Local negative circuits and fixed points in non-expansive Boolean networks

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Abstract: Given a Boolean function $F : \{0,1\}^n \to \{0,1\}^n$, and a point x in $\{0,1\}^n$, we represent the discrete Jacobian matrix of F at point x by a signed directed graph $G_F(x)$. We then focus on the following open problem: Is the absence of a negative circuit in $G_F(x)$ for every x in $\{0,1\}^n$ a sufficient condition for F to have at least one fixed point? As result, we give a positive answer to this question under the additional condition that F is non-expansive with respect to the Hamming distance.

Keywords: Boolean network, fixed point, discrete Jacobian matrix, interaction graph, negative circuit.

1 Introduction

We are interested in the relationships between the fixed points and the discrete Jacobian matrix of a Boolean function $F : \{0,1\}^n \to \{0,1\}^n$,

$$x = (x_1, \dots, x_n) \mapsto F(x) = (f_1(x), \dots, f_n(x)).$$

The discrete Jacobian matrix of F is here defined to be the map F' associating to each point x in $\{0,1\}^n$ the $n \times n$ matrix $F'(x) = (f_{ij}(x))$ over $\{-1, 0, 1\}$ defined by

$$f_{ij}(x) = f_i(x_1, \dots, \underset{\substack{\uparrow \\ j\text{th component}}}{1}, \dots, x_n) - f_i(x_1, \dots, \underset{\substack{\uparrow \\ j\text{th component}}}{0}, \dots, x_n) \qquad (i, j = 1, \dots, n).$$

In order to use graph theoretic notions (instead of matrix theoretic notions), we represent $F'(x)$ under the form of directed graph with signed arcs, called the *local interaction graph* of F evaluated at point x, and denoted by $G_F(x)$: the vertex-set is $\{1, \dots, n\}$, and there exists a positive (resp. negative) arc from j to i if $f_{ij}(x)$ is positive (resp. negative) $(i, j = 1, \dots, n)$. The *global interaction graph* of F, denoted by $G(F)$, is then defined to be

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the union of all the local interaction graphs: the vertex-set is $\{1,\ldots,n\}$, and there exists a positive (resp. negative) arc from j to i if f_{ij} is somewhere positive (resp. negative) (the presence of both a positive and a negative arc from one vertex to another is allowed). A *positive* (resp. *negative*) *circuit* in such signed directed graphs is an elementary directed cycle containing an even (resp. odd) number of negative arcs.

Our starting point is the following fixed point theorem of Robert [5, 6, 7]:

Theorem 1 **[5]** *If $G(F)$ has no circuit, then F has a unique fixed point.*

What interests us here is the fact that, by considering the signs of the circuits of $G(F)$, both the uniqueness and the existence part of this theorem can be obtained under conditions weaker than the absence of circuit. Indeed, on one side, the uniqueness part has been proved by Remy, Ruet and Thieffry under the absence of positive circuit:

Theorem 2 **[1]** *If $G(F)$ has no positive circuit, then F has at most one fixed point.*

And on the other side, the existence part has been proved under the absence of negative circuit:

Theorem 3 **[4]** *If $G(F)$ has no negative circuit, then F has at least one fixed point.*

[Theorems 2 and 3 can be seen as discrete versions of two general rules on dynamical systems stated by the biologist René Thomas, see [1, 4].]

Now, consider the following local version of Theorem 1, stated by Shih and Ho in [8] as a Boolean analog of the Jacobian conjecture in algebraic geometry, and proved by Shih and Dong:

Theorem 4 **[9]** *If $G_F(x)$ has no circuit for all x in $\{0,1\}^n$, then F has a unique fixed point.*

This theorem is a sensible generalization of the theorem of Robert: since each local interaction graph $G_F(x)$ is a subgraph of the global interaction graph $G(F)$, it is clear that if $G(F)$ has no circuit, then $G_F(x)$ has no circuit for all x in $\{0,1\}^n$.

