

LABORATOIRE



INFORMATIQUE, SIGNAUX ET SYSTÈMES
DE SOPHIA ANTIPOLIS
UMR 6070

CHOOSABILITY OF THE SQUARE OF PLANAR SUBCUBIC GRAPHS WITH LARGE GIRTH

F. Havet

Projet MASCOTTE

Rapport de recherche
ISRN I3S/RR-2006-01-FR

Janvier 2006

RÉSUMÉ :

Nous montrons que le nombre de choix du carré d'un graphe subcubique de degré moyen maximum inférieur à $18/7$ est au plus 6. Ceci implique que le carré d'un graphe planaire subcubique de maille au moins 9 est 6-choissable. Ensuite, nous montrons que le nombre de choix du carré d'un graphe planaire subcubique de maille au moins 13 est au plus 5.

MOTS CLÉS :

coloration, coloration par listes, graphe planaire, maille

ABSTRACT:

We first show that the choose number of the square of a subcubic graph with maximum average degree less than $18/7$ is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

KEY WORDS :

colouring, list colouring, planar graph, girth

Choosability of the square of planar subcubic graphs with large girth

F. Havet*

Projet Mascotte, CNRS/UNSA/INRIA
INRIA Sophia-Antipolis,
2004 route des Lucioles BP 93,
06902 Sophia-Antipolis Cedex
France
e-mail:fhavet@sophia.inria.fr

January 10, 2006

Abstract

We show that the choose number of the square of a subcubic graph with maximum average degree less than $18/7$ is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

1 Introduction

Let G be a (simple) graph.

Let v be a vertex of G . Its *neighbourhood*, denoted $N_G(v)$ is the set of its *neighbours*, i.e. is vertices y such that xy is an edge. The *degree* of a vertex v in G , denoted $d_G(v)$, is its number of neighbours. Often, when the graph G is clearly understood from the context, we omit the subscript G .

A graph is *subcubic* if every vertex has degree at most 3.

Let $p : V(G) \rightarrow \mathbb{N}$. A *p-list-assignment* is a list-assignment L such that $L(v) = p(v)$ for any $v \in V(G)$. G is *p-choosable* if it is L -colourable for any p -list-assignment. By extension, if k is an integer, we say that G is *k-choosable* if it is p -choosable when p is the constant function with value k (i. e. $p(v) = k$ for all $v \in V$). The *choose number* of G is the smallest integer k such that G is k -choosable. Clearly the choose number of G is at least as large as $\chi(G)$ the *chromatic number* of G .

The *square* of G , is the graph G^2 with vertex set $V(G)$ such that two vertices are linked by an edge of G^2 if and only if x and y are at distance at most 2 in G (either xy is an edge or x and y share a neighbour). Formally, $E(G^2) = \{xy, (xy \in E(G) \text{ or } \exists z \in N(x) \cap N(y))\}$.

Wegner [4] proved that the square of a subcubic planar is 8-colourable. He also conjectured it is 7-colourable.

Conjecture 1 (Wegner [4]) *Let G be a subcubic planar graph. Then $\chi(G^2) \leq 7$.*

*Partially supported by the Integrated Project IST-15964 *AEOLUS*

The *average degree* of G , denoted $Ad(G)$ is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.

The *maximum average degree* of G , denoted $Mad(G)$, is $\max\{Ad(H), H \text{ subgraph of } G\}$.

In [1], Dvořák, Škrekovski and Tancer proved that the choose number of the square of a subcubic graph G is at most 4 if $Mad(G) < 24/11$ and G has no 5-cycle, at most 5 if $Mad(G) < 7/3$ and at most 6 if $Mad(G) < 5/2$.

A graph is called *planar* if it can be embedded in the plane. The *girth* of a graph is the smallest length of a cycle in G . Planar graph with prescribed girth have bounded maximum average degree:

Proposition 2 *Every planar graph with girth at least g has maximum average degree less than $2 + \frac{4}{g-2}$.*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choose number of the square of a planar graph with girth g is at most 6 if $g \geq 13$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two later results had been previously proved by Montassier and Raspaud [3].

In this paper, we improve some of these results. We first show (Theorem 4) that the choose number of the square of a subcubic graph with maximum average degree less than $18/7$ is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show (Theorem 9) that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

2 The main results

The general frame of the proofs are classical. We consider a *k-minimal graph*, that is a subcubic graph such that its square is not *k-choosable*, but the square of every proper subgraph is *k-choosable*. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction.

An *i-vertex* is a vertex of degree i . We denote by V_i the set of i -vertices of G and by v_i its cardinality. Let v be a vertex. An *i-neighbour* of v is a neighbour of v with degree i . The *i-neighbourhood* of v is $N_i(v) = N(v) \cap V_i$ and its *i-degree* is $d_i(v) = |N_i(v)|$.

Some properties of 6- and 5-minimal graphs have already been proved in [1]. The easy first one is that $V_0 \cup V_1 = \emptyset$, so G has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

Definition 3 Let G be a subcubic graph with minimum degree 2.

A *thread* of G is a maximal induced path of G . In other words, it is a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices.

The *kernel* of G is the weighted graph K such that $V(K) = V_3(G)$ and xy is an edge in K with weight l if and only if x and y are connected by a thread of length l in G . An edge of weight l is also called *l-edge*.

Let x be a 3-vertex of G . The *type* of x is the triple (l_1, l_2, l_3) such that $l_1 \leq l_2 \leq l_3$ and the three edges (A loop is counted twice.) incident to x have weight l_1 , l_2 and l_3 in K . We denote by Y_{l_1, l_2, l_3} the set of 3-vertices of type (l_1, l_2, l_3) and y_{l_1, l_2, l_3} its cardinality. Moreover, for every integer i , we define $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1, l_2, l_3}$ and $z_i = |Z_i|$.

