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DESIGN OF FAULT-TOLERANT ON-BOARD NETWORKS

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Design of fault-tolerant on-board networks

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Abstract

An (n,k,r)-*network* is a triple $N = (G, in, out)$ where $G = (V,E)$ is a graph and in, out are integral functions defined on V called *input* and *output* functions, such that for any $v \in V$, $in(v) + out(v) + deg(v) \leq 2r$ with $deg(v)$ the degree of v in the graph G. The total number of inputs is $in(V) = \sum_{v\in V} in(v) = n$, and the total number of outputs is $out(V) = \sum_{v\in V} out(v) = n+k$.

An (n,k,r)-network is *valid*, if for any *faulty* output function out' (that is such that $out'(v) \leq out(v)$ for any $v \in V$, and $out'(V) = n$), there are n edge-disjoint paths in G such that each vertex $v \in V$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out'(v)$ paths.

We investigate the design problem of determining the minimum number of vertices in a valid (n,k,r)-network and of constructing minimum (n,k,r)-networks, or at least valid (n,k,r)-networks with a number of vertices close to the optimal value.

We first show $\frac{3n+k}{2r-2+\frac{3r^2}{k}} \leq \mathcal{N}(n,k,r) \leq \left\lceil \frac{k+2}{2r-2} \right\rceil \frac{n}{2}$. We prove a better upper bound when $r \geq k/2$: $\mathcal{N}(n,k,r) \leq \frac{r-2+k/2}{r^2-2r+k/2} n + O(1)$. Finally, we give the exact value of $\mathcal{N}(n,k,r)$ when $k \leq 6$ and exhibit the corresponding networks.

1 Introduction

In this paper, we study networks having the following informal fault tolerance property: the network interconnects a set of input ports (where signals enter the network) with a set of output ports (where signals leave the network) and for any set of at most k output port failures, there exists a set of edge-disjoint paths connecting the input ports to the non faulty output ports. The network is built with degree $2r$ switches, links are bidirectional (undirected graph) and the paths connecting inputs to outputs are link-disjoint.

This problem was originally motivated by a design problem raised by Alcatel Space Industries : For this application, the interconnection network is embarqued on a satellite and connects the satellite up links to the satellite down links. Before being transmitted downward, the signals must be amplified, so the output ports are indeed connected to amplifiers. Along the time, amplifiers are subject to failure, and the designer must be able to guarantee that the network will tolerate some given number k of amplifiers failures (which correspond to a high enough probability of survival during the satellite life). Since the satellite cannot be maintained, this is done by adding redundant amplifiers. Because each switching device induces a high over-cost, the aim is to minimize the number of switches.

The problem was initially studied in [BDD02] (degree 4 switches, $k \leq 4$ failures), and then in [BPT01] (degree 4 switches, up to 12 failures), some variation of the problem in which there are two

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kinds of inputs is considered in [BHT06, Hav06]. In this paper, we solve the problem for switches with a general (even) number of ports.

Generalizing the definitions (n, k)-networks introduced in [BDD02] and [BPT01], we define (n, k, r)-networks as follows : An (n, k, r)-*network* is a triple $N = (G, in, out)$ where $G = (V, E)$ is a graph and in, out are integral functions defined on V called *input* and *output* functions, such that for any $v \in V$, its number of ports $por(v)$ defined by $por(v) = in(v) + out(v) + deg(v)$ is at most $2r$. ($deg(v)$ denotes the degree of v in the graph G, that is the number of edges of G incident to v.) An $i|o$-*switch* or *switch of type* $i|o$ is a switch s connected to i inputs and o outputs, that is $in(s) = i$ and $out(s) = o$. The total number of inputs is $in(V) = \sum_{v \in V} in(v) = n$, and the total number of outputs is $out(V) = \sum_{v \in V} out(v) = n + k$.

Any integral function out' defined on V such that $out'(v) \leq out(v)$ for any $v \in V$, and $out'(V) = n$ is called a *faulty output function*. Note that $out(v) - out'(v)$ is the number of faults at vertex v. An (n, k, r)-network is *valid*, if for any faulty output function out', there are n edge-disjoint paths in G such that each vertex $v \in V$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out'(v)$ paths.

Let us denote the minimum number of vertices in a valid (n, k, r)-network by $\mathcal{N}(n, k, r)$. A valid (n, k, r)-network with exactly $\mathcal{N}(n, k, r)$ vertices is called a *minimum* (n, k, r)-network.

The design problem consists of determining $\mathcal{N}(n, k, r)$ and of constructing minimum (n, k, r)-networks, or at least valid (n, k, r)-networks with a number of vertices close to the optimal value.

We present now an example: We would like to construct valid $(4, 4, 2)$-networks. A first solution is depicted in Figure 1. The network N_1 is composed of eight switches u_i, v_i for $1 \leq i \leq 4$. The associated graph $G = (V, E)$ is the grid 4×2. The input and ouput functions are defined as follows: So, $in(v_i) = 1$, $in(u_i) = 0$ for $1 \leq i \leq 4$ and $out(v_2) = out(v_3) = 0$, $out(v_1) = out(u_2) = out(u_3) = out(v_4) = 1$, $out(u_1) = out(u_4) = 2$.

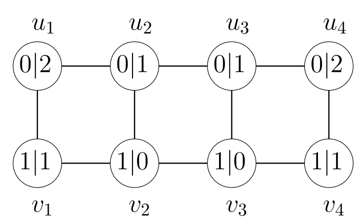

Figure 1: A first solution: the network N_1.

For any faulty ouput function out', it is easy to see that there are 4 edge-disjoint paths in G such that each vertex $v \in V$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out'(v)$ paths. This implies that this network is valid. It follows that $\mathcal{N}(4, 4, 2) \leq 8$. But this solution is not best possible. The network depicted in Figure 2 is valid and contains only five switches. Moreover we can prove that $\mathcal{N}(4, 4, 2) = 5$.

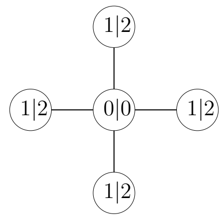

Figure 2: A best solution: the network N_2.

In this paper, we first give some general lower and upper bounds of $\mathcal{N}(n,k,r)$. We then give some exact bounds when $k \leq 6$ and some minimum networks. We prove the following bounds:

1. For any n, k, r, $\mathcal{N}(n,k,r) \leq \left\lceil \frac{k+2}{2r-2} \right\rceil \frac{n}{2}$.
2. For $k \geq 3$ and $r \geq k/2$, $\mathcal{N}(n,k,r) \leq \frac{r-2+k/2}{r^2-2r+k/2}n + O(1)$.
3. For any n, k, r, $\frac{3n+k}{2r-2+\frac{3r^2}{k}} \leq \mathcal{N}(n,k,r)$.
4. For $k \in \{1,2\}$ and $r \geq 1$, $\mathcal{N}(n,k,r) = \left\lceil \frac{n}{r-1} \right\rceil$.
5. For $k \in \{3,4\}$ and $r \geq 3$, $\mathcal{N}(n,k,r) = \frac{r}{r^2-2r+2}n + \Theta(1)$.
6. For $k \in \{5,6\}$ and $r \geq 7$, $\mathcal{N}(n,k,r) = \frac{r+1}{r^2-2r+3}n + \Theta(1)$.

