Adaptive Design and Control

Luc Pronzato

Laboratoire I3S, CNRS-Univ. Nice Sophia Antipolis, France

Outline

1 Optimal input design
- 1.1 Time-domain input design
- 1.2 Frequency-domain input design (linear system)

2 Adaptive control

3 Sequential design
- 3.1 D-optimal design
- 3.2 Penalized D-optimal design

4 Examples

5 Design with dynamical constraints

6 Conclusions

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

1 Optimal input design

$y_k = \mathbf{f}^\top(x_k)\theta + \varepsilon_k\,,\ k = 1, 2 \ldots$
Dynamical system (linear): x_k may depend (linearly)

- on previous y_{k-i}, $i = 1, 2 \ldots$
- on inputs u_{k-i}, $i = 1, 2 \ldots$
- on errors ε_{k-i}, $i = 1, 2 \ldots$

Choose a sequence $u_1, u_2 \ldots$ to estimate θ
$\mathbf{M}(\xi, \theta)$ the information matrix with ξ=input sequence

$\rightarrow$ an algorithmic problem [Mehra 1974; Goodwin & Payne 1977; Zarrop 1979...]

1.1 Time-domain input design

Example 1: Finite Impulse Response model
$y_k = B(\bar{\theta}, z)u_k + \varepsilon_k$, $k = 1, \ldots, N$, (ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$,
$B(\theta, z) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}$

$$\rightarrow\ y_k = \bar{b}_1 u_{k-1} + \bar{b}_2 u_{k-2} + \cdots + \bar{b}_n u_{k-n} + \varepsilon_k$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Choose the input sequence to estimate $\theta = (b_1, \ldots, b_n)^\top$, with

$$\frac{1}{N}\sum_{k=1}^{N} u_{k-\tau}^2 \leq P_{u\max} \;\; \tau = 1, \ldots, n$$

$\rightarrow$ linear regression $y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with
$\mathbf{f}_k = (u_{k-1}, \ldots, u_{k-n})^\top$ ($\rightarrow \mathbf{f}_{k+1} = h(\mathbf{f}_k, u_k)$: dynamically constrained design)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Choose the input sequence to estimate $\theta = (b_1, \ldots, b_n)^\top$, with

$$\frac{1}{N}\sum_{k=1}^{N} u_{k-\tau}^2 \leq P_{u\max} \;\; \tau = 1, \ldots, n$$

$\rightarrow$ linear regression $y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with
$\mathbf{f}_k = (u_{k-1}, \ldots, u_{k-n})^\top$ ($\rightarrow \mathbf{f}_{k+1} = h(\mathbf{f}_k, u_k)$: dynamically constrained design)

$\mathrm{Var}[\hat{\theta}_{LS}^N] = \sigma^2(\mathbf{R}^\top\mathbf{R})^{-1}$ where

$$\mathbf{R} = \begin{pmatrix} u_0 & u_{-1} & \cdots & u_{1-n} \\ u_1 & u_0 & \cdots & u_{2-n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N-1} & u_{N-2} & \cdots & u_{N-n} \end{pmatrix}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

D-optimum input: maximize $\det(\mathbf{R}^\top \mathbf{R}) \rightarrow$ choose u_k such that

$$(1/N) \sum_{k=1}^{N} u_{k-i} u_{k-j} = P_{u\max} \delta_{i,j}, \; i,j = 1, \ldots, n$$

for N large: white noise sequence

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

D-optimum input: maximize $\det(\mathbf{R}^\top \mathbf{R}) \to$ choose u_k such that

$$(1/N) \sum_{k=1}^{N} u_{k-i} u_{k-j} = P_{u_{\max}} \delta_{i,j}, \; i,j = 1, \ldots, n$$

for N large: white noise sequence

Example 2 (more difficult): model with AR part, e.g., Box & Jenkins

$$y_k = F(\bar{\theta}, z) u_k + G(\bar{\theta}, z) \varepsilon_k$$

with (ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$, $F(\theta, z)$ and $G(\theta, z)$ rational fractions in z^{-1}, G stable with stable inverse, σ^2 unknown $\to$ $\theta_e = \begin{pmatrix} \theta \\ \sigma^2 \end{pmatrix}$, and $G(\theta, \infty) = 1$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

For a suitable input sequence (persistence of excitation), $N\mathrm{Var}(\hat{\theta}_e^N) \rightarrow \mathbf{M}^{-1}(\xi, \bar{\theta}_e)$ with

$$\mathbf{M}(\xi, \theta_e) = \mathbb{E}\left\{ \frac{1}{N} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e^\top} |\theta_e \right\}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

For a suitable input sequence (persistence of excitation), $N\mathrm{Var}(\hat{\theta}_e^N) \to \mathbf{M}^{-1}(\xi, \bar{\theta}_e)$ with

$$\mathbf{M}(\xi, \theta_e) = \mathbb{E}\left\{ \frac{1}{N} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e^\top} |\theta_e \right\}$$

prediction error: $e(\theta, k) = G^{-1}(\theta, z)[y_k - F(\theta, z)u_k]$
$\to$ log likelihood

$$\log \pi(\mathbf{y}|\theta_e) = -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{k=1}^{N} e^2(\theta, k)$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

For a suitable input sequence (persistence of excitation), $N\mathrm{Var}(\hat{\theta}_e^N) \to \mathbf{M}^{-1}(\xi, \bar{\theta}_e)$ with

$$\mathbf{M}(\xi, \theta_e) = \mathbb{E}\left\{ \frac{1}{N} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e} \frac{\partial \log \pi(\mathbf{y}|\theta_e)}{\partial \theta_e^\top} |\theta_e \right\}$$

prediction error: $e(\theta, k) = G^{-1}(\theta, z)[y_k - F(\theta, z)u_k]$
$\to$ log likelihood

$$\log \pi(\mathbf{y}|\theta_e) = -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{k=1}^{N} e^2(\theta, k)$$

Differentiate, use independence of $e(\theta, i)$ and $e(\theta, j)$ conditionally on θ for $i \neq j$, use normality

$$\mathbb{E}\{e^2(\theta, k)|\theta_e\} = \sigma^2,\ \mathbb{E}\{e^3(\theta, k)|\theta_e\} = 0,\ \mathbb{E}\{e^4(\theta, k)|\theta_e\} = 3\sigma^4$$

σ^2 does not depend on θ ...

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

We obtain $\mathbf{M}(\xi, \theta_e) = \begin{pmatrix} \mathbf{M}(\xi, \theta) & \mathbf{0} \\ \mathbf{0}^\top & \frac{1}{2\sigma^4} \end{pmatrix}$

with $\mathbf{M}(\xi, \theta) = \mathbb{E}\left\{ \frac{1}{N\sigma^2} \sum_{k=1}^{N} \frac{\partial e(\theta, k)}{\partial \theta} \frac{\partial e(\theta, k)}{\partial \theta^\top} | \theta \right\}$

$\Rightarrow$ σ^2 unknown does not influence the estimation of θ

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

We obtain $\mathbf{M}(\xi, \theta_e) = \begin{pmatrix} \mathbf{M}(\xi, \theta) & \mathbf{0} \\ \mathbf{0}^\top & \frac{1}{2\sigma^4} \end{pmatrix}$

with $\mathbf{M}(\xi, \theta) = \mathbb{E}\left\{ \frac{1}{N\sigma^2} \sum_{k=1}^{N} \frac{\partial e(\theta, k)}{\partial \theta} \frac{\partial e(\theta, k)}{\partial \theta^\top} | \theta \right\}$

$\Rightarrow$ σ^2 unknown does not influence the estimation of θ

Now, $\frac{\partial e(\theta,k)}{\partial \theta} = -G^{-1}(\theta, z) \left[\frac{\partial G(\theta,z)}{\partial \theta} e(\theta, k) + \frac{\partial F(\theta,z)}{\partial \theta} u_k \right]$ and $G(\theta, \infty) = 1$

$\Rightarrow$ $\frac{\partial G(\theta,z)}{\partial \theta} e(\theta, k)$ only depends on $e(\theta, k - i)$ for $i \geq 1$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

We obtain $\mathbf{M}(\xi, \theta_e) = \begin{pmatrix} \mathbf{M}(\xi,\theta) & \mathbf{0} \\ \mathbf{0}^\top & \frac{1}{2\sigma^4} \end{pmatrix}$

with $\mathbf{M}(\xi,\theta) = \mathbb{E}\left\{ \frac{1}{N\sigma^2} \sum_{k=1}^{N} \frac{\partial e(\theta,k)}{\partial \theta} \frac{\partial e(\theta,k)}{\partial \theta^\top} | \theta \right\}$

$\Rightarrow$ σ^2 unknown does not influence the estimation of θ

Now, $\frac{\partial e(\theta,k)}{\partial \theta} = -G^{-1}(\theta,z) \left[\frac{\partial G(\theta,z)}{\partial \theta} e(\theta,k) + \frac{\partial F(\theta,z)}{\partial \theta} u_k \right]$ and $G(\theta,\infty) = 1$

$\Rightarrow$ $\frac{\partial G(\theta,z)}{\partial \theta} e(\theta,k)$ only depends on $e(\theta,k-i)$ for $i \geq 1$

- Assume open loop identification: (no feedback)

$\Rightarrow$ $\mathbb{E}\{e(\theta,j)u_k|\theta\} = 0$ $k \geq j$ $\rightarrow$ no cross-product terms $e \times u$

- Assume F and G have no common parameters:

$$\theta = \begin{pmatrix} \theta_F \\ \theta_G \end{pmatrix}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\mathbf{M}(\xi,\theta) = \begin{pmatrix} \mathbf{M}^F(\xi,\theta) & \mathbf{O} \\ \mathbf{O} & \mathbf{M}^G(\xi,\theta) \end{pmatrix}$$

$$\text{with } \mathbf{M}^F(\xi,\theta) = \frac{1}{N\sigma^2}\sum_{k=1}^{N}\left[G^{-1}(\theta,z)\frac{\partial F(\theta,z)}{\partial\theta_F}u_k\right] \times \left[G^{-1}(\theta,z)\frac{\partial F(\theta,z)}{\partial\theta_F^\top}u_k\right]$$

$$\mathbf{M}^G(\xi,\theta) = \frac{1}{\sigma^2}\mathbb{E}\left\{\left[G^{-1}(\theta,z)\frac{\partial G(\theta,z)}{\partial\theta_G}e(\theta,k)\right] \times \left[G^{-1}(\theta,z)\frac{\partial G(\theta,z)}{\partial\theta_G^\top}e(\theta,k)\right]|\theta\right\}$$

$\rightarrow$ u has no effect on the precision on θ_G

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\begin{aligned} D\text{-optimality: } j_D(u) &= \det \mathbf{M}^F(\xi, \theta) \\ &= \det \frac{1}{N\sigma^2} \sum_{k=1}^{N} \mathbf{f}_k \mathbf{f}_k^\top \end{aligned}$$

where $\mathbf{f}_k = G^{-1}(\theta, z) \frac{\partial F(\theta, z)}{\partial \theta_F} u_k$
($\rightarrow$ dynamically constrained design)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\begin{aligned} D\text{-optimality: } j_D(u) &= \det \mathbf{M}^F(\xi, \theta) \\ &= \det \frac{1}{N\sigma^2} \sum_{k=1}^{N} \mathbf{f}_k \mathbf{f}_k^\top \end{aligned}$$

where $\mathbf{f}_k = G^{-1}(\theta, z) \frac{\partial F(\theta,z)}{\partial \theta_F} u_k$
($\rightarrow$ dynamically constrained design)