The number of vertices and edges and thus the average degree of a subcubic graph G with minimum degree 2 may be easily expressed in terms of the z_i :

$$|V(G)| = \sum_{i \geq 3} \frac{i-1}{2} z_i$$

$$\begin{aligned}
2|E(G)| &= \sum_{i \geq 3} i \cdot z_i \\
Ad(G) &= \frac{\sum_{i \geq 3} i \cdot z_i}{\sum_{i \geq 3} \frac{i-1}{2} z_i} \tag{1}
\end{aligned}$$

2.1 6-choosability

The aim of this subsection is to prove the following result.

Theorem 4 *Let G be a subcubic graph of maximum average degree $d < 18/7$. Then G^2 is 6-choosable.*

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

Lemma 5 (Dvořák, Škrekovski and Tancer [1]) *Let G be a 6-minimal graph. Then the following hold:*

- 1) all the edges of K have weight at most 2;
- 2) every 3-cycle of G has its vertices in V_3 ;
- 3) every 4-cycle of G has at least three vertices in V_3 ;
- 4) a vertex of $Y_{2,2,2}$ is not adjacent to a vertex of $Y_{1,2,2} \cup Y_{2,2,2}$.

We will prove in Subsection 3.2 some new properties.

Lemma 6 *Let G be a 6-minimal graph. Then the following hold:*

- 5) if $(v_1, v_2, v_3, v_4, v_1)$ is a 4-cycle with $v_2 \in V_2$ then v_1 or v_3 is not in Z_5 ;
- 6) a vertex of $Y_{1,2,2}$ is adjacent to at most one vertex of $Y_{1,2,2}$ by 2-edges.

Proof of Theorem 4. Let G be a 6-minimal planar graph. G has minimum degree 2, so its kernel K is defined. Moreover by Lemma 5-1), Z_i is empty for $i \geq 7$ and $Z_6 = Y_{2,2,2}$ and $W_5 = Y_{1,2,2}$.

Let us consider a vertex of $Z_4 = Y_{1,1,2}$. Its neighbour via the 2-edge is in $Z_4 \cup Z_5 \cup Z_6$ because a vertex of $Z_3 = Y_{1,1,1}$ is incident to no edge of weight 2.

For $i = 4, 5, 6$, let Z_4^i be the set of vertices of Z_4 which are incident to a vertex of Z_i by their unique 2-edge and z_4^i its cardinality. (Z_4^4, Z_4^5, Z_4^6) is a partition of Z_4 so $z_4 = z_4^4 + z_4^5 + z_4^6$.

Hence Equation (1) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 5-4), the three neighbours of a vertex of Z_6 is not in $Z_6 \cup Z_5$. So they must be in Z_4^6 . It follows that $3z_6 = z_4^6$. So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 6-6), a vertex of Z_5 is adjacent to at least one vertex of Z_4^5 . Thus $z_5 \leq z_4^5$.

But $Ad(G)$ is decreasing as a function of z_5 since z_4^6, z_4^5, z_4^4 and z_3 are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□

Remark 7 Theorem 4 is tight. Indeed, the graph J_7 depicted Figure 1 has average degree $18/7$ and its square is the complete graph on seven vertices K_7 which is not 6-choosable (nor 6-colourable).

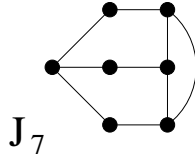


Figure 1: The graph J_7

Theorem 4 and Proposition 2 yield that the square of a planar graph with girth 9 is 6-choosable.

Corollary 8 *The square of a planar graph with girth 9 is 6-choosable.*

2.2 5-choosability

Dvořák, Škrekovski and Tancer [1] proved that the square of a subcubic graph G with maximum average degree less than $7/3$ is 5-choosable. This result is tight since the graph J_6 depicted Figure 2 has average degree $7/3$ and its square is the complete graph on six vertices K_6 which is not 5-choosable (nor 5-colourable).

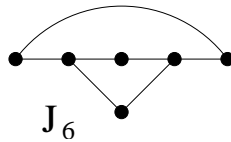


Figure 2: The graph J_6

However, we will prove that the square of a planar graph with girth at least 13 is 5-choosable, which improves the result of Montassier and Raspaud.

Theorem 9 *The square of a planar graph with girth 13 is 5-choosable.*

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

Lemma 10 (Dvořák, Škrekovski and Tancer [1]) *Let G be a 5-minimal graph of girth 13. Then the following hold:*

- 1) *all the edges of K have weight at most 3;*
- 2) *if $i \geq 8$, Z_8 is empty.*

We will prove in Subsection 3.3 some new properties.

Lemma 11 *Let K be the G be a 5-minimal graph of girth at least 13. Then the following holds:*

- 3) *A vertex of $Y_{2,2,3}$ and a vertex of $Y_{1,2,3} \cup Y_{2,2,3}$ are not linked by a 2-edge.*

4) A vertex of $Y_{1,3,3}$ and a vertex of $Y_{1,2,3} \cup Y_{1,3,3}$ are not linked by a 1-edge.

5) A vertex of $Y_{2,2,2}$ is not adjacent to three vertices of $Y_{2,2,3}$ (by 2-edges).