Seeing the proof by dichotomy (positive/negative case) of the global theorem of Robert, it is natural to think about a proof by dichotomy of the local theorem of Shih and Dong. In this direction, the uniqueness part has been obtained by Remy, Ruet and Thieffry, who proved the following local version of Theorem 2:

Theorem 5 **[1]** *If $G_F(x)$ has no positive circuit for all x in $\{0,1\}^n$, then F has at most one fixed point.*

However, the existence part is an open problem: there is no proof or counter example to the local version of Theorem 3. We have thus the following question:

Question 1 *Is the absence of a negative circuit in $G_F(x)$ for all x in $\{0,1\}^n$ a sufficient condition for F to have at least one fixed point?*

Theorems 1-5 remain valid in the general discrete case, that is, when F sends into itself a product of n finite interval of integers (see [5, 2, 3, 4]), but the previous question has a negative answer in the non-Boolean discrete case [4] (the counter example is a map from $\{0,1,2,3\}^2$ to itself). Therefore, the situation is clear in the non-Boolean discrete case, and to have a clear situation in the general discrete case, it remains to answer to Question 1 in the Boolean case.

In this note, we positively answer to Question 1 under the additional condition that F is *non-expansive with respect to the Hamming distance* d, that is, under the condition that

$$\forall x, y \in \{0,1\}^n, \qquad d(F(x), F(y)) \leq d(x, y).$$

[In the following, the mention "with respect to the Hamming distance" is omitted.]

Theorem 6 *Let F be a non-expansive map from $\{0,1\}^n$ to itself. If $G_F(x)$ has no negative circuit for all x in $\{0,1\}^n$, then F has at least one fixed point.*

The non-expansive condition is rather strong (among the $(2^n)^{2^n}$ maps from $\{0,1\}^n$ to itself, at most $(n+1)^{2^n+n}$ are non expansive (rough upper bound)). However, this partial answer is a first result about Question 1, and more generally, a first result about negative circuits in local interaction graphs. [And it is not, a priori, an obvious exercise. To see this, one can refer to the technical arguments used by Shih and Ho [8, pages 75-88] to prove that a non-expansive map F has a fixed point if $G_F(x)$ has no circuit for all x in $\{0,1\}^n$.]

The proof of Theorem 6 is given in Section 3. In section 2, we state additional definitions and preliminary results.

2 Additional definitions and preliminary results

As usual, we set $\overline{0} = 1$ and $\overline{1} = 0$. For all $x \in \{0,1\}^n$ and $I \subseteq \{1, \ldots, n\}$, we denote by $\overline{x}^I$ the point y of $\{0,1\}^n$ defined by: $y_i = \overline{x_i}$ if $i \in I$, and $y_i = x_i$ otherwise $(i = 1, \ldots, n)$. We write $\overline{x}$ instead of $\overline{x}^{\{1,\ldots,n\}}$, and $\overline{x}^i$ instead of $\overline{x}^{\{i\}}$. So, for instance, $d(x, y) = n$ if and only if $y = \overline{x}$, and $d(x, y) = 1$ if and only if there exists an index i such that $y = \overline{x}^i$.

Let $F : \{0,1\}^n \to \{0,1\}^n$. With the previous notations, for all $x \in \{0,1\}^n$, we have

$$f_{ij}(x) = \frac{f_i(\overline{x}^j) - f_i(x)}{\overline{x_j} - x_j} \qquad (i, j = 1, \ldots, n).$$

In the following, we write $j \to i \in G_F(x)$ to mean that $G_F(x)$ has a positive or a negative arc from j to i, *i.e.* to mean that $f_{ij}(x) \neq 0$.

Proposition 1 *If F is non-expansive then, for all $x \in \{0,1\}^n$,*

$$j \to i \in G_F(x) \iff F(\overline{x}^j) = \overline{F(x)}^i.$$

Proof – It is sufficient to observe that $j \to i \in G_F(x)$ if and only if $f_i(\overline{x}^j) = \overline{f_i(x)}$ and to use the non-expansiveness of F. $\square$

Proposition 2 *F is non-expansive if and only if, for all $x \in \{0,1\}^n$, the maximal out-degree of $G_F(x)$ is at most one.*

Proof – Indeed, by definition, $d(F(x), F(\overline{x}^i))$ is the out-degree of i in $G_F(x)$. So if F is non-expansive, then $d(F(x), F(\overline{x}^i)) \le d(x, \overline{x}^i) = 1$, and one direction is proved. For the converse, suppose that, for all $x \in \{0,1\}^n$, the out-degree of each vertex of $G_F(x)$ is at most one. Then $d(F(x), F(y)) \le 1$ if $d(x,y) = 1$, and from this it is easy to show, by induction on $d(x,y)$, that $d(F(x), F(y)) \le d(x,y)$ for all $x, y \in \{0,1\}^n$. $\square$

So, if the maximal out-degree of $G(F)$ is one, then F is non expansive.