$\rightarrow$ optimal control problem
(deterministic but nonlinear. . . , non trivial)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\begin{aligned} D\text{-optimality: } j_D(u) &= \det \mathbf{M}^F(\xi, \theta) \\ &= \det \frac{1}{N\sigma^2} \sum_{k=1}^{N} \mathbf{f}_k \mathbf{f}_k^\top \end{aligned}$$

where $\mathbf{f}_k = G^{-1}(\theta, z) \frac{\partial F(\theta, z)}{\partial \theta_F} u_k$
($\rightarrow$ dynamically constrained design)

$\rightarrow$ optimal control problem
(deterministic but nonlinear. . . , non trivial)

Nonlinear dynamical system:
$\rightarrow$ additional difficulty = construction of $\mathbf{M}(\xi, \theta)$

$\rightarrow$ Simpler if on-line characterization of parameter uncertainty

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 3: ARX model: $A(\bar{\theta}_A, z)y_k = B(\bar{\theta}_B, z)u_k + \varepsilon_k$, (ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$

$\rightarrow y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with $\bar{\theta} = \begin{pmatrix} \bar{\theta}_A \\ \bar{\theta}_B \end{pmatrix}$,

$A(\theta_A, z) = 1 - \sum_{i=1}^{n_A} a_i z^{-i}$ and $B(\theta_B, z) = \sum_{i=1}^{n_B} b_i z^{-i}$

$$\mathbf{f}_k = [y_{k-1} \ldots y_{k-n_A} \; u_{k-1} \; u_{k-2} \cdots u_{k-n_B}]^\top$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 3: ARX model: $A(\bar{\theta}_A, z)y_k = B(\bar{\theta}_B, z)u_k + \varepsilon_k$, (ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$

$\rightarrow y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with $\bar{\theta} = \begin{pmatrix} \bar{\theta}_A \\ \bar{\theta}_B \end{pmatrix}$,

$A(\theta_A, z) = 1 - \sum_{i=1}^{n_A} a_i z^{-i}$ and $B(\theta_B, z) = \sum_{i=1}^{n_B} b_i z^{-i}$

$$\mathbf{f}_k = [y_{k-1} \ldots y_{k-n_A} \; u_{k-1} \; u_{k-2} \cdots u_{k-n_B}]^\top$$

Bayesian estimation: posterior cov. matrix for θ (recursive LS) (after observation of y_{k+1} and u_k applied)

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \frac{\mathbf{P}_k \mathbf{f}_{k+1} \mathbf{f}_{k+1}^\top \mathbf{P}_k}{\sigma^2 + \mathbf{f}_{k+1}^\top \mathbf{P}_k \mathbf{f}_{k+1}}$$

$\rightarrow$ choose u_k so that $\det \mathbf{P}_{k+1}$ minimum

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 3: ARX model: $A(\bar{\theta}_A, z)y_k = B(\bar{\theta}_B, z)u_k + \varepsilon_k$, (ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$

$\rightarrow y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with $\bar{\theta} = \begin{pmatrix} \bar{\theta}_A \\ \bar{\theta}_B \end{pmatrix}$,

$A(\theta_A, z) = 1 - \sum_{i=1}^{n_A} a_i z^{-i}$ and $B(\theta_B, z) = \sum_{i=1}^{n_B} b_i z^{-i}$

$$\mathbf{f}_k = [y_{k-1} \ldots y_{k-n_A} \; u_{k-1} \; u_{k-2} \cdots u_{k-n_B}]^\top$$

Bayesian estimation: posterior cov. matrix for θ (recursive LS) (after observation of y_{k+1} and u_k applied)

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \frac{\mathbf{P}_k \mathbf{f}_{k+1} \mathbf{f}_{k+1}^\top \mathbf{P}_k}{\sigma^2 + \mathbf{f}_{k+1}^\top \mathbf{P}_k \mathbf{f}_{k+1}}$$

$\rightarrow$ choose u_k so that $\det \mathbf{P}_{k+1}$ minimum

$$\det \mathbf{P}_{k+1} = \det \mathbf{P}_k \frac{\sigma^2}{\sigma^2 + \mathbf{f}_{k+1}^\top \mathbf{P}_k \mathbf{f}_{k+1}}$$

$\rightarrow$ maximize $\mathbf{f}_{k+1}^\top \mathbf{P}_k \mathbf{f}_{k+1}$, quadratic in u_k, easy

but. . . <u>only one-step ahead optimal</u>

Example 2 (continued): Box & Jenkins model

$$\mathbf{M}^F(\xi,\theta) = \frac{1}{N\sigma^2}\sum_{k=1}^{N}\mathbf{f}_k\mathbf{f}_k^\top$$

with $\mathbf{f}_k = G^{-1}(\theta,z)\frac{\partial F(\theta,z)}{\partial\theta_F}u_k$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

1.2 Frequency-domain input design (linear system)

Example 2 (continued): Box & Jenkins model

$$\mathbf{M}^F(\xi,\theta) = \frac{1}{N\sigma^2}\sum_{k=1}^{N} \mathbf{f}_k\mathbf{f}_k^\top$$

with $\mathbf{f}_k = G^{-1}(\theta,z)\frac{\partial F(\theta,z)}{\partial \theta_F} u_k$

Uniform sampling, period T: $\underline{\mathbf{M}}^F(\xi,\theta) = \lim_{N\to\infty} \mathbf{M}^F(\xi,\theta)/T$

= average Fisher information matrix per time unit

$$\underline{\mathbf{M}}^F(\xi,\theta) = \frac{1}{2\pi\sigma^2}\int_{-\pi}^{\pi} \mathcal{P}_f(\omega)d\omega \text{ (Fourier)}$$

= integral of power spectral density of $\mathbf{f}_k$

$$= \frac{1}{\pi}\int_0^{\pi} \underline{\tilde{\mathbf{M}}}^F(\omega,\theta)\mathcal{P}_u(\omega)d\omega$$

with $\mathcal{P}_u(\omega)$ the power spectral density of u and

$$\underline{\tilde{\mathbf{M}}}^F(\omega,\theta) = \frac{1}{\sigma^2}\mathcal{R}_e\left\{\frac{\partial F(\theta,e^{j\omega})}{\partial \theta_F} G^{-1}(\theta,e^{j\omega})G^{-1}(\theta,e^{-j\omega})\frac{\partial F(\theta,e^{-j\omega})}{\partial \theta_F^\top}\right\}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same framework as "approximate" design theory:

- experimental domain $\mathcal{X} \to$ frequency domain $\mathbb{R}^+$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same framework as "approximate" design theory:

- experimental domain $\mathcal{X} \rightarrow$ frequency domain $\mathbb{R}^+$
- design measure $\xi(dx) \rightarrow$ power spectral density $\mathcal{P}_u(\omega)$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same framework as "approximate" design theory:

- experimental domain $\mathcal{X} \rightarrow$ frequency domain $\mathbb{R}^+$
- design measure $\xi(dx) \rightarrow$ power spectral density $\mathcal{P}_u(\omega)$
- the optimum measure ξ^* is discrete
  $\rightarrow$ the optimum spectrum is discrete
  - support points $\rightarrow$ frequencies
  - weights $\rightarrow$ power

$\Leftrightarrow$ optimal input $u^* =$ combination of sinusoids

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same framework as "approximate" design theory:

- experimental domain $\mathcal{X} \to$ frequency domain $\mathbb{R}^+$
- design measure $\xi(dx) \to$ power spectral density $\mathcal{P}_u(\omega)$
- the optimum measure ξ^* is discrete
  $\to$ the optimum spectrum is discrete
  - support points $\to$ frequencies
  - weights $\to$ power

$\Leftrightarrow$ optimal input $u^* =$ combination of sinusoids

Same algorithms as for "approximate" design theory (without approximation)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same framework as "approximate" design theory:

- experimental domain $\mathcal{X} \rightarrow$ frequency domain $\mathbb{R}^+$
- design measure $\xi(dx) \rightarrow$ power spectral density $\mathcal{P}_u(\omega)$
- the optimum measure ξ^* is discrete
  $\rightarrow$ the optimum spectrum is discrete
  - support points $\rightarrow$ frequencies
  - weights $\rightarrow$ power

$\Leftrightarrow$ optimal input $u^* =$ combination of sinusoids

Same algorithms as for "approximate" design theory (without approximation)

... but only for linear dynamical systems

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

2 Adaptive control

Example 3 (continued): ARX model:
$A(\bar{\theta}_A, z)y_k = B(\bar{\theta}_B, z)u_k + \varepsilon_k$,
(ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$
$\rightarrow y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with $\bar{\theta} = \begin{pmatrix} \bar{\theta}_A \\ \bar{\theta}_B \end{pmatrix}$,
$A(\theta_A, z) = 1 - \sum_{i=1}^{n_A} a_i z^{-i}$ and $B(\theta_B, z) = \sum_{i=1}^{n_B} b_i z^{-i}$

$$\mathbf{f}_k = [y_{k-1} \ldots y_{k-n_A} \; u_{k-1} \; u_{k-2} \cdots u_{k-n_B}]^\top$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 3 (continued): ARX model:
$A(\bar{\theta}_A, z) y_k = B(\bar{\theta}_B, z) u_k + \varepsilon_k$,
(ε_k) i.i.d. $\mathcal{N}(0, \sigma^2)$
$\rightarrow y_k = \mathbf{f}_k^\top \bar{\theta} + \varepsilon_k$ with $\bar{\theta} = \begin{pmatrix} \bar{\theta}_A \\ \bar{\theta}_B \end{pmatrix}$,
$A(\theta_A, z) = 1 - \sum_{i=1}^{n_A} a_i z^{-i}$ and $B(\theta_B, z) = \sum_{i=1}^{n_B} b_i z^{-i}$

$$\mathbf{f}_k = [y_{k-1} \ldots y_{k-n_A} \; u_{k-1} \; u_{k-2} \cdots u_{k-n_B}]^\top$$

Self-tuning regulator
Minimum-variance control: minimize regret
$R_N = \sum_{k=1}^{N} (y_k - \varepsilon_k)^2$
(u_k) "globally convergent" if $R_N / N \overset{\text{a.s.}}{\rightarrow} 0$ [Lai & Wei, 1986]

INI
20/07/2011

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

If θ known (and $b_1 \neq 0$): optimal controller
$u_k^*(\theta) =$
$-(a_1 y_k + \cdots + a_{n_A} y_{k+1-n_A} + b_2 u_{k-1} + \cdots + b_{n_B} u_{k+1-n_B})/b_1$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

If θ known (and $b_1 \neq 0$): optimal controller
$u_k^*(\theta) =$
$-(a_1 y_k + \cdots + a_{n_A} y_{k+1-n_A} + b_2 u_{k-1} + \cdots + b_{n_B} u_{k+1-n_B})/b_1$
but then $\mathbf{f}_k^\top \theta = 0$ for all k and $\mathbf{M}_N = \sum_{k=1}^N \mathbf{f}_k \mathbf{f}_k^\top$ is singular ($\theta^\top \mathbf{M}_N \theta = 0$) and θ not estimable!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

If θ known (and $b_1 \neq 0$): optimal controller
$u_k^*(\theta) =$
$-(a_1 y_k + \cdots + a_{n_A} y_{k+1-n_A} + b_2 u_{k-1} + \cdots + b_{n_B} u_{k+1-n_B})/b_1$
but then $\mathbf{f}_k^\top \theta = 0$ for all k and $\mathbf{M}_N = \sum_{k=1}^N \mathbf{f}_k \mathbf{f}_k^\top$ is singular ($\theta^\top \mathbf{M}_N \theta = 0$) and θ not estimable!