2 Preliminaries

If $k \leq k'$, we can easily obtain an (n,k,r)-network from an (n,k',r)-network by removing an arbitrary set of $k'-k$ outputs.

Proposition 1 *If $k \leq k'$ then $\mathcal{N}(n,k,r) \leq \mathcal{N}(n,k',r)$.*

Before we proceed with the lower and upper bounds on $\mathcal{N}(n,k,r)$, we make an observation on the structure of (n,k,r)-networks. Free to add an edge between two unused ports as long as there are two of them, we can assume without loss of generality that in an (n,k,r)-network all switches have $2r$ ports, with an exception of one having $2r-1$ ports, if k is odd. Let $\epsilon(k) = 1$ if k is odd and 0 otherwise.

Proposition 2 *There is a minimum (n,k,r)-network with $\epsilon(k)$ switch with $2r-1$ ports and all the others with $2r$ ports.*

A switch with $2r-1$ ports is called *defective*. In the remainder of this paper, we assume that all the switches of an (n,k,r)-network have $2r$ ports except has $\epsilon(k)$ which are defective.

2.1 Cut criterion

All the results that will be proved in this paper rely on Lemma 3, which gives us a necessary and sufficient condition, called the *cut criterion*, for an (n, k, r)-network to be valid. It extends a result of [BPT01] for $r = 2$ and easily follows from the Ford-Fulkerson Theorem [FF62](Theorem 1.1 p.38).

Let W be a set of switches of an (n, k, r)-network. $in(W)$ and $out(W)$ are its number of inputs and outputs respectively: $in(W) = \sum_{v\in W} in(v)$ and $out(W) = \sum_{v\in W} out(v)$. $\Gamma(W)$ denotes the edges with one endvertex in W and the other in $\overline{W} = V \setminus W$ and $deg(W) = |\Gamma(W)|$.
The *excess* of W, denoted $exc(W)$, is defined by $exc(W) = deg(W) - in(W) + out(W) - \min\{out(W), k\}$.

Lemma 3 (Cut criterion) *An (n, k, r)-network is valid if and only if for every set of vertices $W \subset V$ has non-negative excess, that is $deg(W) - in(W) + out(W) - \min\{out(W), k\} \geq 0$.*

The proof of this lemma is identical to the proof of [BPT01]. We give it here for sake of completeness.

Proof Let out' be a fixed faulty output function, then a supply/demand flow problem is defined by an integral (not necessarily positive) demand at each vertex v. In our case, the demand of a vertex $v \in V$ is $demand(v) = out'(v) - in(v)$. (Note that $demand(V) = 0$, which is always the case for supply/demand problems.) A variant of the Ford-Fulkerson Theorem states that the supply/demand problem is feasible if and only if

$$\forall W \subset V : \quad deg(W) \geq demand(\overline{W}) = out'(\overline{W}) - in(\overline{W}) = in(W) - out'(W).$$

It follows that the (n, k, r) network is valid if and only if

$$\forall W \subset V : \quad deg(W) \geq in(W) - \min\{out'(W) \mid out' \text{ faulty output function}\} \qquad (1)$$

By definition, $\min\{out'(W) \mid out' \text{ faulty output function}\}$ is the minimum number of non-faulty outputs in W. This minimum is attained either by choosing all the outputs in W to be faulty when $out(W) \leq k$, or by choosing k outputs in W to be faulty when $out(W) \geq k$.

Hence, $\min\{out'(W) \mid out' \text{ faulty output function}\} = out(W) - \min\{out(W), k\}$. The property follows then from Equation (1). □

In order to prove that a network is valid, by Lemma 3, we need to prove that every set of switches has non-negative excess. We now prove that it is indeed sufficient to prove it for connected sets.

Lemma 4 *If W is not connected and $exc(W) < 0$ then W has a connected component W_1 such that $exc(W_1) < 0$. Hence a network is valid if and only if every connected subset has non-negative excess.*

Proof Let W_i, $1 \leq i \leq l$, be the connected component of W. Then $exc(W) = \sum_{i=1}^{l} exc(W_i)$. So if $exc(W) < 0$, one of the W_i has also negative excess. □

We now strengthen again Lemma 4, by showing that to check that an (n, k, r)-network is valid it is only necessary to check the cut criterion for *essential* set of vertices.
Let N be an (n, k, r)-network. Let X be a set of S-switches. We denote by $\mathcal{B}(X)$ the set of blocks adjacent to X. A set W of vertices of N is *essential* if there exists a proper subset X of S (i.e. $X \neq \emptyset$ and $X \neq S$) such that $W = X \cup \bigcup_{B\in\mathcal{B}(X)} B$ and W is connected.

Lemma 5 *An (n, k, r)-network is valid if and only if every essential set of vertices has non-negative excess.*

Proof The proof follows easily the following assertion whose proof is straightforward:
Let W be a set of vertices of an (n, k, r)-network. Assume that W is adjacent to a vertex $v \notin W$ such that $deg(v) \leq in(v) - out(v) + 2$ then $exc(W \cup \{v\}) \leq exc(W)$.

Indeed every block-switch v satisfies $deg(v) \leq in(v) - out(v) + 2$. □

3 Upper bounds

In this section, we present two constructive processes which, from some specific valid networks, allow us to construct some bigger valid networks.

3.1 First constructive process

In this process, we distinguish two cases following the parity of k.

3.1.1 First constructive process for even k

Construction. Let k be even. For $i = 1, 2$, let $N_i = (G_i, in_i, out_i)$ be an (n_i, k, r)-network with a set $A_i = \{v_i^1, \ldots, v_i^{k/2}\}$ of $k/2$ switches in N_i connected to at least two outputs (i.e. $out_i(v_i^j) \geq 2$ for $1 \leq j \leq k/2$). We construct the $(n_1 + n_2, k, r)$-network $N_1 \oplus N_2 = (G, in, out)$ from N_1 and N_2 as follows: we remove on each v_i^j one output ($out(v_i^j) = out_i(v_i^j) - 1$) and we add a link between v_1^j and v_2^j for $1 \leq j \leq k/2$. Let $M = \{v_1^j v_2^j : 1 \leq j \leq k/2\}$ be the set of added links. The network $N_1 \oplus N_2$ has $s_1 + s_2$ switches.