Now, we associate with F two maps $F^0, F^1 : \{0,1\}^{n-1} \to \{0,1\}^{n-1}$, which will be used as inductive tools in the proof of Theorems 6. Let $b \in \{0,1\}$. If $x \in \{0,1\}^{n-1}$, we denote by (x,b) the point $(x_1, \dots, x_{n-1}, b)$ of $\{0,1\}^n$. Then, we define F^b by:

$$\forall x \in \{0,1\}^{n-1}, \qquad f_i^b(x) = f_i(x,b) \qquad (i = 1, \dots, n-1).$$

Proposition 3 *For all $x \in \{0,1\}^{n-1}$, $G_{F^b}(x)$ is a subgraph of $G_F(x,b)$. In other words, if $G_{F^b}(x)$ has a positive (resp. negative) arc from j to i, then $G_F(x,b)$ has a positive (resp. negative) arc from j to i.*

Proof – It is sufficient to observe that $f_{ij}^b(x) = f_{ij}(x,b)$ for $i, j = 1, \dots, n-1$. $\square$

As an immediate consequence of Propositions 2 and 3, we have the following proposition:

Proposition 4 *If F is non-expansive, then F^0 and F^1 are non-expansive.*

3 Proof of Theorem 6

Lemma 1 *Let $F : \{0,1\}^n \to \{0,1\}^n$, and let $x \in \{0,1\}^n$. If $d(x, F(x)) = 1$, then every Hamiltonian circuit of $G_F(x)$ is negative.*

Proof – Suppose that $d(x, F(x)) = 1$ and that $i_1 \to i_2 \to \cdots \to i_n \to i_1$ is an Hamiltonian circuit of $G_F(x)$ (that is, an elementary directed cycle of length n). Since $d(x, F(x)) = 1$, we can suppose, without loss of generality, that $F(x) = \overline{x}^{i_1}$. Then,

$$f_{i_1 i_n}(x) = \frac{f_{i_1}(\overline{x}^{i_n}) - f_{i_1}(x)}{\overline{x_{i_n}} - x_{i_n}} = \frac{f_{i_1}(\overline{x}^{i_n}) - \overline{x_{i_1}}}{\overline{x_{i_n}} - x_{i_n}}.$$

Since $i_n \to i_1 \in G_F(x)$, we have $f_{i_1 i_n}(x) \neq 0$, and we deduce that

$$f_{i_1 i_n}(x) = \frac{x_{i_1} - \overline{x_{i_1}}}{\overline{x_{i_n}} - x_{i_n}}.$$

Furthermore, for $k = 1, \dots, n-1$ we have $f_{i_{k+1}}(x) = x_{i_{k+1}}$ so

$$f_{i_{k+1} i_k}(x) = \frac{f_{i_{k+1}}(\overline{x}^{i_k}) - f_{i_{k+1}}(x)}{\overline{x_{i_k}} - x_{i_k}} = \frac{f_{i_{k+1}}(\overline{x}^{i_k}) - x_{i_{k+1}}}{\overline{x_{i_k}} - x_{i_k}}.$$

Since $i_k \to i_{k+1} \in G_F(x)$, we have $f_{i_{k+1} i_k}(x) \neq 0$, and we deduce that

$$f_{i_{k+1} i_k}(x) = \frac{\overline{x_{i_{k+1}}} - x_{i_{k+1}}}{\overline{x_{i_k}} - x_{i_k}}.$$

By definition, the sign of the circuit $i_1 \to i_2 \to \dots \to i_n \to i_1$ is the sign of

$$s \;=\; f_{i_2 i_1}(x) \cdot f_{i_3 i_2}(x) \cdot f_{i_4 i_3}(x) \cdots f_{i_n i_{n-1}}(x) \cdot f_{i_1 i_n}(x).$$