(Forced-) Certainty-Equivalence control with LS estimation:
use $u_k = u_k^*(\hat{\theta}_{LS}^k)$
$\rightarrow$ additional perturbations must be added to guarantee that $\hat{\theta}_{LS}^N \overset{\text{a.s.}}{\rightarrow} \bar{\theta}$ [Åström & Wittenmark, 1973]

- Persistently exciting input ($\lambda_{\min}(\mathbf{M}_N) = \mathcal{O}(N)$): $R_N > \lambda_{\min}(\mathbf{M}_N)\|\bar{\theta}\|^2$ and $\liminf_{N\to\infty} R_N/N > 0$
- Best possible regret: $R_N = \mathcal{O}(\log(N))$ [Lai & Wei, 1987; Guo 1994]

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\rightarrow$ a statistical problem

LS estimation in regression

$Y_i = \eta(x_i, \bar{\theta}) + \varepsilon_i$, $x_i \in \mathcal{X} \subset \mathbb{R}^d$, $\bar{\theta} \in \Theta \subset \mathbb{R}^p$, $\{\varepsilon_i\}$ i.i.d., variance σ^2

LS estimation: $\hat{\theta}^n = \arg\min_{\theta \in \Theta} S_n(\theta)$ with $S_n(\theta) = \sum_{i=1}^n [Y_i - \eta(x_i, \theta)]^2$

Conditions for strong consistency of $\hat{\theta}^n$ ($\hat{\theta}^n \stackrel{\text{a.s.}}{\rightarrow} \bar{\theta}$) very much differ depending whether the x_k are constants or depend on ε_i, $i < k$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\{x_i\}$ = deterministic sequence

➟ Linear regression: NSC for the strong consistency of $\hat{\theta}^n$:
$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\{x_i\}$ = deterministic sequence

➟ Linear regression: NSC for the strong consistency of $\hat{\theta}^n$:
$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

➟ Nonlinear regression:
Define $D_n(\theta, \bar{\theta}) = \sum_{k=1}^{n} [\eta(x_k, \theta) - \eta(x_k, \bar{\theta})]^2$
[Jennrich 1969]: $\hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ when $D_n(\theta, \theta')/n \to J(\theta, \theta')$ (uniformly) with $J(\theta, \theta')$ continuous and > 0 for all $\theta \neq \theta'$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\{x_i\}$ = deterministic sequence

➟ Linear regression: NSC for the strong consistency of $\hat{\theta}^n$:
$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

➟ Nonlinear regression:
Define $D_n(\theta, \bar{\theta}) = \sum_{k=1}^{n} [\eta(x_k, \theta) - \eta(x_k, \bar{\theta})]^2$
[Jennrich 1969]: $\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}$ when $D_n(\theta, \theta')/n \to J(\theta, \theta')$ (uniformly) with $J(\theta, \theta')$ continuous and > 0 for all $\theta \neq \theta'$
$\hookrightarrow$ in linear models, $\eta(x, \theta) = \mathbf{f}^\top(x)\theta$: condition equivalent to $(1/n)[\mathbf{X}_n^\top \mathbf{X}_n] \to \mathbf{M}$ positive-definite
with $\mathbf{X}_n = [\mathbf{f}(x_1), \ldots, \mathbf{f}(x_n)]^\top$ (persistency of excitation)
➟ much stronger than $\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\{x_i\}$ = deterministic sequence

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

➟ Linear regression: NSC for the strong consistency of $\hat{\theta}^n$: $\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

➟ Nonlinear regression:
Define $D_n(\theta, \bar{\theta}) = \sum_{k=1}^n [\eta(x_k, \theta) - \eta(x_k, \bar{\theta})]^2$
[Jennrich 1969]: $\hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ when $D_n(\theta, \theta')/n \to J(\theta, \theta')$ (uniformly) with $J(\theta, \theta')$ continuous and > 0 for all $\theta \neq \theta'$
$\hookrightarrow$ in linear models, $\eta(x, \theta) = \mathbf{f}^\top(x)\theta$: condition equivalent to $(1/n)[\mathbf{X}_n^\top \mathbf{X}_n] \to \mathbf{M}$ positive-definite
with $\mathbf{X}_n = [\mathbf{f}(x_1), \ldots, \mathbf{f}(x_n)]^\top$ (persistency of excitation)
➟ much stronger than $\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \to \infty$

One would expect something like $D_n(\theta, \theta') \to \infty \ \forall \theta \neq \theta'$
([Wu 1981]: sufficient for $\hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ when Θ is finite, additional conditions required otherwise, necessary for the existence of a weakly consistent estimator)

x_i depends on $\varepsilon_{i-1}, \varepsilon_{i-2} \ldots$

In particular: sequential design (x_i depends on $\hat{\theta}^{i-1}$)

➟ Linear regression

[Lai & Wei, 1982]: SC

$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} \infty$ and

$\{\log \lambda_{\max}[\mathbf{X}_n^\top \mathbf{X}_n]\}^\rho / \lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} 0$ for some $\rho > 1$

$\Rightarrow \hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ ($\sim$ weakest possible condition)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

x_i depends on $\varepsilon_{i-1}, \varepsilon_{i-2} \ldots$

In particular: sequential design (x_i depends on $\hat{\theta}^{i-1}$)

➟ Linear regression

[Lai & Wei, 1982]: SC

$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} \infty$ and

$\{\log \lambda_{\max}[\mathbf{X}_n^\top \mathbf{X}_n]\}^\rho / \lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} 0$ for some $\rho > 1$

$\Rightarrow \hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ ($\sim$ weakest possible condition)

➟ Nonlinear regression

[Lai 1994] give a SC

equivalent in a linear context to [Christopeit & Helmes, 1980] :

$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} \infty$ and

$\lambda_{\max}[\mathbf{X}_n^\top \mathbf{X}_n] = \mathcal{O}\{\lambda_{\min}^\rho[\mathbf{X}_n^\top \mathbf{X}_n]\}$ a.s. for some $\rho \in (1, 2)$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

x_i depends on $\varepsilon_{i-1}, \varepsilon_{i-2} \ldots$

In particular: sequential design (x_i depends on $\hat{\theta}^{i-1}$)

➟ Linear regression

[Lai & Wei, 1982]: SC

$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} \infty$ and

$\{\log \lambda_{\max}[\mathbf{X}_n^\top \mathbf{X}_n]\}^\rho / \lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} 0$ for some $\rho > 1$

$\Rightarrow \hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ ($\sim$ weakest possible condition)

➟ Nonlinear regression

[Lai 1994] give a SC

equivalent in a linear context to [Christopeit & Helmes, 1980] :

$\lambda_{\min}[\mathbf{X}_n^\top \mathbf{X}_n] \stackrel{\text{a.s.}}{\to} \infty$ and

$\lambda_{\max}[\mathbf{X}_n^\top \mathbf{X}_n] = \mathcal{O}\{\lambda_{\min}^\rho[\mathbf{X}_n^\top \mathbf{X}_n]\}$ a.s. for some $\rho \in (1, 2)$

Results used

➟ in adaptive control: e.g., self-tuning regulator such that $R_n = \mathcal{O}(\log(n))$ [Lai & Wei, 1987; Guo 1994]

➟ . . . and in sequential design

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

3 Sequential design

3.1 *D*-optimal design

Regression model $\eta(x,\theta)$, nonlinear in θ

$\Rightarrow$ the optimal design for estimating θ depends on θ!

Information matrix $\mathbf{M}(\xi,\theta) = \int_{\mathcal{X}} \mathbf{f}_\theta(x)\mathbf{f}_\theta^\top(x)\,\xi(dx)$ with

- ξ a probability measure on $\mathcal{X}$
- $\mathbf{f}_\theta(x) = \frac{1}{\sigma}\,\frac{\partial\eta(x,\theta)}{\partial\theta}$

$\xi_D^*(\theta)$ is *D*-optimal for θ: $\xi_D^*(\theta)$ maximizes $\log\det[\mathbf{M}(\xi,\theta)]$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

full sequential design: choose $x_1, \ldots, x_{n_0}$, estimate $\hat{\theta}^{n_0}$, set $k = n_0$ then

- design x_{k+1}
- observe Y_{k+1}
- re-estimate $\hat{\theta}^{k+1}$ (LS)
- $k \leftarrow k+1 \ldots$

D-optimality:

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \det\left[\sum_{i=1}^{k} \mathbf{f}_{\hat{\theta}^k}(x_i)\mathbf{f}_{\hat{\theta}^k}^\top(x_i) + \mathbf{f}_{\hat{\theta}^k}(x)\mathbf{f}_{\hat{\theta}^k}^\top(x)\right]$$

or equivalently

$$\boxed{x_{k+1} = \arg\max_{x\in\mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x)\mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k}(x)}$$ with

$\xi_k = \sum_{i=1}^{k} \delta_{x_i}$ the empirical measure defined by $x_1, \ldots, x_k$

$\rightarrow \mathbf{M}(\xi_k, \theta) = \frac{1}{k}\sum_{i=1}^{k} \mathbf{f}_\theta(x_i)\mathbf{f}_\theta^\top(x_i)$

full sequential design: choose $x_1, \ldots, x_{n_0}$, estimate $\hat{\theta}^{n_0}$, set $k = n_0$ then

- design x_{k+1}
- observe Y_{k+1}
- re-estimate $\hat{\theta}^{k+1}$ (LS)
- $k \leftarrow k+1 \ldots$

D-optimality:

$x_{k+1} = \arg\max_{x \in \mathcal{X}} \det \left[\sum_{i=1}^{k} \mathbf{f}_{\hat{\theta}^k}(x_i) \mathbf{f}_{\hat{\theta}^k}^\top(x_i) + \mathbf{f}_{\hat{\theta}^k}(x) \mathbf{f}_{\hat{\theta}^k}^\top(x) \right]$

or equivalently

$$\boxed{x_{k+1} = \arg\max_{x \in \mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x) \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k}(x)}$$ with

$\xi_k = \sum_{i=1}^{k} \delta_{x_i}$ the empirical measure defined by $x_1, \ldots, x_k$

$\rightarrow \mathbf{M}(\xi_k, \theta) = \frac{1}{k} \sum_{i=1}^{k} \mathbf{f}_\theta(x_i) \mathbf{f}_\theta^\top(x_i)$

we hope that $\hat{\theta}^n \stackrel{\text{a.s.}}{\rightarrow} \bar{\theta}$ and $\sqrt{n}(\hat{\theta}^n - \bar{\theta}) \stackrel{\text{d}}{\rightarrow} \mathcal{N}(\mathbf{0}, \mathbf{M}^{-1}[\xi_D^*(\bar{\theta}), \bar{\theta}])$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

How to prove that $\hat{\theta}^n \stackrel{\text{a.s.}}{\rightarrow} \bar{\theta}$ and $\sqrt{n}(\hat{\theta}^n - \bar{\theta}) \stackrel{\text{d}}{\rightarrow} \mathcal{N}(\mathbf{0}, \mathbf{M}^{-1}[\xi_D^*(\bar{\theta}), \bar{\theta}])$?

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

How to prove that $\hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ and $\sqrt{n}(\hat{\theta}^n - \bar{\theta}) \stackrel{\text{d}}{\to} \mathcal{N}(\mathbf{0}, \mathbf{M}^{-1}[\xi_D^*(\bar{\theta}), \bar{\theta}])$?