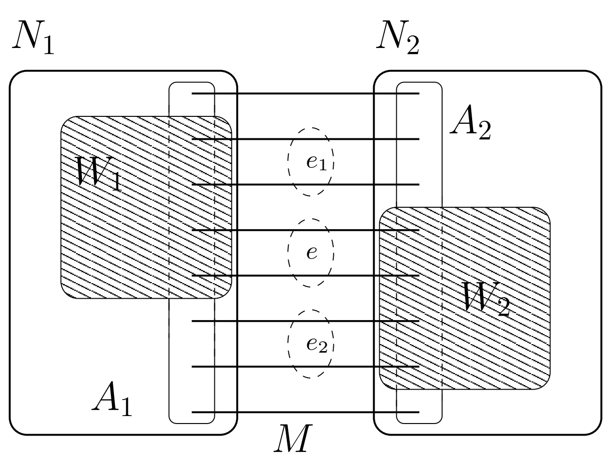

Figure 3: The first constructive process for k even.

The following theorem is an extension of Theorem 9 of [BDD02].

Theorem 6 *Let k be even. Let N_1 be a valid (n_1, k, r)-network with s_1 switches and let N_2 be a valid (n_2, k, r)-network with s_2 switches both containing at least $k/2$ switches connected to at least two ouputs. Then, $N_1 \oplus N_2$ is a valid $(n_1 + n_2, k, r)$-network containing $s_1 + s_2$ switches.*

Proof

By Lemma 4 we need to prove that for any set W of switches of N has non-negative excess that is

$$deg(W) \geq in(W) - out(W) + \min\{out(W), k\}$$

Let W be a connected set of switches and for $i = 1, 2$ let $W_i = V(G_i) \cap W$. We denote by e the number of links of M between W_1 and W_2 and e_1 (resp. e_2) the number of links of M between W_1 (resp. W_2) and the switches of N_2 (resp. N_1) not in W_2 (resp. W_1).

By construction, we have the following:

$$out(W) = out_1(W_1) + out_2(W_2) - e_1 - e_2 - 2e \quad (2)$$

$$in(W) = in_1(W_1) + in_2(W_2) \quad (3)$$

$$deg(W) = deg_1(W_1) + deg_2(W_2) + e_1 + e_2 \quad (4)$$

Since N_i is valid, the cut criterion implies that :

$$deg_i(W_i) \geq in_i(W_i) - out_i(W_i) + \min\{out_i(W_i), k\}$$

where $deg_i(W_i)$ is the degree of W_i in N_i. We consider the cases following the value of $\min\{out_i(W_i), k\}$.

Case 1. Suppose that $out_1(W_1) \geq k$ and $out_2(W_2) \geq k$. Hence the cut criterion implies that for $i = 1, 2$, $deg_i(W_i) \geq in_i(W_i) + k - out_i(W_i)$ and so,

$$deg_1(W_1) + deg_2(W_2) \geq in_1(W_1) + in_2(W_2) + 2k - out_1(W_1) - out_2(W_2).$$

Hence, by (2), (3) and (4), we obtain

$$deg(W) \geq in(W) - out(W) + 2k - 2e \geq in(W) - out(W) + k.$$

Case 2. Suppose that $out_1(W_1) < k$ and $out_2(W_2) < k$. For $i = 1, 2$, we have $deg_i(W_i) \geq in_i(W_i)$, so

$$deg_1(W_1) + deg_2(W_2) \geq in_1(W_1) + in_2(W_2).$$

So, by (3) and (4), $deg(W) \geq in(W) \geq in(W) - out(W) + \min\{out(W), k\}$.

Case 3. Suppose that $out_1(W_1) < k$ and $out_2(W_2) \geq k$. Then, $deg_1(W_1) \geq in_1(W_1)$ and $deg_2(W_2) \geq in_2(W_2) + k - out(W_2)$. So,

$$deg_1(W_1) + deg_2(W_2) \geq in_1(W_1) + in_2(W_2) + k - out_2(W_2).$$

By (2), (3), et (4), we obtain:

$$deg(W) \geq in(W) + k - out(W) + out_1(W_1) - 2e.$$

Moreover, by construction, $out_1(W_1) \geq 2e$ since each vertex of W_1 incident to an edge of M satisfi es $out_1 \geq 2$. Hence,

$$deg(W) \geq in(W) - out(W) + k.$$

□

3.1.2 First constructive process for odd k

Construction. Let $k = 2p+1$ be odd. For $i = 1, 2$, let $N_i = (G_i, in_i, out_i)$ be an (n_i, k, r)-network with a set $A_i = \{v_i^1, \ldots, v_i^p\}$ of p switches in N_i connected to at least two outputs (i.e. $out_i(v_i^j) \geq 2$ for $1 \leq j \leq p$). Let w_1 be a switch of $V(G_1) \setminus A_1$ connected to an output ($out_1(w_1) \geq 1$) or a vertex of A_1 with at least 3 outputs ($out_1(w_1) \geq 3$). Let z_2 be the unique switch of N_2 with $2r - 1$ ports. We construct the $(n_1 + n_2, k, r)$-network $N_1 \oplus N_2$ from N_1 and N_2 as follows: we remove on each v_i^j one output and we add a link between v_1^j and v_2^j for $1 \leq j \leq k/2$. Moreover we remove an output to w_1 and connect w_1 to z_2. The network $N_1 \oplus N_2$ has $s_1 + s_2$ switches.

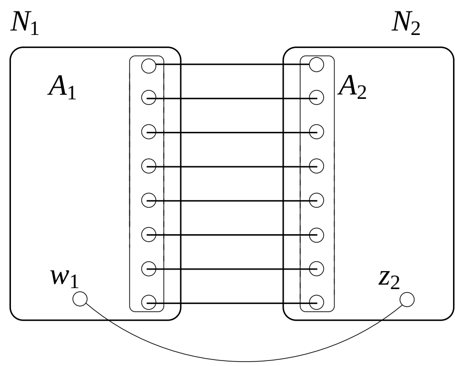

Figure 4: The first constructive process for k odd.

We will now prove an analog of Theorem 6 for odd k. Therefore, we need the following well-known lemma:

Lemma 7 (folklore) *Let u and v be two vertices of a graph G. If u and v have both odd degree and all the other vertices have even degree, then there is a path in G with endvertices u and v.*

Theorem 8 *Let $k = 2p + 1$ be odd. Let N_1 be a valid (n_1, k, r)-network with s_1 switches and let N_2 be a valid (n_2, k, r)-network with s_2 switches containing both at least p switches connected to at least two ouputs. Then, $N_1 \oplus N_2$ is a valid $(n_1 + n_2, k, r)$-network containing $s_1 + s_2$ switches.*

Proof Let out' be a faulty output function such that $out'(N_1 \oplus N_2) = n_1 + n_2$.

Let A'_2 be the set of vertices v of A_2 such that $out(v) \geq 3$ and $A''_2 = A_2 \setminus A'_2$.

