PRECISION AND SENSITIVITY OF COHERENT AND INCOHERENT INVERSION METHODS: A COMPARATIVE STUDY

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ABSTRACT This paper presents a comparative study of the precision and sensitivity of coherent and incoherent inversion methods to modeling errors. It is based on the use of performance analysis tools: (i) the Cramer Rao lower bound, that establishes a bound on the precision of parametric estimates; (ii) a generalization of the classical ambiguity function that assesses the global performance under incorrect modeling assumptions, e.g. introduction of biases. Results showing the behaviour of the two methods are presented for specific examples.

1. Introduction

Acoustic tomography systems exploit the structure of the underwater propagation channel to estimate oceanographic parameters of interest. The accuracy of the propagation model used is crucial to the performance of these systems. In this paper, we consider two kinds of methods. *(i) Fully coherent* methods, that use directly the signals captured on the sensors of the receiving array: the so-called "matched field" approach. These methods are certainly the most powerful, taking advantage of the whole information contained in the acoustic field as sampled by the sensors. *(ii)* Methods that use only a set of selected features of the acoustic field, such as the time delays between the different paths that propagate from the source to the receiving points. These methods are *a priori* less performing than the matched field method, they are often used, however, since they are expected to be less sensitive to propagation modeling mismatches.

This paper addresses this question, comparing the **sensitivity to modeling errors** of the **two approaches** to inversion problems. The sensitivity of the first method is assessed by applying directly the ambiguity function defined in [4]. To study the second method, we use large sample results to derive the statistical characteristics of the estimated features.

The paper is organized in the following way. Section 2 states the problem and defines notation and nomenclature. Section 3 briefly decribes the performance analysis tools used

*on leave from Instituto Superior Técnico, Universidade Técnica de Lisboa, Portugal, under an ECE Fellowship (Human Capital and Mobility).

and section 4 defines the statistical distribution of the estimated inter-path delays used by the incoherent method. Finally, Section 5 applies the expressions previously developed to specific examples of ocean tomography, comparing the sensitivity of the two methods.

2. Problem Statement

The paper addresses the sensitivity study in the framework of a controlled acoustic experiment in the ocean. The signal emitted by a source at a *known* position is observed at a distant array of *known* geometry and localization, during a time interval T. The source signal $s(t), t \in T$, is usually known, providing in this way absolute time reference. Here, we address the case where the signal $s(t)$ is not exactly known at the receiver side, but is rather a pseudo-random Gaussian series, of *known* spectral characteristics, $S_s(\omega)$.

The transfer function of the ocean between source and the outputs of the receiving array is denoted throughout the paper by $\mathcal{H}^0[s(t)]$. This operator combines the effect of the propagation channel and of the receiving operator. To make our analysis concrete, we assume that the spatial variation of the sound speed induces multipath propagation, i.e., that $\mathcal{H}^0[s(t)]$ can be decomposed in the following way:

$$\mathcal{H}^0[s(t)] = \sum_{p=1}^{P^0} \mathcal{H}_p^0[s(t)], \tag{1}$$

where $\mathcal{H}_p^0[s(t)]$ denotes the transfer function along path p. We further assume that the medium is non-dispersive (linear), and that the signal received through each path is an attenuated and delayed version of the original signal, i.e.: $\mathcal{H}_p^0[s(t)] = a_p^0 s(t-\tau_p^0)$.

The signal radiated by the source is observed at the receiver side superimposed to a zero mean stationary Gaussian observation noise with spectral density matrix $\mathbf{S}_n(\omega)$. For simplicity, we assume that the noise is spatially white, i.e., $\mathbf{S}_n(\omega) = S_n(\omega)I_K$, where $S_n(\omega)$ is a real valued function, and I_K stands for the identity matrix of order K.