1. suitable deterministic choice of x_k when $k \in \{k_1, k_2, \ldots\}$, with $k_i = i^\alpha$, $\alpha \in (1, 2)$ [Lai 1994]

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

How to prove that $\hat{\theta}^n \stackrel{\text{a.s.}}{\to} \bar{\theta}$ and $\sqrt{n}(\hat{\theta}^n - \bar{\theta}) \stackrel{\text{d}}{\to} \mathcal{N}(\mathbf{0}, \mathbf{M}^{-1}[\xi_D^*(\bar{\theta}), \bar{\theta}])$?

1. suitable deterministic choice of x_k when $k \in \{k_1, k_2, \ldots\}$, with $k_i = i^\alpha$, $\alpha \in (1, 2)$ [Lai 1994]
2. let n_0 tend to ∞ when $n \to \infty$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

How to prove that $\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}$ and $\sqrt{n}(\hat{\theta}^n - \bar{\theta}) \overset{\text{d}}{\to} \mathcal{N}(\mathbf{0}, \mathbf{M}^{-1}[\xi_D^*(\bar{\theta}), \bar{\theta}])$?

1. suitable deterministic choice of x_k when $k \in \{k_1, k_2, \ldots\}$, with $k_i = i^\alpha$, $\alpha \in (1, 2)$ [Lai 1994]
2. let n_0 tend to ∞ when $n \to \infty$
3. suppose that $\mathcal{X}$ is a finite set
   $\Rightarrow$ replication of observations

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\underline{\mathcal{X} \text{ is a finite set}}$ $\mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

Convergence of $\hat{\theta}^n$ to $\bar{\theta}$: everything is fine if $D_n(\theta, \bar{\theta}) = \sum_{k=1}^{n} [\eta(x_i, \theta) - \eta(x_i, \bar{\theta})]^2$ grows to ∞ fast enough for all $\theta \neq \bar{\theta}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\mathcal{X}$ is a finite set $\mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

Convergence of $\hat{\theta}^n$ to $\bar{\theta}$: everything is fine if $D_n(\theta, \bar{\theta}) = \sum_{k=1}^n [\eta(x_i, \theta) - \eta(x_i, \bar{\theta})]^2$ grows to ∞ fast enough for all $\theta \neq \bar{\theta}$

Theorem 1: convergence [LP, S&P Letters, 2009]

If $D_n(\theta, \bar{\theta}) = \sum_{k=1}^n [\eta(x_i, \theta) - \eta(x_i, \bar{\theta})]^2$ satisfies

$$\text{for all } \; \delta > 0 \, , \; \left[\inf_{\|\theta - \bar{\theta}\| \geq \delta/\tau_n} D_n(\theta, \bar{\theta}) \right] / (\log \log n) \stackrel{\text{a.s.}}{\to} \infty$$

with $\mathcal{X}$ finite and $\{\tau_n\}$ a non-decreasing sequence of positive constants, then $\hat{\theta}^n$ satisfies $\tau_n \|\hat{\theta}^n - \bar{\theta}\| \stackrel{\text{a.s.}}{\to} 0$

(Replace $\log \log n$ by $(\log n)^\rho$, $\rho > 1$, if $\{\varepsilon_i\}$ forms a martingale difference sequence)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Theorem 2: asymptotic normality [LP, S&P Letters, 2009]

If there exists a sequence of matrices $\mathbf{C}_n$ symmetric pos. def. such that $\mathbf{C}_n^{-1}\mathbf{M}^{1/2}(\xi_n,\bar{\theta}) \stackrel{\mathrm{p}}{\rightarrow} \mathbf{I}$
with $c_n = \lambda_{\min}(\mathbf{C}_n)$ and $D_n(\theta,\bar{\theta})$ satisfying $n^{1/4}c_n \rightarrow \infty$ and

$$\text{for all } \ \delta > 0\,, \ \left[\inf_{\|\theta-\bar{\theta}\| \geq c_n^2 \delta} D_n(\theta,\bar{\theta}) \right] /(\log\log n) \stackrel{\mathrm{a.s.}}{\rightarrow} \infty$$

then $\hat{\theta}^n$ satisfies $\sqrt{n}\,\mathbf{M}^{1/2}(\xi_n,\hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \stackrel{\mathrm{d}}{\rightarrow} \mathcal{N}(\mathbf{0},\mathbf{I})$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Theorem 2: asymptotic normality [LP, S&P Letters, 2009]

If there exists a sequence of matrices $\mathbf{C}_n$ symmetric pos. def. such that $\mathbf{C}_n^{-1}\mathbf{M}^{1/2}(\xi_n,\bar{\theta}) \xrightarrow{\text{p}} \mathbf{I}$
with $c_n = \lambda_{\min}(\mathbf{C}_n)$ and $D_n(\theta,\bar{\theta})$ satisfying $n^{1/4}c_n \to \infty$ and

$$\text{for all } \delta > 0\,, \; \left[\inf_{\|\theta-\bar{\theta}\|\geq c_n^2\delta} D_n(\theta,\bar{\theta})\right] / (\log\log n) \xrightarrow{\text{a.s.}} \infty$$

then $\hat{\theta}^n$ satisfies $\sqrt{n}\,\mathbf{M}^{1/2}(\xi_n,\hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \xrightarrow{\text{d}} \mathcal{N}(\mathbf{0},\mathbf{I})$

☞ Apply that to

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x)\mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k}(x)$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Theorem 2: asymptotic normality [LP, S&P Letters, 2009]

If there exists a sequence of matrices $\mathbf{C}_n$ symmetric pos. def. such that $\mathbf{C}_n^{-1}\mathbf{M}^{1/2}(\xi_n,\bar{\theta}) \stackrel{\mathrm{p}}{\to} \mathbf{I}$
with $c_n = \lambda_{\min}(\mathbf{C}_n)$ and $D_n(\theta,\bar{\theta})$ satisfying $n^{1/4}c_n \to \infty$ and

$$\text{for all} \quad \delta > 0\,, \quad \left[\inf_{\|\theta-\bar{\theta}\|\geq c_n^2\delta} D_n(\theta,\bar{\theta})\right] /(\log\log n) \stackrel{\mathrm{a.s.}}{\to} \infty$$

then $\hat{\theta}^n$ satisfies $\sqrt{n}\,\mathbf{M}^{1/2}(\xi_n,\hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \stackrel{\mathrm{d}}{\to} \mathcal{N}(\mathbf{0},\mathbf{I})$

☞ Apply that to

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x)\mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k}(x)$$

➡ $\hat{\theta}^n \stackrel{\mathrm{a.s.}}{\to} \bar{\theta}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Theorem 2: asymptotic normality [LP, S&P Letters, 2009]

If there exists a sequence of matrices $\mathbf{C}_n$ symmetric pos. def. such that $\mathbf{C}_n^{-1}\mathbf{M}^{1/2}(\xi_n, \bar{\theta}) \stackrel{\mathrm{p}}{\to} \mathbf{I}$
with $c_n = \lambda_{\min}(\mathbf{C}_n)$ and $D_n(\theta, \bar{\theta})$ satisfying $n^{1/4} c_n \to \infty$ and

$$\text{for all} \quad \delta > 0\,, \quad \left[\inf_{\|\theta - \bar{\theta}\| \geq c_n^2 \delta} D_n(\theta, \bar{\theta}) \right] / (\log \log n) \stackrel{\mathrm{a.s.}}{\to} \infty$$

then $\hat{\theta}^n$ satisfies $\sqrt{n}\, \mathbf{M}^{1/2}(\xi_n, \hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \stackrel{\mathrm{d}}{\to} \mathcal{N}(\mathbf{0}, \mathbf{I})$

☞ Apply that to

$$x_{k+1} = \arg\max_{x \in \mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x) \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k}(x)$$

➟ $\hat{\theta}^n \stackrel{\mathrm{a.s.}}{\to} \bar{\theta}$

➟ $\mathbf{M}(\xi_n, \hat{\theta}^n) \stackrel{\mathrm{a.s.}}{\to} \mathbf{M}[\xi_D^*(\bar{\theta}), \bar{\theta}]$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Theorem 2: asymptotic normality [LP, S&P Letters, 2009]

If there exists a sequence of matrices $\mathbf{C}_n$ symmetric pos. def. such that $\mathbf{C}_n^{-1}\mathbf{M}^{1/2}(\xi_n,\bar{\theta}) \xrightarrow{\mathrm{p}} \mathbf{I}$
with $c_n = \lambda_{\min}(\mathbf{C}_n)$ and $D_n(\theta,\bar{\theta})$ satisfying $n^{1/4}c_n \to \infty$ and

$$\text{for all } \ \delta > 0\,, \ \left[\inf_{\|\theta-\bar{\theta}\|\geq c_n^2\delta} D_n(\theta,\bar{\theta})\right]/(\log\log n) \stackrel{\mathrm{a.s.}}{\to} \infty$$

then $\hat{\theta}^n$ satisfies $\sqrt{n}\,\mathbf{M}^{1/2}(\xi_n,\hat{\theta}^n)(\hat{\theta}^n-\bar{\theta}) \xrightarrow{\mathrm{d}} \mathcal{N}(\mathbf{0},\mathbf{I})$

☞ Apply that to

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \mathbf{f}_{\hat{\theta}^k}^\top(x)\mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k}(x)$$

➟ $\hat{\theta}^n \stackrel{\mathrm{a.s.}}{\to} \bar{\theta}$

➟ $\mathbf{M}(\xi_n,\hat{\theta}^n) \stackrel{\mathrm{a.s.}}{\to} \mathbf{M}[\xi_D^*(\bar{\theta}),\bar{\theta}]$

➟ $[n\mathbf{M}(\xi_n,\hat{\theta}^n)]^{1/2}(\hat{\theta}^n-\bar{\theta}) \xrightarrow{\mathrm{d}} \mathcal{N}(\mathbf{0},\mathbf{I})$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

3.2 Penalized D-optimal design

Maximize $\log\det \mathbf{M}(\xi,\theta)$ under the contraint $\Phi(\xi,\theta) \leq C$
with $\Phi(\xi,\theta) = \int_{\mathcal{X}} \phi(x,\theta)\,\xi(dx)$
($\phi(x,\theta)$ = cost of one observation at x)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

3.2 Penalized D-optimal design

Maximize $\log\det \mathbf{M}(\xi,\theta)$ under the contraint $\Phi(\xi,\theta) \leq C$
with $\Phi(\xi,\theta) = \int_{\mathcal{X}} \phi(x,\theta)\, \xi(dx)$
($\phi(x,\theta)$ = cost of one observation at x)

NSC for the optimality of ξ^*:

$\Phi(\xi^*,\theta) \leq C$ and there exists $\lambda^* = \lambda^*(C,\theta) \geq 0$ such that

$$\begin{cases} \lambda^*[C - \Phi(\xi^*,\theta)] = 0 \\ \forall x \in \mathcal{X}\,,\ \mathbf{f}_\theta^\top(x)\mathbf{M}^{-1}(\xi^*,\theta)\mathbf{f}_\theta(x) \leq p + \lambda^*[\phi(x,\theta) - \Phi(\xi^*,\theta)] \end{cases}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

3.2 Penalized D-optimal design

Maximize $\log\det \mathbf{M}(\xi,\theta)$ under the contraint $\Phi(\xi,\theta) \leq C$
with $\Phi(\xi,\theta) = \int_{\mathcal{X}} \phi(x,\theta)\,\xi(dx)$
($\phi(x,\theta)$ = cost of one observation at x)
NSC for the optimality of ξ^*:
$\Phi(\xi^*,\theta) \leq C$ and there exists $\lambda^* = \lambda^*(C,\theta) \geq 0$ such that

$$\begin{cases} \lambda^*[C - \Phi(\xi^*,\theta)] = 0 \\ \forall x \in \mathcal{X}\,,\ \mathbf{f}_\theta^\top(x)\mathbf{M}^{-1}(\xi^*,\theta)\mathbf{f}_\theta(x) \leq p + \lambda^*[\phi(x,\theta) - \Phi(\xi^*,\theta)] \end{cases}$$

