 4$. Every vertex of G^+ has $col^+ \leq 2$ otherwise all the arcs of R_5 in G^- must be coloured with three colours, a contradiction to Proposition 32. Now, all the vertices of R_7 have distinct and non-empty Col^+. So at least three vertices of R_7 have $col^+ = 2$ and $|\mathcal{S}| \geq 3$. Thus, without loss of generality, we are in one these three following cases:

(a) $\mathcal{S} \subset \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ and $Col^+(v_1^+) = \{1,2\}$;

(b) $\mathcal{S} \subset \{\{1,2\},\{2,3\},\{1,3\},\{1,4\}\}$ and $Col^+(v_1^+) = \{2,3\}$;

(c) $\mathcal{S} \subset \{\{1,2\},\{2,3\},\{1,3\},\{1,4\}\}$ and $Col^+(v_1^+) = \{1,4\}$.

Let $x_1, \dots, x_5$ be the vertices of R_5^1 such that x_ix_j is an arc if and only if $j = i+1 \bmod 5$ or $j = i+2 \bmod 5$ and $\mathcal{F} = \{Col^+(x_i) : 1 \leq i \leq 5\}$. Recall that $|\mathcal{F}| = 5$ since R_5^1 is a tournament and that every element S of $\mathcal{F}$ is not included in $Col^+(v_1^+)$ since $x_i \rightarrow v_1^+$ for every $1 \leq i \leq 5$.

Case (a): We have $\mathcal{F} = \{\{3\},\{4\},\{2,3\},\{3,4\},\{1,4\}\}$. So we may assume that $Col^+(x_1) = \{3\}$. Now because $x_1 \rightarrow x_2$, $x_1 \rightarrow x_3$ and $x_2 \rightarrow x_3$, $Col^+(x_1) \not\subset Col^+(x_2)$, $Col^+(x_1) \not\subset Col^+(x_3)$ and $Col^+(x_2) \not\subset Col^+(x_3)$. It follows that $Col^+(x_2) = \{1,4\}$ and $Col^+(x_3) = \{4\}$. Hence, none of $Col^+(x_4)$ and $Col^+(x_5)$ is $\{3,4\}$ since $x_3 \rightarrow x_4$ and $x_3 \rightarrow x_5$, a contradiction.

Case (b): We have $\mathcal{F} = \{\{1\},\{4\},\{1,2\},\{1,3\},\{1,4\}\}$. So we may assume that $Col^+(x_1) = \{1\}$. Since $x_1 \rightarrow x_2$, $Col^+(x_1) \not\subset Col^+(x_2)$, so $Col^+(x_2) = \{4\}$. Similarly, $Col^+(x_3) = \{4\}$ which is a contradiction.

Case (c): We have $\mathcal{F} = \{\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$. So we may assume that $Col^+(x_1) = \{2\}$. It follows that $Col^+(x_2) = \{1,3\}$ and $Col^+(x_3) = \{3\}$. Hence, none of $Col^+(x_4)$ and $Col^+(x_5)$ is $\{2,3\}$ since $x_3 \rightarrow x_4$ and $x_3 \rightarrow x_5$, a contradiction. □

We will now prove that $\Phi^\vee(3,3) \leq 5$. As in the proof of Theorem 35, in order to exhibit a homomorphism from a (3∨3)-digraph D to $\overline{H}_5$, we first show that there are two homomorphisms, h^+ from D^+ into a subdigraph S_3^+ of $\overline{H}_5$ and h^- from D^- into another subdigraph S_3^- of $\overline{H}_5$, with specific properties.

Definition 39 Let S_3^+ be the complete digraph with vertex-set $\{s_1^+, \dots, s_7^+\}$.
Let S_3^- be the digraph with vertex-set $\{s_1^-, \dots, s_9^-\}$ and arc-set $\{s_i^- s_j^- : i \neq j\} \setminus \{s_2^- s_1^-, s_3^- s_1^-\}$.

Lemma 40 *Let D be a 3-digraph. There exists a homomorphism h^+ from D to S_3^+ such that the vertices x with $h^+(x) \in \{s_6^+, s_7^+\}$ have outdegree 3.*

Proof. Let us prove it by induction on $n = |V(D)|$. If there exists a vertex x with $d^+(x) + d^-(x) \leq 4$ then we obtain the desired homomorphism h^+ from $D - x$ to S_3^+ and extend it with a suitable choice of $h^+(x)$ in $\{s_1^+, \dots, s_5^+\}$.
Assume now that $d^+(x)+d^-(x) \geq 5$ for every x. Let n_i be the number of vertices with outdegree i. Clearly, $n = n_0 + n_1 + n_2 + n_3$. Moreover, we have:

$$3n \geq \sum_{x \in V} d^+(x) = \sum_{x \in V} d^-(x) = \sum_{d^+(x)=0} d^-(x) + \sum_{d^+(x)=1} d^-(x) + \sum_{d^+(x)=2} d^-(x) + \sum_{d^+(x)=3} d^-(x)$$

Then, by assumption:

$$3n \geq 5n_0 + 4n_1 + 3n_2 + \sum_{d^+(x)=3} d^-(x)$$

If there is no vertex with outdegree 3, then D is 5-colourable and there is an homomorphism h^+ from D to $S_3^+[\{s_1^+, \dots, s_5^+\}]$. Suppose now that there exists a vertex with outdegree 3. Then, there exists a vertex with outdegree 3 and indegree at most 3. If not, $d^-(x) \geq 4$ for every x with $d^+(x) = 3$ and the previous inequality implies $3n \geq 5n_0 + 4n_1 + 3n_2 + 4n_3$ with $n_3 \neq 0$, what contradicts $n = n_0 + n_1 + n_2 + n_3$.
Finally, let x be a vertex with outdegree 3 and indegree at most 3. By induction hypothesis, there is a homomorphism h^+ from $D - x$ to S_3^+ with the required property. As x has at most 6 neighbours, we extend h^+ with a suitable choice for $h^+(x)$ in $\{s_1^+, \dots, s_7^+\}$. □