With the preceding we have

$$\begin{aligned} s &= \frac{\overline{x_{i_2}} - x_{i_2}}{\overline{x_{i_1}} - x_{i_1}} \cdot \frac{\overline{x_{i_3}} - x_{i_3}}{\overline{x_{i_2}} - x_{i_2}} \cdot \frac{\overline{x_{i_4}} - x_{i_4}}{\overline{x_{i_3}} - x_{i_3}} \cdots \frac{\overline{x_{i_n}} - x_{i_n}}{\overline{x_{i_{n-1}}} - x_{i_{n-1}}} \cdot \frac{x_{i_1} - \overline{x_{i_1}}}{\overline{x_{i_n}} - x_{i_n}} \\ &= \frac{\cancel{\overline{x_{i_2}} - x_{i_2}}}{\overline{x_{i_1}} - x_{i_1}} \cdot \frac{\cancel{\overline{x_{i_3}} - x_{i_3}}}{\cancel{\overline{x_{i_2}} - x_{i_2}}} \cdot \frac{\cancel{\overline{x_{i_4}} - x_{i_4}}}{\cancel{\overline{x_{i_3}} - x_{i_3}}} \cdots \frac{\cancel{\overline{x_{i_n}} - x_{i_n}}}{\cancel{\overline{x_{i_{n-1}}} - x_{i_{n-1}}}} \cdot \frac{x_{i_1} - \overline{x_{i_1}}}{\cancel{\overline{x_{i_n}} - x_{i_n}}} \\ &= \frac{x_{i_1} - \overline{x_{i_1}}}{\overline{x_{i_1}} - x_{i_1}} = -1, \end{aligned}$$

and the lemma is proved. $\square$

Remark 1 *One can prove the following more general property: every circuit of $G_F(x)$ is positive (resp. negative) if it contains an even (resp. odd) number of vertices i such that $f_i(x) \neq x_i$.*

The rest of the proof is based on the following notion of opposition: given two points $x, y \in \{0,1\}^n$, and an index $i \in \{1, \dots, n\}$, we say that x and y are in *opposition (with respect to i in F)* if

$$F(x) = \overline{x}^i, \qquad F(y) = \overline{y}^i \qquad \text{and} \qquad x_i \neq y_i.$$

Lemma 2 *Let F be a non-expansive map from $\{0,1\}^n$ to itself. If F has two points in opposition, then F has no fixed point.*

Proof – Suppose that α and β are two points in opposition with respect to i in F, and suppose that x is a fixed point of F. If $x_i = \alpha_i$, then $d(F(x), F(\alpha)) = d(x, \overline{\alpha}^i) > d(x, \alpha)$ and this contradicts the non-expansiveness of F. Otherwise, $x_i = \beta_i$, thus $d(F(x), F(\beta)) = d(x, \overline{\beta}^i) > d(x, \beta)$ and we arrive to the same contradiction. $\square$

Lemma 3 *Let F be a non-expansive map from $\{0,1\}^n$ to itself. If F has two points in opposition, then there exists two distinct points x and y in $\{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit.*

Proof – We proceed by induction on n. The lemma being obvious for $n = 1$, we suppose that $n > 1$ and that the lemma holds for the dimension $n - 1$. We also suppose that F is non-expansive and has at least two points in opposition.

Suppose that α and β are two points in opposition with respect to i in F such that $\alpha \neq \overline{\beta}$. Then there exists $j \neq i$ such that $\alpha_j = \beta_j$ and, without loss of generality, we can suppose that $\alpha_n = \beta_n = b$. We set $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ and $\tilde{\beta} = (\beta_1, \ldots, \beta_{n-1})$ so that $\alpha = (\tilde{\alpha}, b)$ and $\beta = (\tilde{\beta}, b)$. Then, $\tilde{\alpha}_i = \alpha_i \neq \beta_i = \tilde{\beta}_i$, and since $F(\alpha) = \overline{\alpha}^i$, we have

$$F^b(\tilde{\alpha}) = (f_1(\alpha), \ldots, f_i(\alpha), \ldots, f_{n-1}(\alpha)) = (\alpha_1, \ldots, \overline{\alpha_i}, \ldots, \alpha_{n-1}) = \overline{\tilde{\alpha}}^i,$$

and we show similarly that $F^b(\tilde{\beta}) = \overline{\tilde{\beta}}^i$. Consequently, $\tilde{\alpha}$ and $\tilde{\beta}$ are in opposition with respect to i in F^b. Since F is non-expansive, F^b is also non-expansive (Proposition 4), and by induction hypothesis, there exists two distinct points $x, y \in \{0,1\}^{n-1}$ such that $G_{F^b}(x)$ and $G_{F^b}(y)$ have a common negative circuit. Since $G_{F^b}(x)$ and $G_{F^b}(y)$ are subgraphs of $G_F(x, b)$ and $G_F(y, b)$ respectively (Proposition 3), we deduce that $G_F(x, b)$ and $G_F(y, b)$ have a common negative circuit and the lemma holds.