Suppose first that there is a switch s_2 of $V(G_2) \setminus A''_2$ such that $out'(s_2) < out(s_2)$. Let $M_1 = (G_1, in_1, sor_1)$ and $M_2 = (G_2, in_2, sor_2)$ with $sor_1(v) = out_1(v)$ if $v \in V(G_1) \setminus \{w_1\}$ and $sor_1(w_1) = out_1(w_1) - 1$, and $sor_2(v) = out_2(v)$ if $v \in V(G_1) \setminus \{s_2\}$ and $sor_2(s_2) = out_2(s_2) - 1$. For $i = 1, 2$, the network M_i is a valid $(n_i, k - 1, r)$-network and every vertex v of A_i satisfies $sor_i(v) \geq 2$ by definition of w_1 and because s_2 is not in A''_2. By Theorem 6, $M_1 \oplus M_2$ is a valid $(n_1 + n_2, k - 1, r)$-network. So we can find $n_1 + n_2$ edge-disjoint paths in $M_1 \oplus M_2$ such that each vertex $v \in V$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out'(v)$ paths and so in $N_1 \oplus N_2$ since the graph of $M_1 \oplus M_2$ is the one of $N_1 \oplus N_2$ minus the edge $w_1 z_2$.

Suppose now that for every vertex v of $V(G_2) \setminus A''_2$, $out'(v) = out(v)$. Let $p_2 = out(V(G_2)) - out'(V(G_2))$, $J_2 = \{j, 1 \leq j \leq p$ and $out'(v_2^j) < out(v_2^j)\}$. Clearly $|J_2| = p_2$. Set $J_1 = \{1, 2, \ldots, p\} \setminus J_2$.

Let us define out_1' by $out_1'(v) = out'(v) + 1$ if $v \in \{v_1^j, j \in J_1\} \cup \{w_1\}$ and $out_1'(v) = out'(v)$ if $v \in V(G_1) \setminus (\{v_1^j, j \in J_1\} \cup \{w_1\})$. Let w_2 be a vertex of $V(G_2) \setminus A_2''$. Let us define out_2' by $out_2'(v) = out'(v) - 1$ if $v \in \{v_2^j, j \in J_1 \setminus \{w_2\}\}$, $out_2'(v) = out'(v)$ if $v \in V(G_2) \setminus (\{v_2^j, j \in J_1\} \cup \{w_2\})$ and $out_2'(w_2) = out(w_2) - 2$ if $w_2 \in \{v_2^j, j \in J_1\}$ and $out_2'(w_2) = out(w_2) - 1$ otherwise.

For $i = 1, 2$, the function out_i' is a faulty output function of N_i. Since N_i is valid, one can find a set $\mathcal{P}_i$ of n_i edge-disjoint paths in N_i such that each vertex $v \in V(G_i)$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out_i'(v)$ paths. $\mathcal{P}_1 \cup \mathcal{P}_2$ is almost the set of desired paths. The only problems are that each vertex of $\{v_1^j, j \in J_1\} \cup \{w_1\}$ is the end of one path too much and each vertex of $v \in \{v_2^j, j \in J_1\} \cup \{w_2\}$ is the end of one path too few (If $w_2 \in \{v_2^j, j \in J_1\}$, then w_2 is the end of two paths too few). For any $j \in J_1$, let P_j be a path $\mathcal{P}_1$ ending in v_1^j and $Q_j = P_j v_1^j v_2^j$ and let P be a path of $\mathcal{P}_1$ ending in w_1.

Consider the graph H_2 obtained from G_2 by removing all the edges of the paths in $\mathcal{P}_2$. Let us show that H_2 has exactly two vertices with odd degree z_2 and w_2 unless $z_2 = w_2$. Let v be a vertex of $V(G_2) \setminus \{w_2, z_2\}$. If $v \in \{v_2^j, j \in J_2\} \setminus \{z_2\}$, then $out_2(v) = 2$ and v is the end of no paths of $\mathcal{P}_2$. So the number $e(v)$ of its incident edges in paths of $\mathcal{P}_2$ has the same parity than $in_2(v)$. Hence $deg_{H_2}(v) = 2r - out_2(v) - in_2(v) - e(v)$ is even. If $v \in \{v_2^j, j \in J_1\}$ then it is the end of $out_2(v) - 2$ paths of $\mathcal{P}_2$. So the number $e(v)$ of its incident edges in paths of $\mathcal{P}_2$ has the same parity than $in_2(v) + out_2(v)$. Hence $deg_{H_2}(v)$ is even. If $v \in V(G_2) \setminus A_2$, v is the end of $out_2(v)$ and the start of $in_2(v)$ paths of $\mathcal{P}_2$. So the number $e(v)$ of its incident edges in paths of $\mathcal{P}_2$ has the same parity than $in_2(v) + out_2(v)$. It follows that $deg_{H_2}(v)$ is even. Analogously, one shows that the degrees of w_2 and z_2 in H_2 are odd unlees $w_2 = z_2$.

Thus by Lemma 7, there is a path Q from z_2 to w_2. Now $(\mathcal{P}_1 \cup \mathcal{P}_2) \setminus (\{P_j, j \in J_1\} \cup \{P\}) \cup (\{Q_j, j \in J_1\} \cup \{Q\})$ is the desired set of paths. □

3.1.3 Derived upper bound

Observe that if N contains k switches connected to at least two outputs, then $N \oplus N$ contains also k such switches and we can apply recursively Theorem 6.

Corollary 9 *Let k be an integer. Let N_1 be a valid (n, k, r)-network with s switches containing k switches connected to at least two outputs. For any integer l, $N_l = N_1 \oplus N_{l-1}$ is a valid (ln, k, r)-network with ls switches.*

Havet [Hav06], showed that $\mathcal{N}(1, k, 2) = \left\lceil \frac{k}{2} \right\rceil$ and $\mathcal{N}(2, k, 2) = \left\lceil \frac{k+2}{2} \right\rceil$. Moreover there are optimum networks having all their switches adjacent to at least two outputs. We generalize these results, for general r:

Proposition 10 $\mathcal{N}(1, k, r) = \left\lceil \frac{k}{2r-2} \right\rceil$ *and* $\mathcal{N}(2, k, r) = \left\lceil \frac{k+2}{2r-2} \right\rceil$. *Moreover, there are optimum networks having all their switches adjacent to at least $2r - 2$ outputs.*

Proof Consider the network N_1 (resp. N_2) with s switches $v_1, \dots, v_s$ such that v_1 is a $0|2r-1$-switch (resp. $1|2r-2$-switch), v_s is a $1|2r-2$-switch, each v_i, $2 \le i \le s-1$ is a $0|2r-2$-switch and $(v_1, v_2, \dots, v_l)$ is a path. It is easy to check that N_1 and N_2 are a valid $(1, (2r-2)s, r)$-network and a valid $(2, (2r-2)s - 2, r)$-network. It follows that $\mathcal{N}(1, k, r) \le \left\lceil \frac{k}{2r-2} \right\rceil$ and $\mathcal{N}(2, k, r) \le \left\lceil \frac{k+2}{2r-2} \right\rceil$ by Proposition 1.