Proof of Theorem 9. Let G be a 5-minimal planar graph with girth at least 13. G has minimum degree 2, so its kernel K is defined. Moreover, by Lemma 10-1), $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$, so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \quad (2)$$

Let us count the number e_2 of 2-edges incident to vertices of $Y_{2,2,3}$. Since 2-edges may not link two vertices of type $(2, 2, 3)$ according to Lemma 11-3), we have $e_2 = 2y_{2,2,3}$. Moreover, the end of such edges which is not in $Y_{2,2,3}$ has to be in $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$ by Lemmas 10 and 11-3). Furthermore, a vertex of $Y_{2,2,2}$ is incident to at most two edges of e_2 according to Lemma 11-5) and a vertex of $Y_{1,2,2}$ (resp. Z_4) is incident to at most two (resp. one) 2-edges. Therefore $e_2 \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4$. So,

$$2y_{2,2,3} \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4 \quad (3)$$

Let us now count the number e_1 of 1-edges incident to vertices of $Y_{1,3,3}$. Since 1-edges may not link two vertices of type $(1, 3, 3)$ according to Lemma 11-4), we have $e_1 = y_{1,3,3}$. Moreover, the end of such edges which is not in $Y_{2,2,3}$ has to be in $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$ by Lemmas 10 and 11-4). Furthermore, vertices of $Y_{1,2,2}$ (resp. $Y_{1,1,3} \cup Z_4, Z_3$) are incident to at most one (resp. two, three) 1-edges. Thus $e_1 \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$. So,

$$y_{1,3,3} \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3 \quad (4)$$

$2 \times (4) + (3)$ yields $2y_{2,2,3} + 2y_{1,3,3} \leq 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$. Hence by Equation 2, $2z_7 \leq 2z_6 + 4z_5 + 5z_4 + 6z_3$, so

$$z_7 \leq z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation 1 the average degree of G is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of z_7 this is a decreasing function (on \mathbb{R}^+), so it is minimum when z_7 is maximum that is equal to $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$. So, :

$$Ad(G) \geq \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \geq \frac{26}{11}.$$

This contradicts the fact that G has girth 13 by Proposition 2. □

It is very likely that using the method below, one can prove that a graph G with maximum average degree less than $\frac{26}{11}$ is 5-choosable unless it contains J_6 as an induced subgraph. However, this will require the tedious study of a large number of configurations.

3 Proofs of Lemmas 6 and 11

In order, to prove Lemmas 6 and 11, we need the following lemma proved in [1]. Let S be a set of vertices of a k -minimal graph G . The function $p_S : S \rightarrow N$ is defined by $p_S(v) = k - |N_{G^2}(v) \setminus S|$. Then $p_S(v)$ represents the minimum number of available colours at a vertex $v \in S$ once we have precoloured the square of $G - S$. Hence if $(G - S)^2$ is 6-choosable, $(G - S)^2 = G^2 - S$ (in particular, it happens when two distinct vertices of $G - S$ have no common neighbour in S) and $G[S]^2$ is p_S -choosable, one can extend any k -list-colouring of $G - H$ into a k -list-colouring of G .

Lemma 12 (Dvořák, Škrekovski and Tancer [1]) *Let S be a set of vertices of a k -minimal graph G . If $(G - S)^2 = G^2 - S$, then $G^2[S]$ is not p_S -choosable.*

In order to use Lemma 12, we need some results on the choosability of some graphs.

3.1 Some choosability tools

Definition 13 Let x and y be two vertices of a graph G . An $(x - y)$ -ordering of G is an ordering such that x is the minimum and y the maximum. An $(x, y - z)$ -ordering is an ordering such that x is minimum, y is the second minimum and z is maximum.

Let $\sigma = (v_1 < v_2, \dots < v_n)$ be an ordering of the vertices of G . σ is p -greedy if for every i , $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$. It is p -nice if for every i except n , $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$ and $d(v_n) = p(v_n)$. It is p -good if for every $3 \leq i \leq n$, $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$ with $\epsilon(v_i) = 1$ if v_i is adjacent to both v_1 and v_2 and $\epsilon(v_i) = 0$ otherwise.

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

Lemma 14 *If G has a p -greedy ordering then G is p -choosable.*

Proof. Applying the greedy algorithm according to the p -greedy ordering gives the desired colouring. \square

Lemma 15 *Let xy be an edge of graph G and L be a p -list-assignment of G . If $L(x) \not\subseteq L(y)$ and G has a p -nice $(x - y)$ -ordering, then G is L -colourable.*

Proof. Let a be a colour in $L(x) \setminus L(y)$. Proceed the greedy algorithm starting by assigning a to x . The only vertex which has not more colour in its list than previously coloured neighbours is y for which $|L(y)| = d(y)$. But since $a \notin L(y)$, at most $d(y) - 1$ colours of $L(y)$ are assigned to the neighbours of y . Hence one can colour y . \square

Lemma 16 *Let x, y and z be three vertices of a graph $G = (V, E)$ such that $xy \notin E$, $xz, yz \in E$. If $L(x) \cap L(y) \neq \emptyset$ and G has a p -good $(x, y - z)$ -ordering, then G is L -colourable.*

Proof. Let a be a colour in $L(x) \cap L(y)$ and $\sigma = (v_1 < v_2, \dots < v_n)$ be a p -good $(x, y - z)$ -ordering. (In particular, $v_1 = x, v_2 = y$ and $v_n = z$.) Proceed the greedy algorithm according to σ starting by assigning a to x and y . For every $3 \leq i \leq n$, the number of colours assigned to already coloured neighbours of v_i is at most $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i)$ since v_1 and v_2 are coloured the same. Hence the greedy algorithm gives an L -colouring. \square

Remark 17 Note that under the condition $xz, yz \in E$, a p -nice ordering is also a p -good ordering.

Definition 18 The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks are either complete graphs or odd cycles.