Lemma 41 *Let D be a digraph with maximal indegree at most 3. There exists a homomorphism h^- from D to S_3^- such that the vertices x with $h^-(x) \in \{s_6^-, \dots, s_9^-\}$ have indegree 3.*

Proof. We prove the result by induction on $|V(D)|$.
If every vertex have indegree at most 2 then, by the dual form of the Lemma 17, there exists a homomorphism from D to $S_3^-[\{s_1^-, \dots, s_5^-\}]$.
Now, let x be a vertex with indegree 3. Let y_1, y_2 and y_3 be the outneighbours of x. By induction, there is a homomorphism h^- from $D - x$ to S_3^- with the required property. In particular, as the vertex y_i, $1 \leq i \leq 3$, has indegree at most 2 in $D - x$, we have $h^-(y_i) \in \{s_1^-, \dots, s_5^-\}$. So, as x has 3 inneighbours, we can extend h^- with a suitable choice for $h^-(x)$ in $\{s_6^-, \dots, s_9^-\}$. □

Theorem 42

$$\Phi^\vee(3,2) = \Phi^\vee(3,3) = 5$$

Proof. By Proposition 38, $5 \leq \Phi^{\vee}(3,2) \leq \Phi^{\vee}(3,3)$. We will prove that $\Phi^{\vee}(3,3) \leq 5$. Let D be a $(3 \vee 3)$-digraph, we will provide a homomorphism from D to $\overline{H}_5$.

By Lemma 40, there is a homomorphism $h^+ : D^+ \to S_3^+$ such that if $h^+(x) \in \{s_6^+, s_7^+\}$ then $d^+_{D^+}(x) = 3$. Moreover by Lemma 41, there is a homomorphism $h^- : D^- \to S_3^-$ such that if $h^-(x) \in \{s_6^-, \dots, s_9^-\}$, then $d^-_{D^-}(x) = 3$.

Let S_3 be the digraph obtained from the disjoint union of S_3^+ and S_3^- by adding the arcs of $\{s_i^- s_j^+ \ : \ 1 \leq i \leq 9, 1 \leq j \leq 7\} \cup \{s_i^+ s_j^- \ : \ i = 1, \dots, 5 \quad j = 1, \dots, 5\}$. The mapping $h : D \to S_3$ defined by $h(x) = h^+(x)$ if $x \in V^+$ and $h(x) = h^-(x)$ if $x \in V^-$ is a homomorphism. Indeed if xy is an arc of D with $x \in V^+$ and $y \in V^-$, conditions on h^+ and h^- imply that $h(x) = h^+(x) \in \{s_1^+, \dots, s_5^+\}$ and $h(y) = h^-(y) \in \{s_1^-, \dots, s_5^-\}$. To conclude, Figure 3 provides a homomorphism g from S_3 to $\overline{H}_5$. Inside S_3^- and S_3^+, only the arcs which are not in a circuit of length 2 are represented, every pair of not adjacent vertices are, in fact, linked by two arcs, one in each way. □

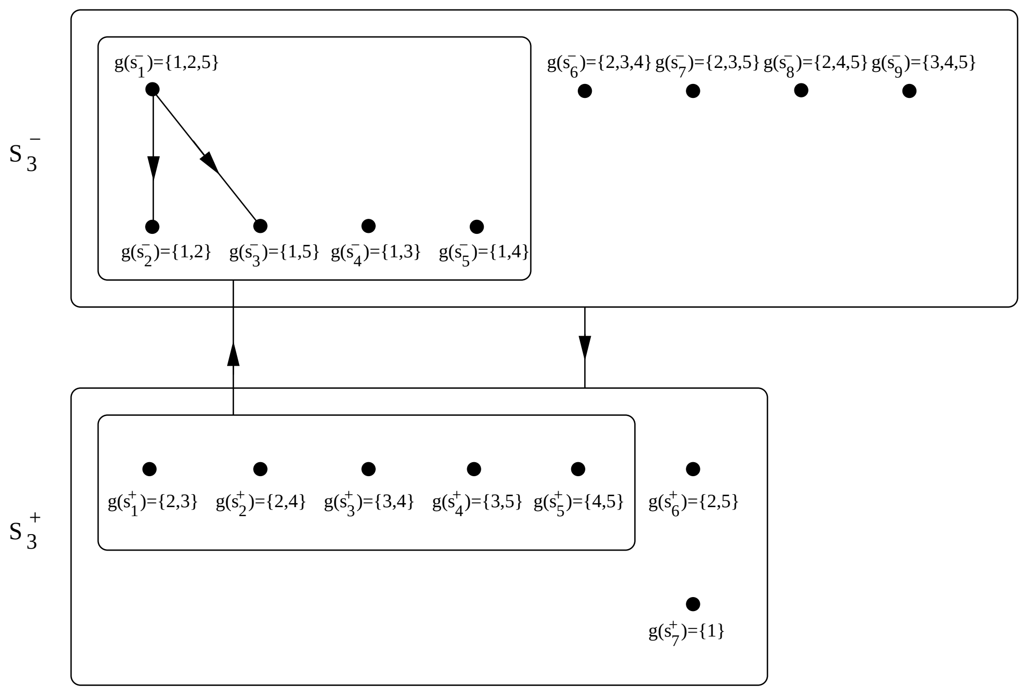

Figure 3: The homomorphism g from S_3 to $\overline{H}_5$.

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