☞ In practice:
maximize $\boxed{H_\theta(\xi,\lambda) = \log\det \mathbf{M}(\xi,\theta) - \lambda\,\Phi(\xi,\theta)}$ (penalized D-optimal design) for an increasing sequence $\{\lambda_i\}$ of Lagrange multipliers, starting from $\lambda_0 = 0$ and stopping at the first λ_i for which $\xi^*(\lambda_i)$ satisfies $\Phi[\xi^*(\lambda_i),\theta] \leq C$ [Mikulecká 1983]
➟ not more complicated than the determination of (a sequence of) D-optimal design(s)!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction:

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x, \hat{\theta}^k) \right\}$$

☞ Use the assumption that $\mathcal{X}$ is finite: $\mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction:

$$x_{k+1} = \arg\max_{x \in \mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \, \phi(x, \hat{\theta}^k) \right\}$$

☞ Use the assumption that $\mathcal{X}$ is finite: $\mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

- If λ_k = constant λ (and $|\phi(x, \theta)|$ bounded) :

➠ $\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}$ and $\sqrt{n}\mathbf{M}^{1/2}(\xi_n, \hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \overset{\text{d}}{\to} \mathcal{N}(\mathbf{0}, \mathbf{I})$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction:

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\,\phi(x,\hat{\theta}^k) \right\}$$

☞ Use the assumption that $\mathcal{X}$ is finite: $\mathcal{X} = \{x^{(1)},\ldots,x^{(K)}\}$

- If λ_k = constant λ (and $|\phi(x,\theta)|$ bounded) :

  ➠ $\hat{\theta}^n \stackrel{\text{a.s.}}{\rightarrow} \bar{\theta}$ and $\sqrt{n}\mathbf{M}^{1/2}(\xi_n,\hat{\theta}^n)(\hat{\theta}^n - \bar{\theta}) \stackrel{\text{d}}{\rightarrow} \mathcal{N}(\mathbf{0},\mathbf{I})$

- If, moreover, $\ldots\phi(x,\theta)$ continuous in θ for all x:

  ➠ $\mathbf{M}(\xi_n,\bar{\theta}) \rightarrow \mathbf{M}^*(\bar{\theta})$, optimal for criterion $\log\det\mathbf{M}(\xi,\bar{\theta}) - \lambda\,\Phi(\xi,\bar{\theta})$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction:

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat\theta^k}^\top \mathbf{M}^{-1}(\xi_k,\hat\theta^k)\mathbf{f}_{\hat\theta^k} - \lambda_k\,\phi(x,\hat\theta^k) \right\}$$

☞ Use the assumption that $\mathcal{X}$ is finite: $\mathcal{X} = \{x^{(1)},\ldots,x^{(K)}\}$

- If λ_k = constant λ (and $|\phi(x,\theta)|$ bounded) :

  ➠ $\hat\theta^n \overset{\text{a.s.}}{\to} \bar\theta$ and $\sqrt{n}\mathbf{M}^{1/2}(\xi_n,\hat\theta^n)(\hat\theta^n - \bar\theta) \overset{\text{d}}{\to} \mathcal{N}(\mathbf{0},\mathbf{I})$

- If, moreover, $\ldots\phi(x,\theta)$ continuous in θ for all x:

  ➠ $\mathbf{M}(\xi_n,\bar\theta) \to \mathbf{M}^*(\bar\theta)$, optimal for criterion $\log\det\mathbf{M}(\xi,\bar\theta) - \lambda\,\Phi(\xi,\bar\theta)$

- Also true if λ_k = bounded measurable function of $x_1, Y_1, \ldots, x_k, Y_k$ (e.g., $\lambda_k = \lambda^*(\hat\theta^k)$ = optimal Lagrange coefficient for minimization of $\log\det\mathbf{M}(\xi,\hat\theta^k)$ under the constraint $\Phi(\xi,\hat\theta^k) \le C$)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

- If $\lambda_k \nearrow \infty$, $(\lambda_k \log\log k)/k \to 0$, then $\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}$
  moreover, convergence to minimum-cost design:
  $$\Phi(\xi_n, \bar{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i, \bar{\theta}) \overset{\text{a.s.}}{\to} \phi^*_{\bar{\theta}} = \min_{x \in \mathcal{X}} \phi(x, \bar{\theta})$$
  ... and $\xi_N(x^{(i^*)}) \overset{\text{a.s.}}{\to} 1$ if $\phi(x, \bar{\theta})$ has a unique minimum at $x^{(i^*)} \in \mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

- If $\lambda_k \nearrow \infty$, $(\lambda_k \log\log k)/k \to 0$, then $\boxed{\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}}$
  moreover, convergence to minimum-cost design:

$$\boxed{\Phi(\xi_n, \bar{\theta}) = \frac{1}{n}\sum_{i=1}^n \phi(x_i, \bar{\theta}) \overset{\text{a.s.}}{\to} \phi^*_{\bar{\theta}} = \min_{x\in\mathcal{X}} \phi(x, \bar{\theta})}$$

  ...and $\boxed{\xi_N(x^{(i^*)}) \overset{\text{a.s.}}{\to} 1}$ if $\phi(x, \bar{\theta})$ has a unique minimum at $x^{(i^*)} \in \mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

$\Rightarrow$ We can thus optimize $\sum_{i=1}^n \phi(x_i, \bar{\theta})$ without knowing $\bar{\theta}$:
self-tuning optimization

$$\boxed{x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}^\top_{\hat{\theta}^k} \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x, \hat{\theta}^k) \right\}}$$

Already suggested for linear regression ($\eta(x, \theta)$ linear in θ) [Åström & Wittenmark, 1989], condition on λ_k in [LP, AS 2000]

Here, LS in nonlinear regression, but with $\mathcal{X}$ finite

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

- If $\lambda_k \nearrow \infty$, $(\lambda_k \log\log k)/k \to 0$, then $\boxed{\hat{\theta}^n \overset{\text{a.s.}}{\to} \bar{\theta}}$
  moreover, convergence to minimum-cost design:
  $$\boxed{\Phi(\xi_n, \bar{\theta}) = \frac{1}{n}\sum_{i=1}^n \phi(x_i, \bar{\theta}) \overset{\text{a.s.}}{\to} \phi^*_{\bar{\theta}} = \min_{x\in\mathcal{X}} \phi(x, \bar{\theta})}$$
  …and $\boxed{\xi_N(x^{(i^*)}) \overset{\text{a.s.}}{\to} 1}$ if $\phi(x, \bar{\theta})$ has a unique minimum at $x^{(i^*)} \in \mathcal{X} = \{x^{(1)}, \ldots, x^{(K)}\}$

$\Rightarrow$ We can thus optimize $\sum_{i=1}^n \phi(x_i, \bar{\theta})$ without knowing $\bar{\theta}$: **self-tuning optimization**

$$\boxed{x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}^\top_{\hat{\theta}^k} \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \, \phi(x, \hat{\theta}^k) \right\}}$$

Already suggested for linear regression ($\eta(x, \theta)$ linear in θ) [Åström & Wittenmark, 1989], condition on λ_k in [LP, AS 2000]

Here, LS in nonlinear regression, but with $\mathcal{X}$ finite

Beware: $x_{k+1} = \arg\min_{x\in\mathcal{X}} \phi(x, \hat{\theta}^k)$ (best intention design) may not work!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

4 Examples

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 4 (self-tuning regulation)

Determine x^* such that $\psi(x, \bar{\theta}) = T \rightarrow$ minimize $\phi(x, \bar{\theta}) = [\psi(x, \bar{\theta}) - T]^2$

- Michaelis-Menten model:
  $Y_i = \frac{\bar{\theta}_1 x}{\bar{\theta}_2 + x} + \varepsilon_i$, $\{\varepsilon_i\}$ i.i.d. $\mathcal{N}(0, 1)$
- Function to be minimized :
  $T = 1/2 \Rightarrow \phi(x, \theta) = \left[\frac{\theta_1 x}{\theta_2 + x} - \frac{1}{2}\right]^2$,
- $x \in [0, 10]$, (grid with 1001 points)
- Simulation with $\bar{\theta} = (1, 1)^\top$
  $\Rightarrow \eta(x, \bar{\theta}) = x/(1 + x)$, $x^* = 1$ (and $\phi(x^*, \bar{\theta}) = 0$)
- $\lambda_k = (\log k)^8$

Here, $\psi(x,\theta) = \eta(x,\theta) \rightarrow$ we directly observe $\psi(x_i,\bar{\theta}) + \varepsilon_i$

$\Psi(x,\bar{\theta})$ estimable if $\{x_n\}$ has a cluster point at x
$\Rightarrow$ estimating $\bar{\theta}$ is not necessary
(it is enough to correctly estimate the sign of the derivative of $\Psi(x,\bar{\theta})$ at x^* such that $\Psi(x^*,\bar{\theta}) = T$)

$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x,\hat{\theta}^k) \right\}$ works provided that $\lambda_k \rightarrow \infty$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Here, $\psi(x,\theta) = \eta(x,\theta) \rightarrow$ we directly observe $\psi(x_i,\bar{\theta}) + \varepsilon_i$

$\Psi(x,\bar{\theta})$ estimable if $\{x_n\}$ has a cluster point at x
$\Rightarrow$ estimating $\bar{\theta}$ is not necessary
(it is enough to correctly estimate the sign of the derivative of $\Psi(x,\bar{\theta})$ at x^* such that $\Psi(x^*,\bar{\theta}) = T$)

$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k,\hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x,\hat{\theta}^k) \right\}$ works provided that $\lambda_k \rightarrow \infty$

In this particular case, one may take
$x_{k+1} = \arg\min_{x\in\mathcal{X}} \phi(x,\hat{\theta}^k)$ (= best intention design = continual reassessment method = forced certainty equivalence)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Alternative approach [Lai & Robbins 1978] :

$$x_{k+1} = x_k - \frac{Y_k - T}{k\,\hat{\beta}^k}$$

with

$$\hat{\beta}^k = \frac{\sum_{i=1}^k (x_i - \bar{x}_k)(Y_i - \bar{Y}_k)}{\sum_{i=1}^k (x_i - \bar{x}_k)^2} \text{ truncated at } [\underline{\beta}, \overline{\beta}]$$

$\hat{\beta}^k$ = LS estimator in $Y_i = T + \beta(x_i - x^*) + \varepsilon_i$
$\hat{\beta}^k$ = constant $\rightarrow$ stochastic approximation
[Robbins-Monro, 1951]

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control

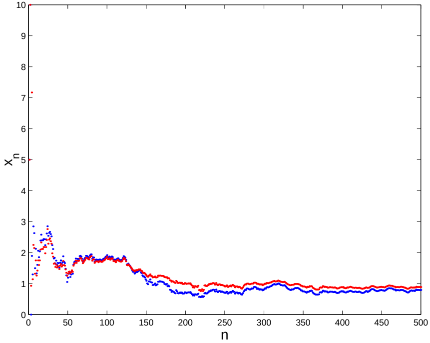
$$x_{k+1} = \arg\max_{x \in \mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \, \phi(x, \hat{\theta}^k) \right\}$$