Theorem 19 (trouver si c'est Lovasz, Borodin, erdos et al) *Let G be a connected graph. Then G is d -choosable if and only if G is not a Gallai tree.*

Lemma 20 *Let $G = (V, E)$ be a graph and $p : V(G) \rightarrow \mathbb{N}$. Let S be a set of vertices such that $p(v) \geq d(v)$ for all $v \in S$. If $G[S]$ is not a Gallai tree and $G - S$ is p -choosable then G is p -choosable.*

Proof. Let L be a p -list-assignment of G . Since $G - S$ is p -choosable, it admits a p -colouring c . Let us now extend it to S . The list $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$ of available colours of a vertex $v \in S$ is of size at least $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$. Since $G[S]$ is not a Gallai tree, by Theorem 19, $G[S]$ is p' -choosable and thus I -colourable. So, G is L -colourable. \square

A 4-regular graph is *cycle+triangles* if it is the edge union of a Hamiltonian cycle and triangles.

Theorem 21 (Fleishner and Stiebitz [2]) *Every cycle+triangles graph is 3-choosable.*

3.2 Proof of Lemma 6

Lemma 22 *Let $q \geq 4$ and $C_{2q} = (v_1, \dots, v_{2q}, v_1)$ be the $2q$ -cycle and p defined by $p(v_i) = 4$ if i is odd and $p(v_i) = 3$ otherwise. Then C_{2q}^2 is p -choosable.*

Proof. The set S of vertices v for which $p(v) \geq d_{C_{2q}^2}(v)$ is the set of v_i with odd indices. $C_{2q}^2[S]$ is a q -cycle and thus is not a Gallai tree if $q \geq 4$. Moreover $C_{2q}^2 - S$ is also a q -cycle and is 3-choosable. Hence Lemma 20 gives the result. \square

Proposition 23 *Let $P_7 = (v_1, \dots, v_7)$ be a path and p the function defined by $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$, $p(v_3) = p(v_5) = 4$ and $p(v_4) = 3$. Then P_7^2 is p -choosable.*

Proof. Since $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p -nice, by Lemma 15, we may assume that $L(v_1) = L(v_2)$, and by symmetry of P_7 and p that $L(v_6) = L(v_7)$.

Since $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p -good, by Lemma 16, we may assume that $L(v_1) \cap L(v_4) = \emptyset$, and by symmetry $L(v_7) \cap L(v_4) = \emptyset$.

Now one can find $c(v_1) \in L(v_1)$, $c(v_2)$ in $L(v_2) \setminus \{c(v_1)\}$, $c(v_6)$ in $L(v_6)$, $c(v_7)$ in $L(v_7) \setminus \{c(v_6)\}$, $c(v_3)$ in $L(v_3) \setminus \{c(v_1), c(v_2)\}$, and $c(v_5)$ in $L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$. Now since $L(v_1) \cap L(v_4) = \emptyset$ and $L(v_1) = L(v_2)$, $c(v_2) \notin L(v_4)$. Analogously, $c(v_6) \notin L(v_4)$. Hence, $L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$. So, one can choose $c(v_4)$ in this set to get an L -colouring c of P_7^2 . \square

Lemma 24 *For $1 \leq i \leq 17$, let F_i be the graphs and p_i be the function depicted Figure 3.*

- a) $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ is p_1 -choosable.
- b) $F_2^2 \cup \{v_1v_4\}$ and $F_2^2 \cup \{v_4v_7\}$ are p_2 -choosable.
- c) $F_3^2 \cup \{v_4v_8\}$ is p_3 -choosable.
- d) F_4^2 is 6-choosable.
- e) $F_5^2 \cup \{v_1v_5\}$ is p_5 -choosable.
- f) $F_6^2 \cup \{v_1v_4\}$ and $F_6^2 \cup \{v_4v_7\}$ are p_6 -choosable.
- g) $F_7^2 \cup \{v_4v_8\}$ is p_7 -choosable.
- h) F_8^2 is 6-choosable.
- i) F_9^2 is p_9 -choosable.
- j) $F_{10}^2 \cup \{v_9v_{10}\}$ is p_{10} -choosable.
- k) F_{11}^2 is p_{11} -choosable.

- l) $F_{12}^2 \cup \{v_2v_9\}$ and $F_{12}^2 \cup \{v_6v_9\}$ are p_{12} -choosable.
- m) $F_{13}^2 \cup \{v_4v_8\}$ is p_{13} -choosable.
- n) $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ is p_{14} -choosable.
- o) $F_{15}^2 \cup \{v_4v_8\}$ is p_{15} -choosable.
- p) F_{16}^2 is 6-choosable.
- q) F_{17}^2 is p_{17} -choosable.

Proof.