So in the following, we suppose that:

$$\text{If } F \text{ has two points } \alpha \text{ and } \beta \text{ in opposition, then } \alpha = \overline{\beta}. \qquad (\mathcal{H})$$

We also use the following notation:

$$\forall x \in \{0,1\}^n, \qquad x^1 = x \qquad \text{and} \qquad x^{k+1} = F(x^k) \qquad (k \in \mathbb{N}).$$

Let us first prove that

$$\text{If } F(\alpha) = \overline{\alpha}^i\text{, then there exists a permutation } \{i_1, \ldots, i_n\} \text{ of } \{1, \ldots, n\} \text{ with } i = i_1 \text{ such that } F(\alpha^k) = \overline{\alpha^k}^{i_k} \text{ for } k = 1, \ldots, n. \qquad (\mathcal{A})$$

Taking $i_1 = i$, we have $F(\alpha^1) = \overline{\alpha^1}^{i_1}$. So there exists a sequence $i_1, i_2, \ldots, i_p$ of $p \geq 1$ distinct indices of $\{1, \ldots, n\}$ with $i_1 = i$ such that $F(\alpha^k) = \overline{\alpha^k}^{i_k}$ for $k = 1, \ldots, p$. If $p = n$ then the property $\mathcal{A}$ is proved. Assume that $p < n$. It is then sufficient to show that there exists a longer "good sequence", that is, an index $i_{p+1} \notin \{i_1, \ldots, i_p\}$ such that $F(\alpha^{p+1}) = \overline{\alpha^{p+1}}^{i_{p+1}}$. Since $\alpha^{p+1} = F(\alpha^p) = \overline{\alpha^p}^{i_p}$, we have $d(\alpha^{p+1}, \alpha^p) = 1$. Since F is non-expansive, we deduce that

$$d(F(\alpha^{p+1}), \alpha^{p+1}) = d(F(\alpha^{p+1}), F(\alpha^p)) \leq d(\alpha^{p+1}, \alpha^p) = 1.$$

Since $F(\alpha^{p+1}) \neq \alpha^{p+1}$ (Lemma 2), we deduce that $d(F(\alpha^{p+1}), \alpha^{p+1}) = 1$. So there exists a unique index of $\{1, \ldots, n\}$, that we denote by i_{p+1}, such that

$$F(\alpha^{p+1}) = \overline{\alpha^{p+1}}^{i_{p+1}}.$$

It remains to prove that $i_{p+1} \notin \{i_1, \ldots, i_p\}$. If not, there exists $k \in \{1, \ldots, p\}$ such that $i_{p+1} = i_k$. Then,

$$F(\alpha^{p+1}) = \overline{\alpha^{p+1}}^{i_k} \quad \text{and} \quad F(\alpha^k) = \overline{\alpha^k}^{i_k}.$$

Furthermore, since

$$\alpha^{p+1} = \overline{\alpha^p}^{\{i_p\}} = \overline{\alpha^{p-1}}^{\{i_{p-1}, i_p\}} = \cdots = \overline{\alpha^k}^{\{i_k, \ldots, i_{p-1}, i_p\}},$$

and since the indices $i_k, \ldots, i_{p-1}, i_p$ are pairwise distinct, we have $\alpha^{p+1}_{i_k} \neq \alpha^k_{i_k}$. Thus, α^k and α^{p+1} are in opposition with respect to i_k in F, and since $\{i_k, \ldots, i_{p-1}, i_p\}$ is strictly included in $\{1, \ldots, n\}$, we have $\alpha^{p+1} \neq \overline{\alpha^k}$ and this contradicts the hypothesis $\mathcal{H}$. This proves $\mathcal{A}$.