$\approx$ similar behavior for $x_{k+1} = \arg\min_{x \in \mathcal{X}} \phi(x, \hat{\theta}^k)$

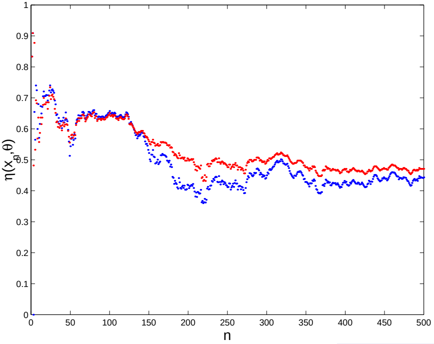
($\hat{\theta}^k$ truncated at $\theta_1 > 1/3$, $\theta_2 > 10^{-2}$)

$x_{k+1} = x_k - \frac{Y_k - T}{k\,\hat{\beta}_k}$ ($\hat{\beta}_k$ truncated at $[10^{-2}, 5]$)

$\rightarrow$ sequence $\{x_n\}$



$\rightarrow$ sequence $\{\eta(x_n, \bar{\theta})\}$



Example 5 (self-tuning regulation — continued)

- $\approx$ Example 4, $Y_i = \frac{\bar{\theta}_1 x}{\bar{\theta}_2 + x} + \varepsilon_i$, $\{\varepsilon_i\}$ i.i.d. $\mathcal{N}(0, 0.1)$
- determine x^* such that $\psi(x, \bar{\theta}) = T \rightarrow$ minimize $\phi(x, \bar{\theta}) = [\psi(x, \bar{\theta}) - T]^2$, with now

$$\Psi(x, \theta) = \theta_1 [1 - \exp(-\theta_2 x/3)] \neq \eta(x, \bar{\theta})$$

$(\bar{\theta} = (1, 1)^\top \Rightarrow \Psi(x^*, \bar{\theta}) = 1/2$ for $x^* = 3 \log(2) \simeq 2.08)$

- More difficult than example 2: we do not observe $\Psi(x, \bar{\theta})$!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 5 (self-tuning regulation — continued)

- $\approx$ Example 4, $Y_i = \frac{\bar{\theta}_1 x}{\bar{\theta}_2 + x} + \varepsilon_i$, $\{\varepsilon_i\}$ i.i.d. $\mathcal{N}(0, 0.1)$
- determine x^* such that $\psi(x, \bar{\theta}) = T \rightarrow$ minimize $\phi(x, \bar{\theta}) = [\psi(x, \bar{\theta}) - T]^2$, with now

$$\Psi(x, \theta) = \theta_1[1 - \exp(-\theta_2 x/3)] \neq \eta(x, \bar{\theta})$$

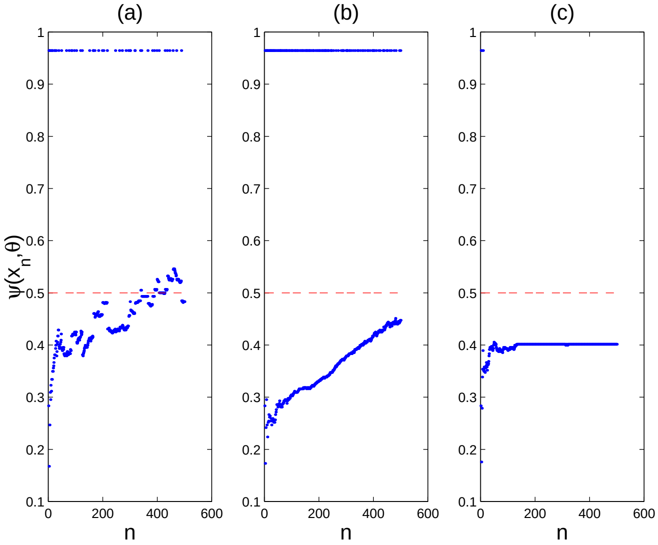
$(\bar{\theta} = (1,1)^\top \Rightarrow \Psi(x^*, \bar{\theta}) = 1/2$ for $x^* = 3\log(2) \simeq 2.08)$

- More difficult than example 2: we do not observe $\Psi(x, \bar{\theta})$!
- $x_1 = 1$, $x_2 = 10$, then $x_{k+1} = \arg\max_{x \in \mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \phi(x, \hat{\theta}^k) \right\}$ for three sequences $\{\lambda_k\}$
    - (a) $\lambda_k = \log^2 k$
    - (b) $\lambda_k = k/(1 + \log^2 k)$
    - (c) $\lambda_k = k^{1.1}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\rightarrow \Psi(x_k, \bar{\theta}),\ k = 1, \ldots, 500$

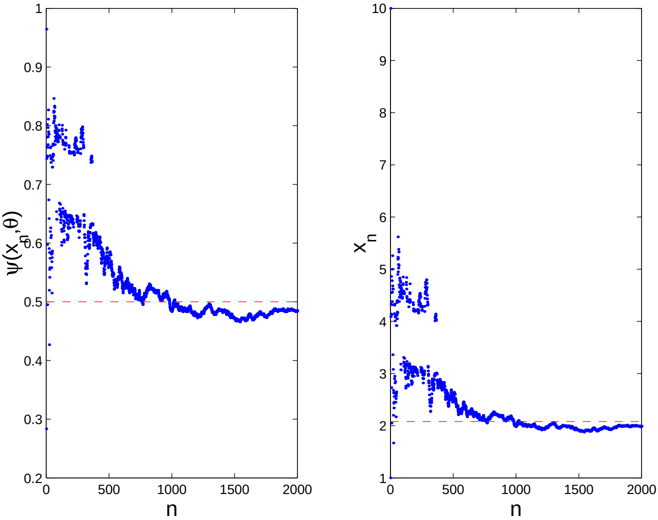
(a) $\lambda_k = \log^2 k$ (b) $\lambda_k = k/(1 + \log^2 k)$ (c) $\lambda_k = k^{1.1}$



1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Replace $\phi(x,\theta) = [\psi(x,\theta) - T]^2$ by $\phi(x,\theta) = [\psi(x,\theta) - T]^4$, take $\lambda_k = 10^3 \log^2 k$

$\rightarrow$ the support points of ξ_k tend to concentrate around x^*

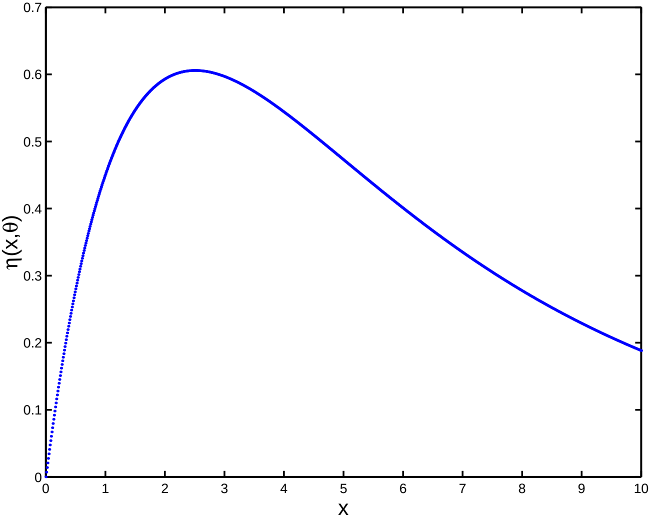


1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Example 6 (self-tuning optimization)

- Model [Box & Lucas, 1959]:
  $Y_i = \eta(x_i, \bar{\theta}) + \varepsilon_i$, $\{\varepsilon_i\}$ i.i.d. $\mathcal{N}(0, \sigma^2)$
  $\eta(x, \theta) = \frac{\theta_1}{\theta_1 - \theta_2} [\exp(-\theta_2 x) - \exp(-\theta_1 x)]$
- We want to maximize $\eta(x, \bar{\theta})$, $x \in [0, 10]$, grid with 1001 points
- $\bar{\theta} = (0.7\,,\ 0.2)^\top$
  $\Rightarrow \xi_D^* = \frac{1}{2}\delta_{x^{(1)}} + \frac{1}{2}\delta_{x^{(2)}}$ with $x^{(1)} \simeq 1.25$ and $x^{(2)} \simeq 6.60$
  $x^* = 2.51$, $\eta(x^*, \bar{\theta}) = \max_{x \in \mathcal{X}} \eta(x, \bar{\theta}) \simeq 0.606$



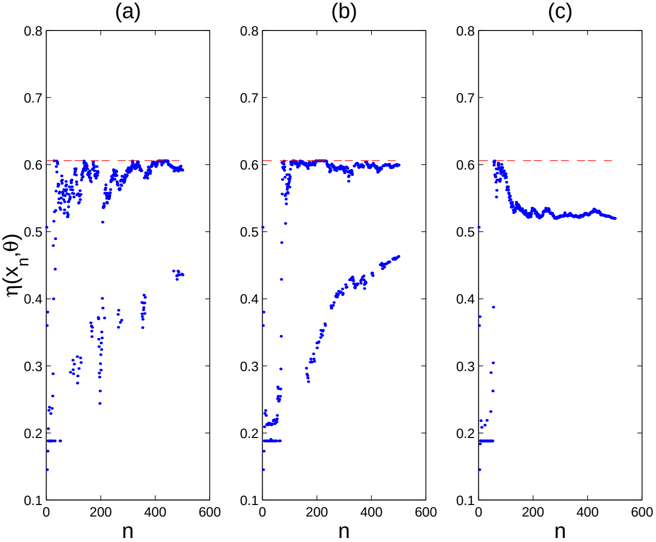
1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

- $x_1 = 1.25$, $x_2 = 6.6$, then
  $x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x, \hat{\theta}^k) \right\}$
  for three sequences $\{\lambda_k\}$
  - (a) $\lambda_k = \log^2 k$
  - (b) $\lambda_k = k/(1+\log^2 k)$
  - (c) $\lambda_k = k^{1.1}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

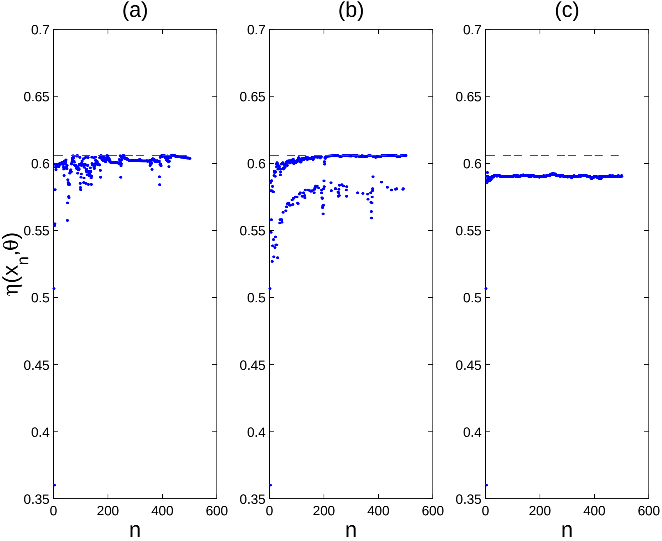
$\rightarrow \eta(x_k, \bar{\theta})$, $k = 1, \ldots, 500$, $\sigma = 1$

(a) $\lambda_k = \log^2 k$ (b) $\lambda_k = k/(1 + \log^2 k)$ (c) $\lambda_k = k^{1.1}$