- a) In $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$, $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$ is p_1 -greedy. So by Lemma 14, $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ is p_1 -choosable.
- b) In $F_2^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_2 -nice and $p_2(v_2) > p_2(v_1)$. So by Lemma 15, $F_2^2 \cup \{v_4v_7\}$ is p_2 -choosable.
By symmetry, one shows that $F_2^2 \cup \{v_1v_4\}$ is p_2 -choosable.
- c) In $F_3^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_3 -nice and $p_3(v_2) > p_3(v_1)$. So by Lemma 15, $F_3^2 \cup \{v_4v_8\}$ is p_3 -choosable.
- d) Let L be a 6-list-assignment of F_4^2 . $(v_3 < v_2 < v_4 < v_6 < v_8 < v_5 < v_7 < v_1)$ is 6-nice. Thus, by Lemma 15, we may assume that $L(v_3) = L(v_1)$. Now, $(v_5 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_3)$ is 6-nice. Hence F_4^2 is L -colourable according to Lemma 16 if $L(v_5) \cap L(v_1) \neq \emptyset$ or Lemma 15 otherwise.
- e) In $F_5^2 \cup \{v_1v_5\}$, $(v_1 < v_5 < v_2 < v_4 < v_3)$ is p_5 -greedy. So by Lemma 14, $F_5^2 \cup \{v_1v_5\}$ is p_5 -choosable.
- f) In $F_6^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_6 -nice and $p_6(v_2) > p_6(v_1)$. So by Lemma 15, $F_6^2 \cup \{v_4v_7\}$ is p_6 -choosable.
By symmetry, one shows that $F_6^2 \cup \{v_1v_4\}$ is p_6 -choosable.
- g) In $F_7^2 \cup \{v_4v_8\}$. $(v_2 < v_4 < v_8 < v_5 < v_7 < v_6 < v_3 < v_1)$ is p_7 -nice and $p_7(v_2) > p_7(v_1)$. So by Lemma 15, $F_7^2 \cup \{v_4v_8\}$ is p_7 -choosable.
- h) Let L be a p_8 -list-assignment of F_8^2 , $(v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5 < v_3)$ is p_8 -greedy, so by Lemma 15, we may assume that $L(v_1) = L(v_3)$. Now $(v_3 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5)$, so by Lemma 16, F_8^2 is L -colourable.
- i) In F_9^2 , $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$ is p_9 -nice and $p_9(v_4) > p_9(v_5)$. So by Lemma 15, F_9^2 is p_9 -choosable.
- j) Let L be a p_{10} -list-assignment of $F_{10}^2 \cup \{v_9v_{10}\}$. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ is p_{10} -nice. Thus, by Lemma 15, we may assume that $L(v_2) \subset L(v_1)$. Analogously, by symmetry, we may assume that $L(v_2) \subset L(v_3)$ and $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_{10} -good, by Lemma 16, $F_{10}^2 \cup \{v_9v_{10}\}$ is L -colourable.
- k) In F_{11}^2 , $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$ is p_{11} -greedy. So by Lemma 14, F_{11}^2 is p_{11} -choosable.

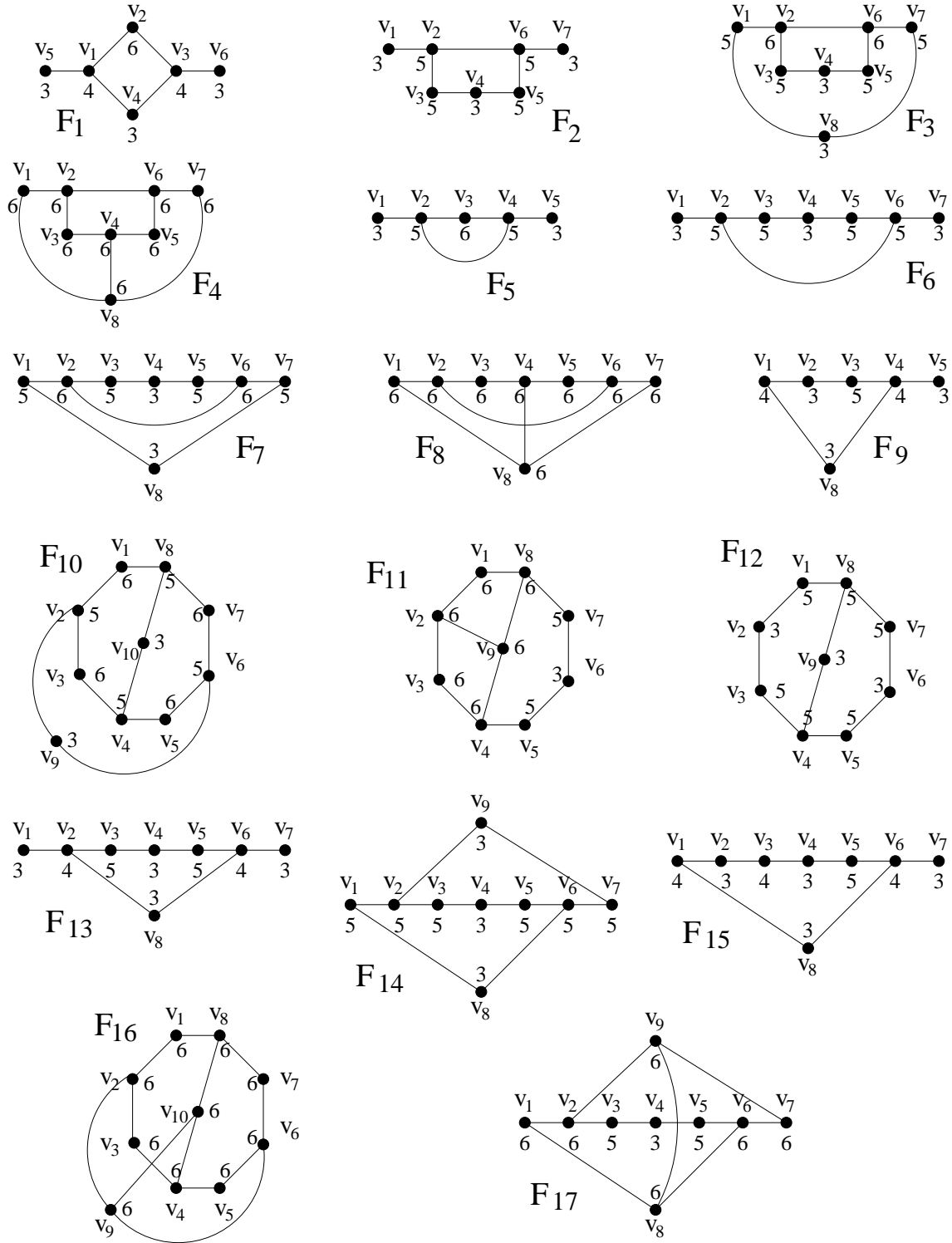


Figure 3: The graphs F_i and functions p_i for $1 \leq i \leq 12$

l) Let L be a p_{12} -list-assignment of $F_{12}^2 \cup \{v_2v_9\}$. Then $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$ and $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$ are p_{12} -nice so by Lemma 15, we may assume that $L(v_2) \subset L(v_3) \cap L(v_1)$. Moreover, $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$ is p_{12} -nice so by Lemma 15, we may assume that $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Thus, by Lemma 16, since $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_{12} -good, $F_{12}^2 \cup \{v_2v_9\}$ is L -colourable.