We denote by $r(t)$ the vector that collects the signals received at the K receiving sensors at time t. With the Gaussian hypothesis on the source signal and observation noise, $r(t)$ is stationary Gaussian, with zero mean and spectral density matrix given by:

$$\mathbf{S}_r(\omega) = S_s(\omega)h^0(\omega)h^{0^H}(\omega) + \mathbf{S}_n(\omega) \tag{2}$$

where $h^0(\omega)$ is the K-dimensional complex *resultant vector* combining the P^0 received paths. We denote the mathematical model used at receiver by $\mathcal{H}_\gamma[s(t)]$, explicitly showing its dependency on the desired ocean parameters γ. We assume that $\mathcal{H}_\gamma[s(t)]$ follows the same general form as $\mathcal{H}^0[s(t)]$.

3. Sensitivity Measure

The sensitivity analysis presented in this paper is based on a definition of ambiguity recently proposed [4]. In this section, we briefly present the definition of the function of interest to this study, referring the interested reader to [4], where a detailed presentation of the tool is given.

Statistically motivated parametric estimation methods are based on the knowledge of a family of conditional pdf's, that describes the dependency of the observed data distribution on the parameter of interest: $\mathcal{G}_\gamma = \{p(r|\gamma), \gamma \in \Gamma\}$. In the case of the *fully coherent methods*

r must be identified with the data collected at the sensors, while for the *incoherent methods* r is the output of a pre-processing estimation step.

When $\mathcal{G}_\gamma$ perfectly reflects the behaviour of the physical environment, the ambiguity function defined in [4] can be used to predict the global performance of the inversion mechanism, identifying possible large errors or configurations of poor observability of the parameter γ. Motivated by the close relationship between Maximum Likelihood estimators and the Kullback-Leibler directed divergence between probability density functions (pdf's) for exponential distributions, it defines ambiguity between two values $\gamma_0, \gamma \in \Gamma$ as a normalized version of the Kullback-Leibler distance between the corresponding elements of $\mathcal{G}_\gamma$.

In this study, we are interested in quantifying the impact of imperfect world knowledge, i.e., the fact that the true world's behaviour is described by a given family of distributions $\mathcal{G}_\gamma^0 = \{p^0(r|\gamma)\}$, and a distinct model is used in the fitting operation done at the receiver site: $\mathcal{G}_\gamma$. The observed data r is governed by a single member of $\mathcal{G}_\gamma^0$, that we denote by $p^0 = p^0(r|\gamma_0)$. The ability to correctly predict γ_0 using r under model mismatch can be analyzed studying a normalized version of the Kullback-Leibler distance between p^0 and the elements of $\mathcal{G}_\gamma$:

$$\mathcal{S}(\gamma) \triangleq 1 - \frac{I(p^0 : p(r|\gamma))}{I_{sup}(p^0)} \tag{3}$$

In this equation $I(p:q) = \mathrm{E}_p\{\ln(p/q)\}$ is the Kullback-Leibler directed divergence between pdf's p and q, where $\mathrm{E}_p\{\cdot\}$ is expectation with respect to the pdf p, and $I_{sup}(p^0)$ is an upper bound on $I(p^0 : p(r|\gamma))$.

The normalization in (3) is suitable to analyze the performance of a single method. However, when comparing several methods, as in this paper, it is more convenient to work directly with the Kullback directed divergence $I(p^0 : p(r|\gamma)$. In this way, we can not only compare the global performance of the different approaches, but also their local performance, using the relation between the gradient of the Kullback divergence and the Fisher information matrix (see [1]). Ideally, $I(p^0 : p(r|\gamma))$ is zero for $\gamma = \gamma_0$, and has large values for all $\gamma \neq \gamma_0$. Modeling errors induce systematic biases in the estimation procedure, that are flagged by the fact that $I(p^0 : p(r|\gamma)$ has its minimum at an erroneous value $\gamma^* \neq \gamma_0$.

3.1. INCOHERENT METHOD

For the incoherent inversion method, the input to the fitting operation are the inter-path delay estimates. Large observation interval assumptions justify the use of a Gaussian distribution to characterize these differential delay estimates: $p^0 = \mathcal{N}(\tau^0, \Sigma^0)$, where τ^0 is a vector collecting the $P^0 - 1$ true inter-path delays measured at a reference sensor, and Σ^0 is the variance of the delay estimates, predicted by the Cramer Rao lower bound.