$\rightarrow \eta(x_k, \bar{\theta})$, $k = 1, \ldots, 500$, $\sigma = 0.1$

(a) $\lambda_k = 10 \log^2 k$ (b) $\lambda_k = 10\, k/(1 + \log^2 k)$ (c) $\lambda_k = 10\, k^{1.1}$



1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\rightarrow$ an algorithmic problem

Example 8: D-optimal design for $\eta(x, \theta) = \theta_1 x + \theta_2 x^2$ (linear regression)

$x \in \mathcal{X} = [\underline{x}, 1] \subset [-1, 1]$, various designs depending on $\underline{x}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

5 Design with dynamical constraints

→ an algorithmic problem

Example 8: D-optimal design for $\eta(x,\theta) = \theta_1 x + \theta_2 x^2$ (linear regression)

$x \in \mathscr{X} = [\underline{x}, 1] \subset [-1, 1]$, various designs depending on $\underline{x}$

- $-1 \leq \underline{x} \leq -0.21685$ or $1/2 \leq \underline{x} \leq 1$ →
  $$\xi_D^* = \left\{ \begin{array}{cc} \underline{x} & 1 \\ 1/2 & 1/2 \end{array} \right\}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

→ an algorithmic problem

Example 8: D-optimal design for $\eta(x,\theta) = \theta_1 x + \theta_2 x^2$ (linear regression)

$x \in \mathcal{X} = [\underline{x}, 1] \subset [-1, 1]$, various designs depending on $\underline{x}$

- $-1 \leq \underline{x} \leq -0.21685$ or $1/2 \leq \underline{x} \leq 1$ → $\xi_D^* = \left\{ \begin{array}{cc} \underline{x} & 1 \\ 1/2 & 1/2 \end{array} \right\}$
- $-1/5 \leq \underline{x} \leq 1/2$ → $\xi_D^* = \left\{ \begin{array}{cc} 1/2 & 1 \\ 1/2 & 1/2 \end{array} \right\}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

5 Design with dynamical constraints

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

→ an algorithmic problem

Example 8: D-optimal design for $\eta(x,\theta) = \theta_1 x + \theta_2 x^2$ (linear regression)

$x \in \mathcal{X} = [\underline{x}, 1] \subset [-1, 1]$, various designs depending on $\underline{x}$

- $-1 \leq \underline{x} \leq -0.21685$ or $1/2 \leq \underline{x} \leq 1 \rightarrow$
  $$\xi_D^* = \left\{ \begin{array}{cc} \underline{x} & 1 \\ 1/2 & 1/2 \end{array} \right\}$$
- $-1/5 \leq \underline{x} \leq 1/2 \rightarrow \xi_D^* = \left\{ \begin{array}{cc} 1/2 & 1 \\ 1/2 & 1/2 \end{array} \right\}$
- $-0.21685 \leq \underline{x} \leq -1/5 \rightarrow$ 3-point measure
  $$\xi_D^* = \left\{ \begin{array}{ccc} \underline{x} & x' & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \right\}$$ with x' and α_3 close to 1/2

Sequential construction of ξ_D^*:
$x_{k+1} = \arg\max_{x\in\mathcal{X}} \mathbf{f}^\top(x)\mathbf{M}_k^{-1}\mathbf{f}(x)$, with
$\mathbf{f}(x) = (x\ x^2)^\top$, $\mathbf{M}_k = (1/k)\sum_{i=1}^{k} \mathbf{f}(x_i)\mathbf{f}^\top(x_i)$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction of ξ_D^*:
$x_{k+1} = \arg\max_{x \in \mathcal{X}} \mathbf{f}^\top(x)\mathbf{M}_k^{-1}\mathbf{f}(x)$, with
$\mathbf{f}(x) = (x\ x^2)^\top$, $\mathbf{M}_k = (1/k)\sum_{i=1}^{k} \mathbf{f}(x_i)\mathbf{f}^\top(x_i)$

Dynamically constrained design:
$x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots$, $x_1 = 1$

❶ $u_k \in \{-\delta, 0, \delta\}$, $\delta = 1/m$, $m \in \mathbb{N}^*$
$x_{k+1} = \arg\max_{u \in \{-\delta, 0, \delta\}} \mathbf{f}^\top(x_k + u)\mathbf{M}_k^{-1}\mathbf{f}(x_k + u)$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Sequential construction of ξ_D^*:
$x_{k+1} = \arg\max_{x \in \mathcal{X}} \mathbf{f}^\top(x)\mathbf{M}_k^{-1}\mathbf{f}(x)$, with
$\mathbf{f}(x) = (x\ x^2)^\top$, $\mathbf{M}_k = (1/k)\sum_{i=1}^k \mathbf{f}(x_i)\mathbf{f}^\top(x_i)$

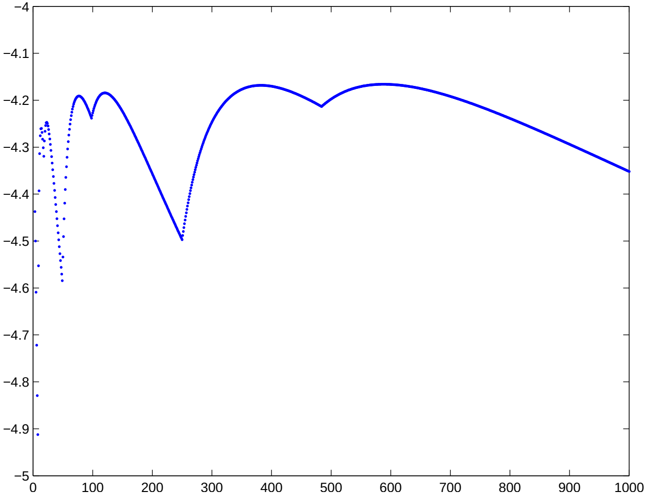
Dynamically constrained design:
$x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots$, $x_1 = 1$

❶ $u_k \in \{-\delta, 0, \delta\}$, $\delta = 1/m$, $m \in \mathbb{N}^*$
$x_{k+1} = \arg\max_{u \in \{-\delta, 0, \delta\}} \mathbf{f}^\top(x_k + u)\mathbf{M}_k^{-1}\mathbf{f}(x_k + u)$

$\mathbf{f}(0) = \mathbf{0} \Rightarrow$ we never cross $x = 0$
$\rightarrow$ same situation if $\underline{x} = 0$, the best we can hope is to reach
$\xi_{1/2} = \left\{ \begin{array}{cc} 1/2 & 1 \\ 1/2 & 1/2 \end{array} \right\}$, D-optimal when $\mathcal{X} = [0, 1]$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$\log\det \mathbf{M}_k\,,\ u_i \in \{-1/4, 0, 1/4\}$



➟ no convergence

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

For $u_k \in \{-\delta, 0, \delta\}$, $\delta = 1/(2q)$, $q \in \mathbb{N}^*$:
$x_k \in [1/2, 1]$ and

- $$\limsup_{n\to\infty} \log\det \mathbf{M}_n = \log\det \mathbf{M}(\xi_{1/2}) = -6 \log 2 \simeq -4.1589$$
  optimal on $\mathscr{X} = [0, 1]$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

For $u_k \in \{-\delta, 0, \delta\}$, $\delta = 1/(2q)$, $q \in \mathbb{N}^*$:
$x_k \in [1/2, 1]$ and

- $\limsup_{n\to\infty} \log\det \mathbf{M}_n = \log\det \mathbf{M}(\xi_{1/2})$
  $= -6 \log 2 \simeq -4.1589$
  optimal on $\mathcal{X} = [0, 1]$
- $\liminf_{n\to\infty} \log\det \mathbf{M}_n = \log\det \mathbf{M}(\xi_{\alpha^*})$ with
  $\xi_\alpha = \left\{ \begin{array}{cc} 1/2 & 1 \\ \alpha & 1-\alpha \end{array} \right\}$ and

$$\alpha = \begin{cases} \frac{16\delta(1-\delta)^2}{6+3\delta-20\delta^2+12\delta^3} & \text{if } \delta < \delta^* \simeq 0.058448 \\ \frac{8(1-2\delta^2)}{9+4\delta-12\delta^2} & \text{otherwise} \end{cases}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

❷ $x_1 = 1$, $x_n = -1$, n given, $u_1, \ldots, u_{n-1}$ such that $P_u = \frac{1}{n-1} \sum_{i=1}^{n-1} u_i^2 \leq P_{u\max}$ and $\log\det \mathbf{M}_n$ maximum

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

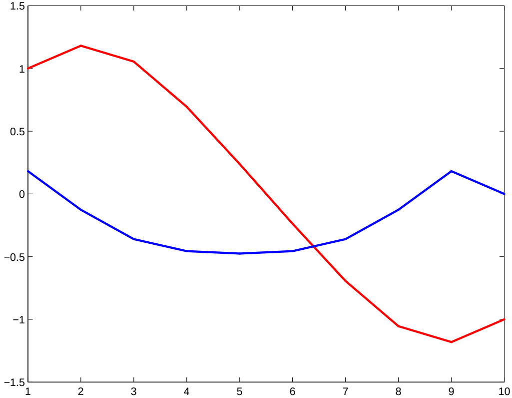
❷ $x_1 = 1$, $x_n = -1$, n given, $u_1, \ldots, u_{n-1}$ such that $P_u = \frac{1}{n-1} \sum_{i=1}^{n-1} u_i^2 \leq P_{u\max}$ and $\log\det \mathbf{M}_n$ maximum

- Direct optimization with respect to $u_1, \ldots, u_{n-1} \rightarrow$ complicated if n large

❷ $x_1 = 1$, $x_n = -1$, n given, $u_1, \ldots, u_{n-1}$ such that $P_u = \frac{1}{n-1}\sum_{i=1}^{n-1} u_i^2 \leq P_{u\max}$ and $\log\det \mathbf{M}_n$ maximum

- Direct optimization with respect to $u_1, \ldots, u_{n-1} \rightarrow$ complicated if n large
- Feedback optimal control: $u_k^* = \sum_{j=0}^{3} a_j x_k^j \rightarrow$ optimization with respect to $a_0, \ldots, a_3$ whatever n is

$n = 10$, $P_{u\max} = 1/9$, (u_k) and (x_k)



1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

❸ Heuristic approach based on unconstrained D-optimal design: $\mathcal{X} = [-1, 1] \Rightarrow \xi_D^* = \left\{ \begin{array}{cc} -1 & 1 \\ 1/2 & 1/2 \end{array} \right\}$ for $x_{k+1} = x_k + u_k$, use

$$u_i = \begin{cases} 0 & \text{for } i = 1, \ldots, n_1 \text{ (stay at } x = 1) \\ -\delta & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 \; (x = 1 \rightarrow x = -1) \\ 0 & \text{for } i = n_1 + n_2 + 1, \ldots, n - 1 \text{ (stay at } x = -1) \end{cases}$$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

❸ Heuristic approach based on unconstrained D-optimal design: $\mathcal{X} = [-1, 1] \Rightarrow \xi_D^* = \left\{ \begin{array}{cc} -1 & 1 \\ 1/2 & 1/2 \end{array} \right\}$ for $x_{k+1} = x_k + u_k$, use

$$u_i = \begin{cases} 0 & \text{for } i = 1, \ldots, n_1 \text{ (stay at } x = 1) \\ -\delta & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 \ (x = 1 \to x = -1) \\ 0 & \text{for } i = n_1 + n_2 + 1, \ldots, n - 1 \text{ (stay at } x = -1) \end{cases}$$

Then $n_2\delta = 2$, $P_u = \frac{n_2\delta^2}{n-1}$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

❸ Heuristic approach based on unconstrained D-optimal design: $\mathcal{X} = [-1, 1] \Rightarrow \xi_D^* = \left\{ \begin{array}{cc} -1 & 1 \\ 1/2 & 1/2 \end{array} \right\}$ for $x_{k+1} = x_k + u_k$, use

$$u_i = \begin{cases} 0 & \text{for } i = 1, \ldots, n_1 \text{ (stay at } x = 1) \\ -\delta & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 \ (x = 1 \rightarrow x = -1) \\ 0 & \text{for } i = n_1 + n_2 + 1, \ldots, n - 1 \text{ (stay at } x = -1) \end{cases}$$

Then $n_2\delta = 2$, $P_u = \frac{n_2\delta^2}{n-1}$

Try to treat separately the design and control problems

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

❸ Heuristic approach based on unconstrained D-optimal design: $\mathcal{X} = [-1, 1] \Rightarrow \xi_D^* = \left\{ \begin{array}{cc} -1 & 1 \\ 1/2 & 1/2 \end{array} \right\}$ for $x_{k+1} = x_k + u_k$, use

$$u_i = \begin{cases} 0 & \text{for } i = 1, \ldots, n_1 \text{ (stay at } x = 1) \\ -\delta & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 \; (x = 1 \to x = -1) \\ 0 & \text{for } i = n_1 + n_2 + 1, \ldots, n - 1 \text{ (stay at } x = -1) \end{cases}$$

Then $n_2\delta = 2$, $P_u = \frac{n_2\delta^2}{n-1}$

Try to treat separately the design and control problems

In nonlinear models, $\mathbf{M}(\xi)$ depends on θ
$\to$ adaptive construction
$\to$ an algorithmic and statistical problem!