By symmetry, one shows that $F_{12}^2 \cup \{v_6v_9\}$ is p_{12} -choosable.

m) In $F_{13}^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p_{13} -nice and $p_{13}(v_2) > p_{13}(v_1)$. So by Lemma 15, $F_{13}^2 \cup \{v_4v_8\}$ is p_{13} -colourable.

n) Let L be a p_{14} -list-assignment of $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ ($v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3$) is p_{14} -nice. So by Lemma 15, $L(v_1) = L(v_3)$. Now, $(v_4 < v_1 < v_8 < v_9 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p_{14} -nice. Hence $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ is L -colourable according to Lemma 16 if $L(v_4) \cap L(v_1) \neq \emptyset$ or Lemma 15 otherwise.

o) In $F_{15}^2 \cup \{v_4v_8\}$. $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$ is p_{15} -nice and $p_{15}(v_6) > p_{15}(v_7)$. So by Lemma 15, $F_{15}^2 \cup \{v_4v_8\}$ is p_{15} -choosable.

p) Let L be a 6-list-assignment of F_{16}^2 . $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ is 6-nice. Thus, by Lemma 15, we may assume that $L(v_2) = L(v_1)$. Analogously, by symmetry, we may assume that $L(v_2) = L(v_3) = L(v_4)$. It follows that $L(v_1) = L(v_4)$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is 6-good, by Lemma 16, F_{16}^2 is L -colourable.

q) In F_{17}^2 . $(v_9 < v_8 < v_2 < v_6 < v_4 < v_5 < v_3 < v_7 < v_1)$ is p_{17} -greedy, so by Lemma 14, F_{17}^2 is p_{17} -choosable.

□

Proof of Lemma 6.

5)] Suppose for a contradiction that v_1 and v_3 are in Z_5 . Let v_5 (resp. v_6) be the neighbour of v_1 (resp. v_3) distinct from v_2 and v_4 . By Lemma 5-2), $v_5 \neq v_6$. $(G[S], p_S) = (F_1, p_1)$ and $G^2[S] \subset F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$. So Lemma 24 contradicts Lemma 12.

Suppose for a contradiction that, in K , a vertex v_4 of $Y_{1,2,2}$ is adjacent to two vertices of $Y_{1,2,2}$ v_2 and v_6 by 2-edges. According to 5), $v_2 \neq v_6$. Let v_3 and v_5 be the 2-neighbours of v_4 common with v_2 and v_6 respectively, and v_1 (resp. v_7) be the neighbour of v_2 (resp. v_6) not adjacent to v_4 . Set $S = \{v_1, \dots, v_6, v_7\}$.

- Assume that v_2v_4 is an edge. For $i \in \{1, 4, 7\}$, let w_i be the neighbour of v_i not in S . $(G[S], p_S) = (F_2, p_2)$ and $G^2[S] \subset F_2^2 \cup \{v_1v_4, v_4v_7, v_1v_7\}$. Thus, by Lemmas 24 and 12, $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $(G[T], p_T) = (F_3, p_3)$ and $G^2[T] \subset F_3^2 \cup \{v_4v_8\}$. So Lemma 24 contradicts Lemma 12. If not then $G[T] = G = F_4$, so G is 6-choosable, by Lemma 24. This is a contradiction.
- Assume that $v_1 = v_7$. Clearly $(G - S)^2 = G^2 - S$ because every vertex of C has at most one neighbour in $G - S$. Moreover C has no chord by Lemma 5-2), $C^2 = G^2[S]$. Finally $p_S(v_i) = 4$ if i is even and $p_S(v_i) = 3$ otherwise. C^2 is a cycle+triangle graph, thus, by Theorem 21, it is 3-choosable and so p_S -choosable. This contradicts Lemma 12.
- Assume that $v_1 \neq v_7$. For $i \in \{1, 2, 4, 6, 7\}$, let w_i be the neighbour of v_i not in S . Let $W = \{w_1, w_2, w_4, w_6, w_7\}$.

Suppose first that $W \cap S \neq \emptyset$. Since G is simple, $w_1 \neq v_2$ and $w_7 \neq v_6$ and by Lemma 5-3), $w_1 \neq v_4$ and $w_7 \neq v_4$. Moreover, by (5), $w_1 \neq v_6$ and $w_7 \neq v_2$. Then, by symmetry, we only need to consider the cases $w_2 = v_4$ and $w_2 = v_6$.

- Suppose that $w_2 = v_4$. Set $R = \{v_1, v_2, v_3, v_4, v_3\}$, then $(G[R], p_R) = (F_2, p_2)$ and $G^2[R] \subset F_2^2 \cup \{v_1 v_5\}$. Thus Lemma 24 contradicts Lemma 12.
 - Suppose that $w_2 = v_6$. Then $(G[S], p_S) = (F_3, p_3)$. If $w_1 \neq w_7$ then $G^2[S] \subset F_3^2 \cup \{v_1 v_4\}$ or $G^2[S] \subset F_3^2 \cup \{v_4 v_7\}$. So Lemma 24 contradicts Lemma 12.
- Hence we may assume that $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $(G[T], p_T) = (F_4, p_4)$ and $G^2[T] \subset F_4^2 \cup \{v_4 v_8\}$. So Lemma 24 contradicts Lemma 12. If $v_8 = w_4$ then $G = G[T] = F_5$ which, according to Lemma 24, is 6-choosable, a contradiction.