Moreover, the above upper bounds is tight since a valid network must be connected. □

Corollary 11

$$\mathcal{N}(n,k,r) \leq \left\lceil \frac{k+2}{2r-2} \right\rceil \frac{n}{2}$$

3.2 Second constructive process

Construction. Let k be an even integer and $r \geq k/2$. For $i = 1, 2$, let $N_i = (G_i, in_i, out_i)$ be an (n_i, k, r)-network containing an $r-k/2|r$-switches u_i. We construct the $(n_1+n_2-(r-k/2), k, r)$-network $N_1 \otimes N_2 = (G, in, out)$ from N_1 and N_2 as follows: we remove on each u_i $k/2$ outputs and we identify u_1 and u_2 in order to obtain an $r-k/2|r-k/2$-switch u^*.

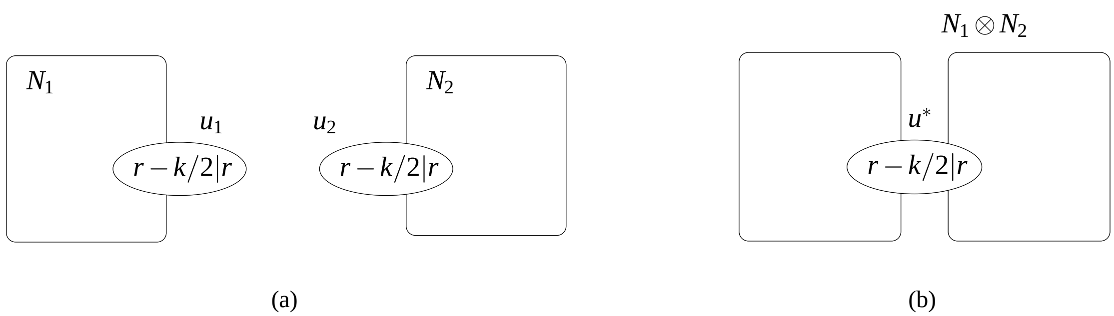

Figure 5: The second constructive process.

Theorem 12 *Let k be an even intger and $r \geq k/2$. Let $N_1 = (G_1, in_1, out_1)$ be a valid (n_1, k, r)-network with s_1 switches and let $N_2 = (G_2, in_2, out_2)$ be a valid (n_2, k, r)-network with s_2 switches both containing at least one $r-k/2|r$. Then, $N_1 \otimes N_2 = (G, in, out)$ is a valid $(n_1+n_2-(r-k/2), k, r)$-network containing s_1+s_2-1 switches.*

Proof Let out' be a faulty output function on $N_1 \otimes N_2$. We will exhibit a set $\mathcal{P}$ of $n_1+n_2-(r-k/2)$ edge-disjoint paths such that any vertex v of $V(G)$ is the initial vertex of $in(v)$ paths and the terminal vertex of $out(v)$ paths. Let f_1 be the number of faults on the vertices of $V(G_1) \setminus \{u_1\}$ and f_2 be the number of faults on the vertices of $V(G_2) \setminus \{u_2\}$. Let us define out'_1 a faulty output function on N_1 such that $out'_1(v) = out'(v)$ for any vertex $v \in V(G_1) \setminus \{u_1\}$ and $out'_1(u_1) = out(u_1) + f_2$. Similarily, we define out'_2 a faulty output function on N_2 such that $out'_2(v) = out'(v)$ for any vertex $v \in V(G_2) \setminus \{u_2\}$ and $out'_2(u_2) = out(u_2) + f_1$. Since for $i = 1, 2$, N_i is valid, there exists a set $\mathcal{P}_i$ of edge-disjoint paths such that any vertex $v \in V(G_i)$ is the initial vertex of $in_i(v)$ paths and the terminal vertex of $out_i(v)$ paths. The set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is almost the desired set. The vertex u^* is the end of $r-k/2$ paths too much and the beginning of $r-k/2$ paths too much. It suffices now to link an entering path with a leaving path. That completes the proof.

□

If N contains two $r-k/2|r$-switches, then $N \otimes N$ contains also two such switches and we can apply recursively Theorem 12.

Corollary 13 *Let k be an integer. Let N_1 be a valid (n, k, r)-network with s switches containing two $r - k/2|r$-switches. For any integer l, $N_l = N_1 \otimes N_{l-1}$ is a valid $(ln - (l-1)(r - k/2), k, r)$-network with $ls - (l - 1)$ switches.*

Corollary 14

$$\mathcal{N}(n, k, r) \leq \left\lceil \frac{n}{r - \lceil k/2 \rceil} \right\rceil$$

Proof By Proposition 1, it suffices to prove the result for even k.

Suppose $r > k/2$ and k is even, the $(2r - k, k, r)$-network consisting of two $(r - k/2, r)$-switches joined by $k/2$ edges is trivially valid. Hence, by Corollary 13, $\mathcal{N}(n, k, r) \leq \left\lceil \frac{n}{r-k/2} \right\rceil$. □

However this upper bound may be improved for $k \geq 3$ using better initial network.

Theorem 15 *For $k \geq 3$ and $r \geq k/2$,*

$$\mathcal{N}(n, k, r) \leq \frac{r - 2 + \lceil k/2 \rceil}{r^2 - 2r + \lceil k/2 \rceil} n + O(1)$$

Proof By Proposition 1, it suffices to prove the result for even k.

Let H be the $(r^2 - r, k, r)$-network depicted in Figure 6 with $r \geq k/2$. It is composed of $r - 1 + k/2$ switches:

- two $(r - k/2|r)$-switches u_1, u_2,
- $k/2$ $(r - 1|2)$-switches $b_1, \ldots, b_{k/2}$ and
- $r - 3$ $(r - k/2|r)$-switches $s_1, \ldots, s_{r-3}$.