$\to$ $\mathcal{X}$ finite may help

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

model-free design

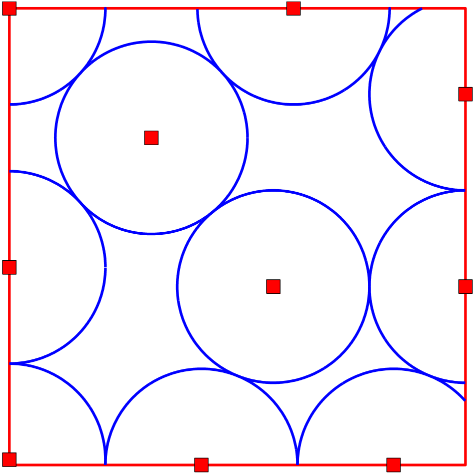
Maximin-optimal design: $\mathbf{X}_n = (X_1, \ldots, X_n)$, $X_i \in [0,1]^d$ for all $i \rightarrow$ maximize $\min_{i \neq j} \|X_i - X_j\|$

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

model-free design

Maximin-optimal design: $\mathbf{X}_n = (X_1, \ldots, X_n)$, $X_i \in [0,1]^d$ for all $i \to$ maximize $\min_{i \neq j} \|X_i - X_j\|$

$n = 10, d = 2$ (from http://www.packomania.com/)



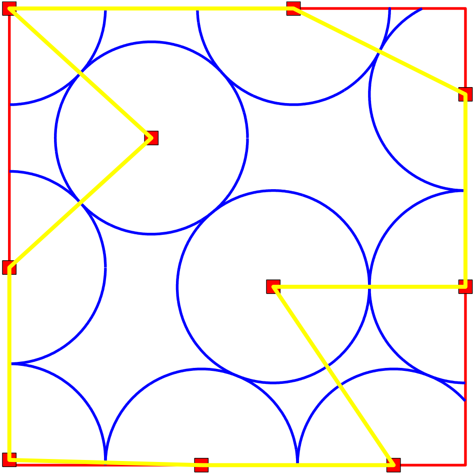
1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same dynamical constraint as previously: move as
$x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots$
Cheapest visit of all X_i in $\mathbf{X}_n$ $\rightarrow$ solve a TSP (shortest tour)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Same dynamical constraint as previously: move as $x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots$

Cheapest visit of all X_i in $\mathbf{X}_n$ → solve a TSP (shortest tour)



INI
20/07/2011

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

[Beardwood, Halton, Hammersley 1959]: X_i i.i.d., p.d.f. φ, TSP graph $\mathcal{G}_{TSP}(\mathbf{X})$

$$\frac{\sum_{e_i \in \mathcal{G}_{TSP}(\mathbf{X})} |e_i|}{n^{(d-1)/d}} \to C(d) \int \varphi^{(d-1)/d}(x)\, dx \text{ a.s.}, \; n \to \infty$$

then [Steele, 1981] for other Euclidean functionals on $\mathbf{X}$, [Redmond & Yukich, 1994] using quasi-additivity

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

[Beardwood, Halton, Hammersley 1959]: X_i i.i.d., p.d.f. φ, TSP graph $\mathcal{G}_{TSP}(\mathbf{X})$

$$\frac{\sum_{e_i \in \mathcal{G}_{TSP}(\mathbf{X})} |e_i|}{n^{(d-1)/d}} \to C(d) \int \varphi^{(d-1)/d}(x)\, dx \text{ a.s.}, \; n \to \infty$$

then [Steele, 1981] for other Euclidean functionals on $\mathbf{X}$, [Redmond & Yukich, 1994] using quasi-additivity

[Redmond & Yukich , 1996], [Yukich, 1998], [Penrose & Yukich 2003...2011], [Wade, 2011]:

$$\frac{\sum_{e_i \in \mathcal{G}(\mathbf{X})} |e_i|^\beta}{n^{(d-\beta)/d}} \to C(\beta, d) \int \varphi^{(d-\beta)/d}(x)\, dx \,, \; n \to \infty$$

with $\mathcal{G}(\mathbf{X})$ Minimum Spanning Tree (MST), NN, TSP, Voronoi, Delaunay, Sphere of Influence, Gabriel... (different types of convergence (L_p), different conditions on φ and β...)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\frac{\sum_{e_i \in \mathcal{G}(\mathbf{X})} |e_i|^\beta}{n^{(d-\beta)/d}} \to C(\beta, d) \int \varphi^{(d-\beta)/d}(x)\, dx\,, \;\; n \to \infty$$

- $\Rightarrow$ estimation of Rényi entropy
  $H_\alpha{}^*(\varphi) = \frac{1}{1-\alpha} \log \int \varphi^\alpha(t)\, dt$ with $\alpha = (d-\beta)/d$
  $1 \leq \beta < d \Rightarrow 0 < \alpha \leq 1 - 1/d$, RHS max for $\varphi =$ ct. (uniform)
  $\to$ maximize the LHS

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

$$\frac{\sum_{e_i \in \mathcal{G}(\mathbf{X})} |e_i|^\beta}{n^{(d-\beta)/d}} \to C(\beta, d) \int \varphi^{(d-\beta)/d}(x)\, dx\,, \;\; n \to \infty$$

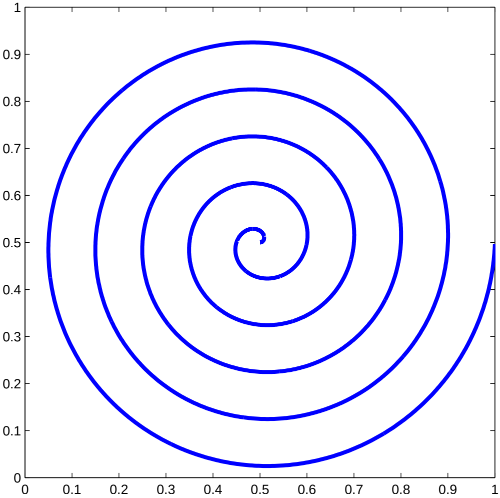
- $\Rightarrow$ estimation of Rényi entropy $H_\alpha{}^*(\varphi) = \frac{1}{1-\alpha} \log \int \varphi^\alpha(t)\, dt$ with $\alpha = (d-\beta)/d$
  $1 \leq \beta < d \Rightarrow 0 < \alpha \leq 1 - 1/d$, RHS max for $\varphi =$ ct. (uniform)
  $\to$ maximize the LHS
- We want a short route (minimize cost) but long routes are good in terms of space-filling!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Different problem if we observe all along $x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots \rightarrow$ space filling curve

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Different problem if we observe all along $x_{k+1} = x_k + u_k$, $k = 1, 2 \ldots \rightarrow$ space filling curve



1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

6 Conclusions

- Input design for dynamical system = control problem; adaptive control (self-tuning regulation/optimization) = design problem

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

6 Conclusions

- Input design for dynamical system = control problem; adaptive control (self-tuning regulation/optimization) = design problem
- sequential penalized D-optimal design

$$\boxed{x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x, \hat{\theta}^k) \right\}}$$

  - consistency and asymptotic normality of $\hat{\theta}^k$ when λ_k bounded
  - consistency when $\lambda_k \to \infty$ not too fast $\to$ self-tuning regulation/optimization

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

6 Conclusions

- Input design for dynamical system = control problem; adaptive control (self-tuning regulation/optimization) = design problem
- sequential penalized *D*-optimal design

$$x_{k+1} = \arg\max_{x \in \mathcal{X}} \left\{ \mathbf{f}^\top_{\hat{\theta}^k} \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \, \phi(x, \hat{\theta}^k) \right\}$$

  - consistency and asymptotic normality of $\hat{\theta}^k$ when λ_k bounded
  - consistency when $\lambda_k \to \infty$ not too fast $\to$ self-tuning regulation/optimization

...assuming that $\mathcal{X}$ is finite (only a technical assumption?)

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

6 Conclusions

- Input design for dynamical system = control problem; adaptive control (self-tuning regulation/optimization) = design problem
- sequential penalized D-optimal design

$$x_{k+1} = \arg\max_{x\in\mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k)\mathbf{f}_{\hat{\theta}^k} - \lambda_k\, \phi(x, \hat{\theta}^k) \right\}$$

  - consistency and asymptotic normality of $\hat{\theta}^k$ when λ_k bounded
  - consistency when $\lambda_k \to \infty$ not too fast $\to$ self-tuning regulation/optimization

  ... assuming that $\mathcal{X}$ is finite (only a technical assumption?)
- Asymptotic normality when $\lambda_n \to \infty$ slowly enough?

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

6 Conclusions

- Input design for dynamical system = control problem; adaptive control (self-tuning regulation/optimization) = design problem
- sequential penalized D-optimal design

$$x_{k+1} = \arg\max_{x \in \mathcal{X}} \left\{ \mathbf{f}_{\hat{\theta}^k}^\top \mathbf{M}^{-1}(\xi_k, \hat{\theta}^k) \mathbf{f}_{\hat{\theta}^k} - \lambda_k \, \phi(x, \hat{\theta}^k) \right\}$$

  - consistency and asymptotic normality of $\hat{\theta}^k$ when λ_k bounded
  - consistency when $\lambda_k \to \infty$ not too fast $\to$ self-tuning regulation/optimization

  ... assuming that $\mathcal{X}$ is finite (only a technical assumption?)
- Asymptotic normality when $\lambda_n \to \infty$ slowly enough?
- Dynamically constrained design $x_{k+1} = h(x_k, u_k) \to$ additional difficulties $\to$ try to visit optimal support points

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 D-optimal design
3.2 Penalized D-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions

Thank you for your attention!

1 Optimal input design
1.1 Time-domain input design
1.2 Frequency-domain input design (linear system)
2 Adaptive control
3 Sequential design
3.1 *D*-optimal design
3.2 Penalized *D*-optimal design
4 Examples
5 Design with dynamical constraints
6 Conclusions