The model used at the receiver in the fitting operation corresponds to applying the same large observation interval hypothesis, but using the wrong ocean/array model: $p(\hat{\tau}|\gamma) = \mathcal{N}(\tau(\gamma), \Sigma(\gamma))$. In Section 4 we give expressions for Σ^0, and $\Sigma(\gamma)$. With the previous assumptions, the Kullback directed divergence between densities p^0 and $p(\hat{\tau}|\gamma)$ is:

$$\mathrm{I}(\mathrm{p}^0 : \mathrm{p}(\hat{\tau}|\gamma)) = \frac{1}{2}\ln\frac{|\Sigma(\gamma)|}{|\Sigma^0|} + \frac{1}{2}\mathrm{tr}\left[\Sigma^0\Sigma(\gamma)^{-1} - I\right] + \frac{1}{2}\left(\tau^0 - \tau(\gamma)\right)^{\mathrm{T}}\Sigma(\gamma)^{-1}\left(\tau^0 - \tau(\gamma)\right). \tag{4}$$

3.2. FULLY COHERENT

The *fully coherent method* requires use of the Kullback directed divergence between the data records themselves. Invoking a large observation interval assumption, we use the asymptotic characterization of the directed divergence between stationary Gaussian processes derived in [2], that gives $I(p^0 : p(r|\gamma)$ in terms of the spectral densities of the observations:

$$\bar{\mathrm{I}}(\mathrm{p}^0 : \mathrm{p}(\mathrm{r}|\gamma)) = \lim_{\mathrm{T}\to\infty} \frac{1}{\mathrm{T}} \mathrm{I}(\mathrm{p}^0 : \mathrm{p}(\mathrm{r}|\gamma)) = \frac{1}{2} \int \left[\mathrm{tr}\left[S^0(\lambda) S_\gamma(\lambda)^{-1}\right] - \mathrm{K} - \ln \frac{|\mathrm{S}^0(\lambda)|}{|\mathrm{S}_\gamma(\lambda)|} \right] \mathrm{d}\lambda. \tag{5}$$

4. Statistical Characterization of Selected Features

We present in this section the expressions of Σ^0 and $\Sigma(\gamma)$. In [3] we derived the following expression for the Fisher information matrix for the estimation of parameters from observations following a multipath model:

$$[\mathcal{F}_\alpha]_{ij} = \frac{N}{2\pi} \int \left\{ K_1 \mathrm{Re} \left\{ \frac{\partial h}{\partial \alpha_i}^H P_{h^\perp} \frac{\partial h}{\partial \alpha_j} \right\} + K_2 \mathrm{Re} \left\{ \frac{\partial h}{\partial \alpha_i}^H h \right\} \mathrm{Re} \left\{ \{ h^H \frac{\partial h}{\partial \alpha_j} \right\} \right\} d\omega \tag{6}$$

where N is the number of snapshots used, $P_{h^\perp}$ denotes the orthogonal projection matrix in the orthogonal complement of the resultant vector h, and the following definitions hold:

$$K_1 = \frac{S_s^2(\omega)}{S_n(\omega) E} \|h\|^2, \qquad K_2 = \frac{S_s^2(\omega)}{E^2}, \qquad E = S_n(\omega) + S_s(\omega)\|h\|^2. \tag{7}$$

Besides the inter-path delays (τ), other parameters enter the multipath model, describing the wavefront shape across the receiving array (δ). The Cramer Rao lower bound on the inter-path delays is obtained by computing the inverse of corresponding block of the partition of the complete Fisher information matrix. Let $\alpha = [\tau \quad \delta]$, and $\mathcal{F}_\tau, \mathcal{F}_\delta$, and $\mathcal{F}_{\tau\delta}, \mathcal{F}_{\delta\tau}$ be the upper and lower diagonal, and the off-diagonal blocks of $\mathcal{F}_\alpha$, respectively. Then, the Cramer Rao lower bound on the inter-path delays is

$$CRB(\tau) = \left[\mathcal{F}_\tau - \mathcal{F}_{\tau\delta} \mathcal{F}_\delta^{-1} \mathcal{F}_{\delta\tau} \right]^{-1}. \tag{8}$$