Each u_i is connected to all b_j. Each s_i is connected to all b_j. Using Lemma 5, it is easy to

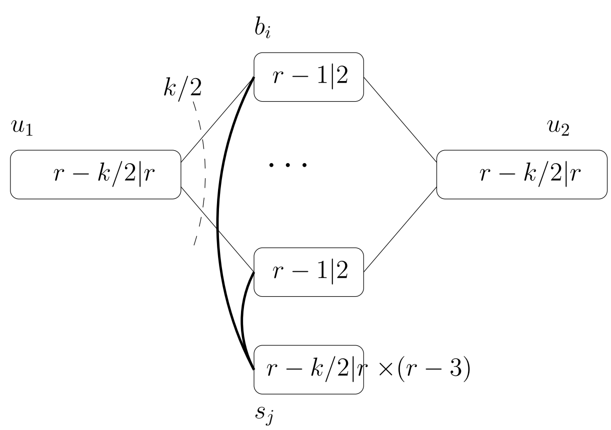

Figure 6: A $(r^2 - r, k, r)$-network with $k \geq 2$ and $r \geq k/2$.

check the validity of the network H. Let W be an essential set of vertices. Let S_W be the set of

$(r-k/2|r)$-switches contained in W. Suppose that $|S_W| = j$ ($1 \leq j \leq r-2$). By the observation made in the proof of Lemma 5, we can assume that W contains all the b_i for $1 \leq i \leq k/2$. Now, $\epsilon(W) = deg(W) - in(W) + out(W) - min\{out(W), k\} = (r-1-j)k/2 - ((r-k/2)j + (r-1)k/2) + j \cdot r + k/2 \cdot 2 - min\{out(w), k\} = k - min\{out(W), k\} \geq 0$. Hence, the network H is valid.

From H and Theorem 12, we can construct the valid (n, k, r)-network depicted in Figure 7.

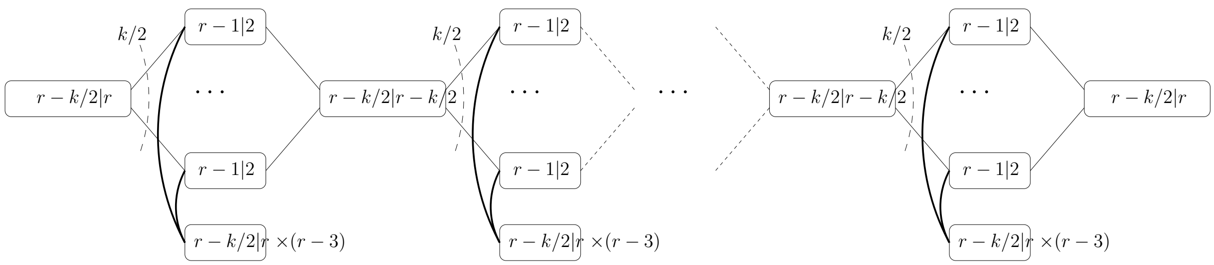

Figure 7: A (n, k, r)-network with $k \geq 2$ and $r \geq k/2$.

Proposition 15 follows. □

4 Lower bounds

4.1 General lower bound

Let $G = (V, E)$ be a graph and p an integer : A *p-quasi-partition* of G is a set $\{A_1, A_2, \ldots, A_m\}$ of subsets of V, such that :

1. For every $1 \leq i \leq m$, the subgraph induced by A_i, $G[A_i]$, is connected;
2. For every $1 \leq i \leq m$, $p/2 \leq |A_i| \leq p$;
3. $V = \bigcup_{i=1}^{m} A_i$ and $\sum_{i=1}^{m} |A_i| < |V| + |\{A_i, |A_i| > \frac{2p}{3}\}|$.

Lemma 16 *Let p be a real and G be a connected graph of order at least $p/2$. Then G admits a p-quasi-partition.*

Proof Since a connected graph has a spanning tree, it suffices to prove the result for trees. We prove it by induction on $|V(A)|$ the result being trivial if $|V(A)| \leq p$.

Let A be a tree of order at least $p/2$. Let E_p be the set of edges of A such that the two components of $A - e$ have at least p vertices. The *p-heart* of A, denoted by H_p, is the subtree induced by the edges of E_p if E_p is not empty, and the tree reduced to the unique vertex x such that all the components of $A - x$ have less than p vertices.

Let u be a leaf of H_p. Let $C_1, \ldots, C_l$ the components of $A - H_p$ which are connected to u. By definition of H_p, each of the C_i has less than p vertices and $\sum |C_i| + 1 \geq p$. Without loss of generality, we may assume that the C_i are numbered in decreasing size $|C_1| \geq |C_2| \geq \cdots \geq |C_l|$.

If C_1 has at least $p/2$ vertices. By induction, the tree $A - C_1$ has a p-quasi-partition $\{A_1, A_2, \ldots, A_m\}$. Setting $A_{m+1} = V(C_1)$, then one verifies easily that $\{A_1, A_2, \ldots, A_m, A_{m+1}\}$ is a p-quasi-partition of A since $A - C_1 \cap C_1 = \emptyset$.

If not, let i be the smallest integer such that $\sum_{j=1}^{i} |C_j| \geq 2p/3$. Since the components are numbered in decreasing size then $\sum_{j=1}^{i} |C_j| < p$. By induction, the tree $A' = A - \bigcup_{j=1}^{i} V(C_j)$ has a p-quasi-partition $\{A_1, A_2, \ldots, A_m\}$. Then setting $A_{m+1} = \bigcup_{j=1}^{i} V(C_j) \cup \{u\}$, one verifies easily that $\{A_1, A_2, \ldots, A_m, A_{m+1}\}$ is a p-quasi-partition of A since $A' \cap A_{m+1} = \{u\}$.

□

Theorem 17

$$\mathcal{N}(n,k,r) \geq \frac{3n+k}{2r-2+\frac{3r^2}{k}}$$

Proof Let $N = (G, in, out)$ be a valid (n,k,r)-network and s its number of switches. Set $p = \frac{k}{r}$ and let $\{A_1, A_2, \ldots, A_m\}$ be a p-quasi-partition of G. Let $m_1 = |\{A_i, |A_i| \leq \frac{2p}{3}\}|$ and $m_2 = |\{A_i, |A_i| > \frac{2p}{3}\}|$.

Since a switch is adjacent to at most r outputs (otherwise, it has negative excess) then by the cut criterion for every i, $deg(A_i) \geq in(A_i)$. Hence $2(r-1)|A_i| + 2 \geq 2in(A_i) + out(A_i)$. Let us sum all these inequalities,

$$\begin{aligned} 2m + 2(r-1)\sum_{i=1}^{m} |A_i| &\geq 2\sum_{i=1}^{m} in(A_i) + \sum_{i=1}^{m} out(A_i) \\ 2m + 2(r-1)(s+m_2) &\geq 2in(G) + out(G) \\ 2m_1 + 2rm_2 + (2r-2)s &\geq 3n+k \end{aligned}$$

Now $s \geq \frac{p}{2}m_1 + \frac{2p}{3}m_2$, so $\dfrac{3r}{p}s \geq \dfrac{3r}{2}m_1 + 2rm_2 \geq 2m_1 + 2rm_2$.

Hence, $\dfrac{3rs}{p} + (2r-2)s \geq 3n+k$, so $s \geq \dfrac{3n+k}{2r-2+\frac{3r}{p}}$.