Using $\mathcal{H}$ and $\mathcal{A}$, we now prove that

$$\text{If } F(\alpha) = \overline{\alpha}^i, \text{ then the in-degree of } i \text{ in } G_F(\alpha) \text{ is at most one.} \tag{$\mathcal{B}$}$$

Let $\{i_1, \ldots, i_n\}$ be a permutation of $\{1, \ldots, n\}$ as in the property $\mathcal{A}$ ($i_1 = i$). Suppose, by contradiction, that i_1 has at least two in-neighbours in $G_F(\alpha)$. Then i_1 has an in-neighbour $i_k \neq i_n$, and using Proposition 1 we deduce that

$$F(\overline{\alpha}^{i_k}) = \overline{F(\alpha)}^{i_1} = \overline{\overline{\alpha}^{i_1}}^{i_1} = \alpha = \overline{\overline{\alpha}^{i_k}}^{i_k} \quad \text{and} \quad F(\alpha^k) = \overline{\alpha^k}^{i_k}.$$

If $k = 1$, then $\alpha^k = \alpha$ and so

$$(\alpha^k)_{i_k} = \alpha_{i_k} \neq (\overline{\alpha}^{i_k})_{i_k} \quad \text{and} \quad \alpha^k_{i_n} = (\overline{\alpha}^{i_k})_{i_n}. \tag{1}$$

Otherwise, $\alpha^k = \overline{\alpha}^{\{i_1, \ldots, i_{k-1}\}}$ and so (1) holds again. So in both cases, α^k and $\overline{\alpha}^{i_k}$ are in opposition with respect to i_k in F and $\alpha^k \neq \overline{\overline{\alpha}^{i_k}}$. This contradicts the hypothesis $\mathcal{H}$. Thus $\mathcal{B}$ is proved.

Using again $\mathcal{H}$ and $\mathcal{A}$, we prove that

> If α and β are in opposition in F, then there exists a permutation $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$ such that α^k and β^k are in opposition with respect to i_k in F, for $k = 1, \ldots, n$. ($\mathcal{C}$)

Suppose that α and β are in opposition in F. Then according to $\mathcal{A}$, there exists a permutation $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$, and a permutation $\{j_1, \ldots, j_n\}$ of $\{1, \ldots, n\}$, such that

$F(\alpha^k) = \overline{\alpha^k}^{i_k}$ and $F(\beta^k) = \overline{\beta^k}^{j_k}$ for $k = 1, \ldots, n$. From this and the hypothesis $\mathcal{H}$, we deduce that

$$\alpha^{n+1} = \overline{\alpha}^{\{i_1,\ldots,i_n\}} = \overline{\alpha} = \beta \qquad \text{and} \qquad \beta^{n+1} = \overline{\beta}^{\{j_1,\ldots,j_n\}} = \overline{\beta} = \alpha. \tag{2}$$

Let us now prove, by recurrence on k decreasing from n to 1, that α^k and β^k are in opposition with respect to i_k in F. From (2) and the non-expansiveness of F, we have

$$d(\alpha^n, \beta^n) \geq d(F(\alpha^n), F(\beta^n)) = d(\alpha^{n+1}, \beta^{n+1}) = d(\beta, \alpha) = n.$$

Thus

$$d(\alpha^n, \beta^n) = n = d(\alpha^{n+1}, \beta^{n+1}) = d(\overline{\alpha^n}^{i_n}, \overline{\beta^n}^{j_n}).$$

So $i_n = j_n$ and $\alpha^n_{i_n} \neq \beta^n_{i_n}$, and it follows that α^n and β^n are in opposition with respect to i_n in F. Now, suppose that α^k and β^k are in opposition with respect to i_k in F $(2 \leq k \leq n)$. Then, following the hypothesis $\mathcal{H}$, $\alpha^k = \overline{\beta^k}$, and since F is non-expansive, we deduce that

$$d(\alpha^{k-1}, \beta^{k-1}) \geq d(F(\alpha^{k-1}), F(\beta^{k-1})) = d(\alpha^k, \beta^k) = n.$$

Thus

$$d(\alpha^{k-1}, \beta^{k-1}) = n = d(\alpha^k, \beta^k) = d(\overline{\alpha^{k-1}}^{i_{k-1}}, \overline{\beta^{k-1}}^{j_{k-1}}).$$

So $i_{k-1} = j_{k-1}$ and $\alpha^{k-1}_{i_{k-1}} \neq \alpha^{k-1}_{i_{k-1}}$ and we deduce that α^{k-1} and β^{k-1} are in opposition with respect to i_{k-1} in F. This completes the recurrence and the proof of $\mathcal{C}$.