In the paper we consider two different ways of modeling the wavefront shape: *(i)* no *a priori* knowledge, i.e., all the inter-path delays defining the shape of the each wavefront accross the array are estimated along with the inter-path delays; *(ii)* planar wavefront approximation, in which case a single parameter defines the orientation of the wavefront accross the array. The two approaches are compared in the simulations presented.

5. Examples

The model used at receiver assumes that the ocean is horizontally stratified, with two distinct homogeneous layers. In the superficial layer (up to depth y_{duct}), the sound speed decreases linearly with depth (with rate g_0), increasing linearly from y_{duct} until the ocean bottom. The ocean boundaries are perfectly flat, with reflection coefficients depending on the grazing angle. The duct and ocean depths, and the sound speed gradient in the deep layer (g_1) are perfectly known, as well as all other parameters relating to source (position, spectrum) and receiver array (geometry, localization, gain).

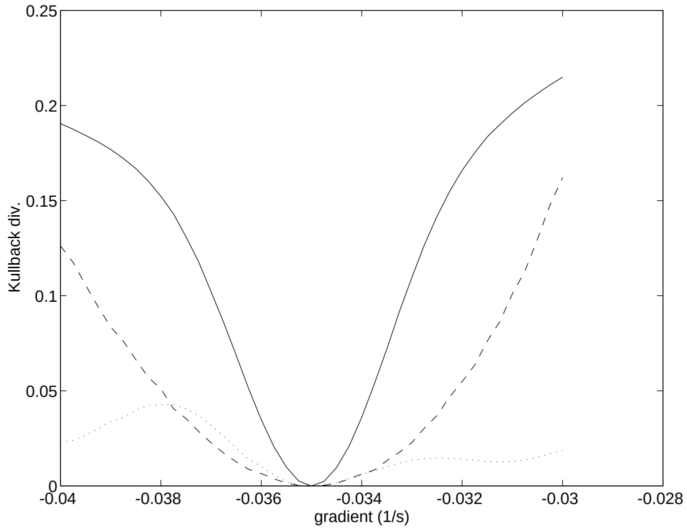

FIGURE 1. Kullback divergence for correct model.

Figure 1 shows the Kullback divergence when the propagation model used at receiver perfectly matches the oceans' transfer function. For this example, the distance between the source and the receiving array is 6 Km, located at the same depth (200 m). The solid curve (f.c.) corresponds to the fully coherent and the dashed (inc. (app)) and dotted (inc.) curves to the incoherent method, under the two different wavefront shape models (dashed line for the planar wavefront approximation, and dotted line for the full parametrization). Using the relation between Kullback divergence and Cramer Rao bound, we can conclude that, as expected, the matched field approach yields the best local performance (larger curvature of the Kullback divergence at the true parameter value), followed by the incoherent method with planar wavefront approximation. The need to estimate a large number of (somewhat redundant) parameters in the detailed wavefront method explains its poor performance. In the following plots, only the plane wave approximation is considered. Except for the incoherent method with no a priori knowledge of the wavefront shape, no secondary lobes are present, and thus no large systematic errors are to be expected.

Figure 2 shows (solid lines) the result of using wrong values for the receiving array's depth *(a)* and for the velocity at the surface *(b)*. To facilitate comparison, we also plot (dashed lines) the curves for no model mismatch. Note that the curvature of the plots around their peak values is only slightly increased, suggesting that local behavior is not significantly affected by model mismatch. Receiver mismatch results in a bias in the gradient estimate. Sensitivity to knowledge of array immersion is larger than to the value of the sound speed at the duct: while a small error in antenna immersion (190 m instead of the actual value of 200 m) induces a considerable systematic bias, a corresponding velocity error (1470 m/s instead of 1480) results in a negligible error.