□

4.2 Optimal lower bounds when $k \leq 6$

Bermond, Pérennes and Tóth [BPT01] proved:

- for $k \in \{1,2\}$, $\mathcal{N}(n,k,2) = n$,
- for $k \in \{3,4\}$, $\mathcal{N}(n,k,2) = \left\lceil \dfrac{5n}{4} \right\rceil$, and
- for $k \in \{5,6\}$, $\mathcal{N}(n,k,2) = n + \dfrac{n}{4} + \sqrt{\dfrac{n}{8}} + \Theta(1)$.

We present now some optimal bounds for $\mathcal{N}(n,k,r)$ with $k \leq 6$ and $r \geq 3$.

Proposition 18 *In a minimum (n,k,r)-network, there is no switch with r (or more) inputs.*

Proof Let N be a valid (n, k, r)-network containing a switch s with $in(v) \geq r$. If $in(v) > r$ or $in(v) = r$ and $out(v) \geq 1$ then $\{v\}$ has negative excess which contradicts the cut criterion. If not, s is incident to r links $e_1, \ldots, e_r$. Then the (n, k, r)-network obtained from N by removing s and adding one input to the endvertex of each e_i is also valid and N is not minimum, a contradiction. □

Corollary 19

$$\mathcal{N}(n, k, r) \geq \left\lceil \frac{n}{r-1} \right\rceil$$

Proof By Proposition 18, in a minimum (n, k, r)-network, the number of inputs n is at most $r - 1$ times the number of switches. □

For $k \in \{1, 2\}$, this lower bound matches the upper one given by Corollary 14.

Theorem 20

$$\mathcal{N}(n, 1, r) = \mathcal{N}(n, 2, r) = \left\lceil \frac{n}{r-1} \right\rceil$$

For larger value of k, we generalize the notion of block introduced in [BPT01]. We call a switch with $r - 1$ inputs, *block switch*, and the non block switches, *S-switches*. We defi ne *blocks* as maximum connected components made of block switches. The block properties presented in [BPT01] immediately extend to our case :

Proposition 21 *Let N be a minimum (n, k, r)-network for $k \geq 3$. Then the following hold:*

- *the blocks of N are trees and contain at most 2 outputs;*
- *for any block B of N, $deg(B) = in(B) + 2 - out(B)$.*

Let us introduce some notations.
Let $0 \leq i \leq r - 1$ and $0 \leq o \leq 2$. We denote by S_i the set of switches with i inputs and s_i its cardinality. We denote $\mathcal{B}_o$ the set of blocks with o outputs and b_o its cardinality. We call *$\mathcal{B}_o$-block* a block in $\mathcal{B}_o$. The number of $\mathcal{B}_o$-switches is denoted t_o. Hence, the total number of block switches is $s_{r-1} = t_0 + t_1 + t_2$.
We call *S-switch* a swicth of S. Let $s_{i|o}$ denote the number of $i|o$-switches, s_S the number of S-switches and e_S the number of edges whose endvertices are both S-switches.
Finally, s denotes the total number of switches of the network.

Let N be an (n, k, r)-network. Then $\epsilon'(N) = 1$ if an S-switch is defective, $\epsilon'(N) = -1$ if a block switch is defective and $\epsilon'(N) = 0$ otherwise.

Proposition 22 *Let N be a minimum (n,k,r)-network.*

$$\sum_{(i,o)=(0,0)}^{(r-2,r)} s_{i|o} + t_0 + t_1 + t_2 = s \quad (5)$$

$$\sum_{(i,o)=(0,0)}^{(r-2,r)} i.s_{i|o} + (r-1)(t_0 + t_1 + t_2) = n \quad (6)$$

$$\sum_{(i,o)=(0,0)}^{(r-2,r)} o.s_{i|o} + b_1 + 2b_2 = n + k \quad (7)$$

$$-\sum_{(i,o)=(0,0)}^{(r-2,r)} (2r - i - o)s_{i|o} + (r-1)(t_0 + t_1 + t_2) + 2e_S + 2b_0 + b_1 = \epsilon'(N) \quad (8)$$

$$b_2 \leq t_2 \quad (9)$$

Proof

Eq.(5), Eq.(6) and Eq.(7) count the number of respectively switches, inputs and outputs in the network.

Eq.(8) counts the number of edges between blocks and S-switches. The number of edges leaving the $\mathcal{B}_0$-blocks (resp. $\mathcal{B}_1$-blocks, $\mathcal{B}_2$-blocks) is $(r-1)t_0 + 2b_0$ (resp. $(r-1)t_1 + b_1$, $(r-1)t_2$) decreased by 1 if a block switch is defective; the number of edges leaving the S-switches is $\sum_{(i,o)=(0,0)}^{(r-2,r)}(2r - i - o)s_{i|o} - 2e_S$ decreased by 1 if an S-switch is defective.

Eq.(9) expresses the fact that a $\mathcal{B}_2$-block contains at least one switch.

□

Proposition 23 *For $r \geq 3$ and $k \geq 3$,*

$$\mathcal{N}(n,k,r) \geq \left\lceil \frac{rn + \frac{1}{2}(k - \epsilon(k))}{r^2 - 2r + 2} \right\rceil$$

Proof Observe that : $(r-2)s_S \geq \sum_{(i,o)=(0,0)}^{r-2,r} i.s_{i|o} \geq n - (r-1)(t_0 + t_1 + t_2)$. Hence,

$$s_S \geq \frac{n - (r-1)s_{r-1}}{(r-2)} \quad (10)$$

Now, detail the ports of the S-switches :

$$\begin{aligned}
2rs_S &= \sum_{(i,o)=(0,0)}^{(r-2,r)}(2r - i - o)s_{i|o} + \sum_{(i,o)=(0,0)}^{(r-2,r)} i.s_{i|o} + \sum_{(i,o)=(0,0)}^{(r-2,r)} o.s_{i|o} \\
&= (r-1)(t_0 + t_1 + t_2) + 2b_0 + b_1 + \sum_{(i,o)=(0,0)}^{(r-2,r)} i.s_{i|o} + \sum_{(i,o)=(0,0)}^{(r-2,r)} o.s_{i|o} && \text{by Eq. (8)} \\
&= n + 2b_0 + b_1 + \sum_{(i,o)=(0,0)}^{(r-2,r)} o.s_{i|o} - \epsilon'(N) && \text{by Eq. (6)} \\
&= n + 2b_0 + n + k - 2b_2 - \epsilon'(N) && \text{by Eq. (7)} \\
&\geq 2n + k - 2s_{r-1} - \epsilon'(N) && \text{by Eq. (9)}
\end{aligned}$$

$$s_S \geq \frac{2n - 2s_{r-1} + k - \epsilon'(N)}{2r} \tag{11}$$

The inequalities (10) and (11) give a lower bound of $s = s_S + s_{r-1}$:

$$s \geq \max\left\{s_{r-1} + \frac{n-(r-1)s_{r-1}}{r-2}, s_{r-1} + \frac{2n - 2s_{r-1} + k - \epsilon'(N)}{2r}\right\}.$$

One of these two functions (of s_{r-1}) increases while the other decreases, thus the minimum is achieved when the two bounds are equal that is when $s_{r-1} = \frac{2n-\frac{1}{2}(r-2)(k-\epsilon'(N))}{r^2-2r+2}$. We obtain $s \geq \dfrac{rn + \frac{1}{2}(k-\epsilon(k))}{r^2-2r+2}$. □

The lower bound of this proposition matches the upper one given by Theorem 15.