Using $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{C}$, we prove that

> If α et β are in opposition in F, then $G_F(\alpha^n)$ and $G_F(\beta^n)$ have a common Hamiltonian circuit. $(\mathcal{D})$

Let $\{i_1, \ldots, i_n\}$ be a permutation of $\{1, \ldots, n\}$ as in the property $\mathcal{C}$. We will show that $i_1 \to i_2 \to \cdots \to i_n \to i_1$ is a circuit of $G_F(\alpha^n)$. For $k = 2, \ldots, n$, we have

$$F\big(\overline{\alpha^k}^{i_{k-1}}\big) = F\left(\overline{\overline{\alpha^{k-1}}^{i_{k-1}}}^{i_{k-1}}\right) = F(\alpha^{k-1}) = \alpha^k = \overline{\overline{\alpha^k}^{i_k}}^{i_k} = \overline{F(\alpha^k)}^{i_k}.$$

Thus

$$i_{k-1} \to i_k \in G_F(\alpha^k) \qquad (k = 2, \ldots, n). \tag{3}$$

In addition, for $k = 1, \ldots, n-1$, we have

$$F\big(\overline{\alpha^k}^{i_k}\big) = F(\alpha^{k+1}) = \overline{\alpha^{k+1}}^{i_{k+1}} = \overline{F(\alpha^k)}^{i_{k+1}}.$$

Thus

$$i_k \to i_{k+1} \in G_F(\alpha^k) \qquad (k = 1, \ldots, n-1).$$

Let $k \in \{1, \dots, n-1\}$, and suppose, by contradiction, that

$$i_k \to i_{k+1} \notin G_F(\alpha^n).$$

Since $i_k \to i_{k+1} \in G_F(\alpha^k)$, there exists $p \in \{k+1, \dots, n\}$ such that

$$i_k \to i_{k+1} \in G_F(\alpha^{p-1}) \qquad \text{and} \qquad i_k \to i_{k+1} \notin G_F(\alpha^p).$$

From (3) we deduce that $k+1 < p$, and from $i_k \to i_{k+1} \in G_F(\alpha^{p-1})$ we deduce that

$$f_{i_{k+1}}(\alpha^{p-1}) \neq f_{i_{k+1}}\big(\overline{\alpha^{p-1}}^{i_k}\big). \tag{4}$$

Furthermore, from $i_k \to i_{k+1} \notin G_F(\alpha^p)$ and $\alpha^p = \overline{\alpha^{p-1}}^{i_{p-1}}$ we deduce that

$$f_{i_{k+1}}\big(\overline{\alpha^{p-1}}^{i_{p-1}}\big) = f_{i_{k+1}}\Big(\overline{\overline{\alpha^{p-1}}^{i_{p-1}}}^{i_k}\Big) = f_{i_{k+1}}\Big(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\Big). \tag{5}$$

If

$$f_{i_{k+1}}(\alpha^{p-1}) \neq f_{i_{k+1}}\big(\overline{\alpha^{p-1}}^{i_{p-1}}\big)$$

then i_{k+1} and i_p are distinct out-neighbours of i_{p-1} in $G_F(\alpha^{p-1})$. So the out-degree of i_{p-1} is at least two, and this contradicts Proposition 2. Thus

$$f_{i_{k+1}}(\alpha^{p-1}) = f_{i_{k+1}}\big(\overline{\alpha^{p-1}}^{i_{p-1}}\big)$$

and from (4) and (5) we deduce that

$$f_{i_{k+1}}\big(\overline{\alpha^{p-1}}^{i_k}\big) \neq f_{i_{k+1}}\Big(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\Big).$$

Thus $i_{p-1} \to i_{k+1} \in G_F\big(\overline{\alpha^{p-1}}^{i_k}\big)$, and following Proposition 1, we have

$$F\big(\overline{\alpha^p}^{i_k}\big) = F\Big(\overline{\overline{\alpha^{p-1}}^{i_{p-1}}}^{i_k}\Big) = F\Big(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\Big) = \overline{F\big(\overline{\alpha^{p-1}}^{i_k}\big)}^{i_{k+1}}$$