Figure 3 shows the Kullback divergence for a larger array-source separation (50 Km). Solid plots correspond to a wrong value of the sound speed gradient in the lower layer ($g_1 = .0165s^{-1}$, instead of .016). The dashed lines show the Kullback divergence for no model mismatch. This figure suggests that the matched field approach is less ambiguous than the incoherent method, the sensitivity of both approaches to model errors being comparable.

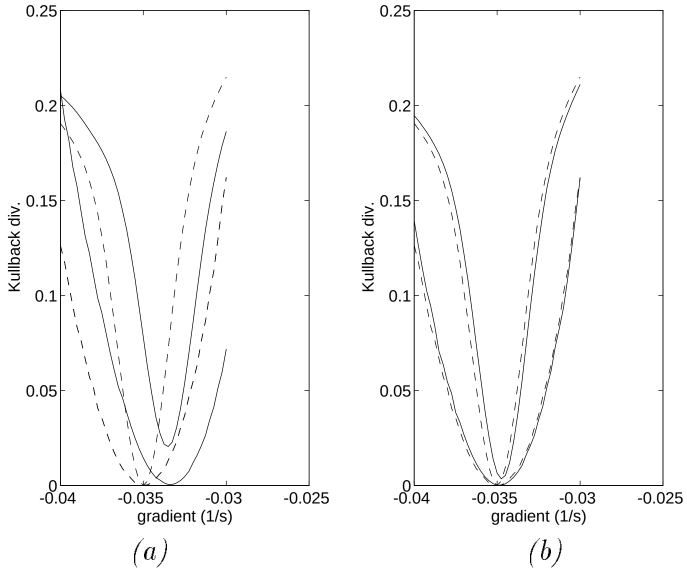

FIGURE 2. Kullback divergence *(a)* wrong array depth; *(b)* wrong velocity at duct.

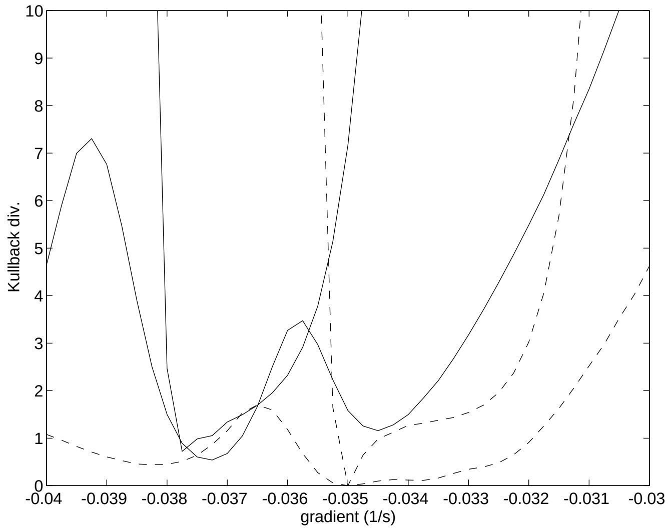

FIGURE 3. Kullback divergence for wrong lower gradient (distant source).

References

1. Solomon Kullback. *Information Theory and Statistics.* Peter Smith, 1978.
2. M. S. Pinsker. *Information and Information Stability of Random Variables and Processes.* Holden-Day, Inc., 1964.
3. M. João D. Rendas and José M. F. Moura. Cramer-Rao bounds for source location systems in multipath environments. *IEEE Trans. on Acoustic, Speech and Signal Processing*, Vol. 39, No. 12:2593–2610, Dec. 1991.
4. M. João D. Rendas and José M. F. Moura. Ambiguity analysis in source location. *Submitted for publication*, March 1994, *also* Research Report no. 94-23, I3S, CNRS-URA 1376, Université de Nice Sophia Antipolis, France.