Corollary 24 *For $k \in \{3,4\}$ and $r \geq 3$, $\mathcal{N}(n,k,r) = \dfrac{r}{r^2-2r+2}n + \Theta(1)$.*

We will now get better lower bounds provided that $k \geq 5$. Therefore we provide new inequalities satisfied by a valid (n,k,r)-network if $k \geq 5$. Therefore, we define $\epsilon''(N) = 1$ if a switch less than $r-2$ inputs is defective, and $\epsilon''(N) = 0$ otherwise. Note that a switch in a $\mathcal{B}_2$-block may not be defective.

Proposition 25 *If $k \geq 5$ and $r \geq 4$, a valid (n,k,r)-network N satisfies the following inequalities :*

$$b_1 \leq t_1 \tag{12}$$

$$(2r+2)s_{r-1} + (2r-6)s_{r-2} - \sum_{j=0}^{r-3} 2j \cdot s_{r-3-j} + k \leq 6s \tag{13}$$

Proof Eq.(12) expresses the fact that a $\mathcal{B}_1$-block contains at least one switch.

Let us now show Eq.(13). Let $H = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup S_{r-2}$ and $H' = H \setminus \mathcal{B}_2$.

We have $out(H) + out(\overline{H}) = in(H) + in(\overline{H}) + k$ so $out(\overline{H}) - in(\overline{H}) = k + in(H) - out(H) = k + (r-1)S_{r-1} - b_1 - 2b_2 + \sum_{o=0}^{4}(r-2-o)s_{r-2|o}$.

Let us now compute $deg(H) = deg(\overline{H})$. Since there is no edge between blocks, $deg(H) = deg(H') + \sum_{B\in\mathcal{B}_2} deg(B) - 2e$ with e the number of edges between $\mathcal{B}_2$-blocks and switches of S_{r-2}.

By the cut criterion, $deg(H') \geq in(H') - out(H') \geq (r-1)(t_1+t_0) - b_1 + \sum_{o=0}^{4}(r-2-o)s_{r-2|o}$.

A $\mathcal{B}_2$-block B is not adjacent to a switch v of type $r-2|3$ or $r-2|4$ otherwise $B \cup \{v\}$ has negative excess which is impossible. Let a_j be the number of links leaving the $\mathcal{B}_2$-blocks and joining the switches of $\mathcal{S}_{r-2}$ connecting to o outputs for $0 \leq o \leq 2$. If $o = 1, 2$, two links leaving $\mathcal{B}_2$-blocks cannot join a same switch of type $r-2|o$ otherwise the union of this switch and the two $\mathcal{B}_2$-blocks connected to it has negative excess, so $a_j \leq s_{r-2|j}$ for $j = 1, 2$. Three links leaving $\mathcal{B}_2$-blocks cannot join a same switch of type $r-2|0$ otherwise the union of this switch and the three $\mathcal{B}_2$-blocks connected to it has negative excess, so $2a_0 \leq s_{r-2|0}$. Thus we have $e \leq s_{r-2|1} + s_{r-2|2} + 2s_{r-2|0}$.

Hence we obtain : $deg(\overline{H}) \geq (r-1)(t_2+t_1+t_0) - b_1 + (r-6)(s_{r-2|4} + s_{r-2|2} + s_{r-2|0}) + (r-5)(s_{r-2|3} + s_{r-2|1})$. So

$$\begin{aligned} deg(\overline{H}) + out(\overline{H}) - in(\overline{H}) \geq\; & k + 2(r-1)s_{r-1} - 2b_1 - 2b_2 + (2r-12)s_{r-2|4} \\ & +(2r-10)(s_{r-2|3} + s_{r-2|2}) + (2r-8)(s_{r-2|1} + s_{r-2|0}) \end{aligned}$$

By (12) and (9), $b_1 \leq t_1$ and $b_2 \leq t_2$, so

$$deg(\overline{H}) + out(\overline{H}) - in(\overline{H}) \quad \geq k + 2(r-2)s_{r-1} + (2r-12)s_{r-2} \tag{14}$$

Now $deg(\overline{H}) + out(\overline{H}) - in(\overline{H}) \leq \sum_{v \in \overline{H}}(deg(v) + out(v) - in(v)) \leq \sum_{v \in \overline{H}}(2r - 2in(v))$.

$$deg(\overline{H}) + out(\overline{H}) - in(\overline{H}) \leq \sum_{j=0}^{r-3}(6+2j)s_{r-3-j} \tag{15}$$

Combining (14) and (15), we obtain (13). □

Proposition 26 *For $r \geq 7$ and $k \geq 5$,*

$$\mathcal{N}(n, 6, r) \geq \frac{(r+1)n + k}{r^2 - 2r + 3}$$

Proof We have $n - (r-3)s = \sum_{i=0}^{r-1}(i - r + 3)s_i = 2s_{r-1} + s_{r-2} - \sum_{j=0}^{r-3} j \cdot s_{r-3-j}$. Hence if $r \geq 7$, by (13), we obtain $6s \geq k + (r+1)(n - (r-3)s)$. So $s \geq \frac{(r+1)n+k}{r^2-2r+3}$. □

We conjecture that this lower bound is also true when $r \leq 7$.
The lower bound of Proposition 26 matches the upper one given by Theorem 15.

Corollary 27 *For $k \in \{5, 6\}$ and $r \geq 7$, $\mathcal{N}(n, k, r) = \dfrac{r+1}{r^2 - 2r + 3}n + \Theta(1)$.*

References

[BDD02] J.-C. Bermond, E. Darrot, and O. Delmas. Design of fault tolerant on-board networks in satellites. *Networks*, 40:202–207, 2002.

[BHT06] J.-C. Bermond, F. Havet, and D. Tóth. Fault tolerant on board networks with priorities. *Networks*, 47, 2006.

[BPT01] J.-C. Bermond, S. Pérennes, and D. Tóth. Design of fault tolerant flow networks. Manuscript, 2001.

[FF62] L.R. Ford and D.R. Fulkerson. *Flows in Networks*. Princeton University Press, 1962. sixth printing, 1974.

[Hav06] F. Havet. Repartitors, selectors and superselectors. *Journal of Interconnection Networks*, 2006.