Hence, we may assume that $W \cap S = \emptyset$.

Note that by Lemma 5-2), $w_1 \neq w_2$ and $w_6 \neq w_7$.

Suppose $w_1 = w_4 = v_8$. Then let $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$ and w_8 the neighbour of v_8 . Then $(G[R], p_R) = (F_6, p_6)$. Recall that $w_2 \neq v_6$. So $G^2[R] = F_6^2$. Thus Lemma 24 contradicts Lemma 12.

Therefore, we may assume that $w_1 \neq w_4$ and, by symmetry, $w_4 \neq w_7$.

Suppose $w_1 = w_7$. Let $T = S \cup \{v_8\}$. Then $G[T]$ is the cycle C_8 and p_T is the function p defined in Lemma 22. So by Lemmas 22 and 12, $G^2[T] \neq C_8^2$. It follows that either $w_2 = w_6$ or $w_4 = w_8$ with w_8 be the neighbour of v_8 not in S .

- Suppose $w_2 = w_6 = v_9$ and $w_4 = w_8 = v_{10}$. Set $W = \{v_1, \dots, v_{10}\}$. If $v_9 v_{10} \notin E(G)$ then $(G[W], p_W) = (F_7, p_7)$ and $G^2[W] \subset F_7^2 \cup \{v_9 v_{10}\}$; so Lemma 24 contradicts Lemma 12. If not, $G = G[W] = F_{13}$, so G^2 is 6-choosable, according to Lemma 24, a contradiction.
 - Suppose $w_2 = w_4 = w_6 = v_9$. Setting $U = \{v_1, \dots, v_9\}$, we have $G[U] = F_8$ and $G^2[U] = F_8^2$. Hence Lemma 24 contradicts Lemma 12.
- By symmetry, we get a contradiction if $w_2 = w_6 = w_8$, $w_2 = w_4 = w_8$ or $w_4 = w_6 = w_8$.
- Suppose $w_4 = w_8 = v_9$, $w_2 \neq v_9$, $w_6 \neq v_9$ and $w_2 \neq w_6$. Setting $U = \{v_1, \dots, v_9\}$, we have $(G[U], p_U) = (F_9, p_9)$ and $G^2[U] \subset F_9^2 \cup \{v_2 v_9\}$ or $G^2[U] \subset F_9^2 \cup \{v_6 v_9\}$. Hence Lemma 24 contradicts Lemma 12.
- By symmetry, we get a contradiction if $w_2 = w_6 = v_9$, $w_4 \neq v_9$, $w_8 \neq v_9$ and $w_4 \neq w_8$.

Therefore, we may assume that $w_1 \neq w_7$.

Suppose that $w_2 = w_6 = v_8$. Let $T = S \cup \{v_8\}$. Then $(G[T], p_T) = (F_{10}, p_{10})$, and $G^2[T] \subset F_{10}^2 \cup \{v_4 v_8\}$, since w_1, w_4 and w_7 are distinct vertices. Hence Lemma 24 contradicts Lemma 12. Therefore, we may assume that $w_2 \neq w_6$.

Suppose that $w_1 = w_6 = v_8$ and $w_2 = w_7 = v_9$. Let $U = S \cup \{v_8, v_9\}$. If $v_8 v_9 \notin E(G)$ then $(G[U], p_U) = (F_{11}, p_{11})$, and $G^2[U] \subset F_{11}^2 \cup \{v_4 v_8, v_8 v_9, v_9 v_4\}$. If not, then $(G[U], p_U) = (F_{14}, p_{14})$, and $G^2[U] = F_{14}^2$. In both cases, Lemma 24 contradicts Lemma 12.

Therefore, we may assume that $w_1 \neq w_6$ or $w_2 \neq w_7$. By symmetry, we may assume that $w_2 \neq w_7$.

Suppose $w_1 = w_6 = v_8$. Let $T = S \cup \{v_8\}$ and let w_8 be the neighbour of v_8 not in S . Then $(G[T], p_T) = (F_{12}, p_{12})$ and $G^2[T] \subset F_{12}^2 \cup \{v_4 v_8\}$. Hence Lemma 24 contradicts Lemma 12.

Therefore, we may assume that $w_1 \neq w_6$.

Hence we have $w_1 \neq w_4, w_6, w_7, w_2 \neq w_6, w_7$ and $w_4 \neq w_7$ then $G[S]^2 = G^2[S]$. Thus Proposition 23 contradicts Lemma 12.

□

3.3 Proof of Lemma 11

Definition 25 For $1 \leq j \leq 4$, let I_j and q_j be the graphs and function depicted Figure 4

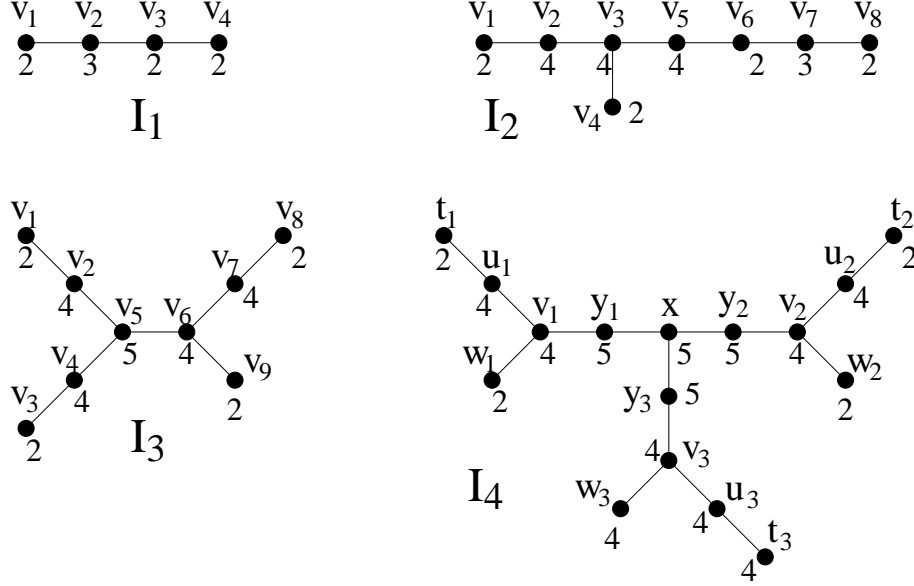


Figure 4: The graphs I_j and functions q_j , $1 \leq j \leq 4$

Lemma 26 For $1 \leq j \leq 4$, I_j^2 is q_j -choosable.

Proof.

- Let L be a q_1 -list-assignment of I_1^2 . $(v_4 < v_3 < v_1 < v_2)$ and $(v_1 < v_3 < v_4 < v_2)$ are q_1 -nice, so by Lemma 15, we may assume that $L(v_1) \cup L(v_4) \subset L(v_2)$. Hence $L(v_1) \cap L(v_4) \neq \emptyset$. But $(v_4 < v_1 < v_3 < v_2)$ is q_1 -good. Thus by Lemma 16, I_1^2 is L -colourable.
- Let L be a q_2 -list-assignment of I_2^2 .
Suppose first that $L(v_3) \not\subset L(v_1) \cup L(v_6)$. Then choose $c(v_3)$ in $L(v_3) \setminus (L(v_1) \cup L(v_6))$ and $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$. Since I_1^2 is q_1 -choosable, one can extend c to $\{v_5, v_6, v_7, v_8\}$. Then one can find $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$ and $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$. So, we may assume that $L(v_3) \subset L(v_1) \cup L(v_6)$, so $L(v_3) = L(v_1) \cup L(v_6)$ and $L(v_1) \cap L(v_6) = \emptyset$.
Now colour v_3 and v_6 the same colour $c_6 \in L(v_6)$. Then proceed greedily according to $(v_4 < v_8 < v_7 < v_5 < v_2 < v_1)$. It is possible since $c_6 \notin L(v_1)$.
- Let L be a q_3 -list-assignment of I_3^2 . Assign to v_5 a colour c_5 in $L(v_5) \setminus (L(v_1) \cup L(v_9))$ and to v_6 a colour in $L(v_6) \setminus (L(v_8) \cup \{c_5\})$. Then colour the remaining vertices greedily according to $(v_3 < v_4 < v_2 < v_1 < v_1 < v_9 < v_7 < v_8)$ to get an L -colouring of I_3^2 .
- Let L be a q_4 -list-assignment of I_4^2 . Pick $c(y_1)$ in $L(y_1) \setminus L(w_1)$, $c(y_2)$ in $L(y_2) \setminus (L(w_2) \cup \{c(v_1)\})$, $c(y_3)$ in $L(y_3) \setminus (L(w_3) \cup \{c(v_1), c(v_2)\})$ and $c(x)$ in $L(x) \setminus \{c(v_1), c(v_2), c(v_3)\}$. Since I_1^2 is q_1 -choosable, one can extend c to a colouring of I_4^2 .

□

Proof of Lemma 6.

- 3) Suppose that a vertex v_3 of $Y_{2,2,3}$ and v_6 of $Y_{1,2,3}$ are adjacent via a 2-edge in K . Then the subgraph of G induced by v_3, v_6 and the 2-vertices of their incident threads is I_2 . Since G has girth at least 13, then $G^2[I_2] = I_2^2$ and $(G - V(I_2))^2 = G^2 - V(I_2)$, so Lemma 26 contradicts Lemma 12.
- 4) Suppose that a vertex v_5 of $Y_{1,3,3}$ and v_6 of $Y_{1,2,3}$ are adjacent via a 1-edge in K . Then the subgraph of G induced by v_5, v_6 and the 2-vertices of their incident threads is I_3 . Then Lemma 26 contradicts Lemma 12.
- 5) Suppose that a vertex x of $Y_{2,2,2}$ is adjacent to three vertices v_1, v_2 and v_3 of $Y_{2,2,3}$ in K . Then the subgraph of G induced by x, v_1, v_2, v_3 and the 2-vertices of their incident threads is I_4 . Then Lemma 26 contradicts Lemma 12.

□

Acknowledgement

The author would like to thank André Raspaud and Riste Škrekovski for stimulating discussions.

References

- [1] Z. Dvořák, R. Škrekovski and M. Tancer, List-colouring squares of sparse subcubic graphs, IMFM Preprint PS-985, 2005.
- [2] H. Fleischner and M. Stiebitz, A solution to a Colouring problem of P. Erdős, *Discrete Math.*, **101** (1992), 39–48.
- [3] M. Montassier and A. Raspaud, A note on 2-facial coloring of plane graphs, *Technical report* RR-1341-05, LaBRI, 2005.
- [4] G. Wegner, Graphs with given diameter and a coloring problem, *Technical Report*, University of Dortmund, Germany, 1977.