Since $i_k \to i_{k+1} \in G_F(\alpha^{p-1})$, we have $F\big(\overline{\alpha^{p-1}}^{i_k}\big) = \overline{F(\alpha^{p-1})}^{i_{k+1}}$ and we deduce that

$$F\big(\overline{\alpha^p}^{i_k}\big) = \overline{\overline{F(\alpha^{p-1})}^{i_{k+1}}}^{i_{k+1}} = F(\alpha^{p-1}) = \alpha^p = \overline{\overline{\alpha^p}^{i_p}}^{i_p} = \overline{F(\alpha^p)}^{i_p}.$$

So i_k and i_{p-1} are in-neighbours of i_p in $G_F(\alpha^p)$, and $i_k \neq i_{p-1}$ since $k+1 < p$. So the in-degree of i_p in $G_F(\alpha^p)$ is at least two, and since $F(\alpha^p) = \overline{\alpha^p}^{i_p}$, this contradicts the property $\mathcal{B}$. We have thus prove that

$$i_k \to i_{k+1} \in G_F(\alpha^n) \qquad (k = 1, \dots, n-1).$$

So to prove that $i_1 \to i_2 \to \cdots \to i_n \to i_1$ is a circuit of $G_F(\alpha^n)$, it is thus sufficient to prove that $i_n \to i_1 \in G_F(\alpha^n)$. Following the hypothesis $\mathcal{H}$, we have $\overline{\alpha} = \beta$, thus

$$F(\alpha^n) = \alpha^{n+1} = \overline{\alpha}^{\{i_1,\dots,i_n\}} = \overline{\alpha} = \beta$$

and we deduce that

$$F\big(\overline{\alpha^n}^{i_n}\big) = F(\alpha^{n+1}) = F(\beta) = \overline{\beta}^{i_1} = \overline{F(\alpha^n)}^{i_1}.$$

We prove similarly that $i_1 \to i_2 \to \cdots \to i_n \to i_1$ is a circuit of $G_F(\beta^n)$, and $\mathcal{D}$ is proved.

We are now in position to prove the lemma. Let α and β be two points in opposition in F. Following $\mathcal{D}$, $G_F(\alpha^n)$ and $G_F(\beta^n)$ have a common Hamiltonian circuit, and following $\mathcal{C}$, α^n and β^n are in opposition in F, so that $d(\alpha, F(\alpha)) = d(\beta, F(\beta)) = 1$ and $\alpha \neq \beta$. Consequently, by Lemma 1, the Hamiltonian circuit, present both in $G_F(\alpha^n)$ and $G_F(\beta^n)$, is negative. This completes the proof of Lemma 3. □

Lemma 4 *Let F be a non-expansive map from $\{0,1\}^n$ to itself. If there is no distinct points $x, y \in \{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit, then F has at least one fixed point.*

Proof – We proceed by induction on n. The lemma being obvious for $n = 1$, we suppose that $n > 1$ and that the lemma holds for the dimension $n-1$. Let F be as in the statement, and let $b \in \{0,1\}$. Since $G_{F^b}(x)$ is a subgraph of $G_F(x,b)$ for all $x \in \{0,1\}^{n-1}$, and since F^b is non-expansive, there is no distinct points $x, y \in \{0,1\}^{n-1}$ such that $G_{F^b}(x)$ and $G_{F^b}(y)$ have a common negative circuit. So, by induction hypothesis, F^b has at least one fixed point, that we denote by ξ^b. Then, for $b \in \{0,1\}$, we have

$$\begin{aligned} F(\xi^b, b) &= (f_1(\xi^b, b), \dots, f_{n-1}(\xi^b, b), f_n(\xi^b, b)) \\ &= (f_1^b(\xi^b), \dots, f_{n-1}^b(\xi^b), f_n(\xi^b, b)) \\ &= (\xi_1^b, \dots, \xi_{n-1}^b, f_n(\xi^b, b)) \\ &= (\xi^b, f_n(\xi^b, b)) \ \in \ \{(\xi^b, b), (\xi^b, \overline{b})\} \end{aligned}$$

So if neither $(\xi^0, 0)$ nor $(\xi^1, 1)$ is a fixed point of F, then $F(\xi^0, 0) = (\xi^0, 1)$, and $F(\xi^1, 1) = (\xi^1, 0)$. Therefore, $(\xi^0, 0)$ and $(\xi^1, 1)$ are in opposition with respect to n in F, and so, by Lemma 3, there exists two distinct points $x, y \in \{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit, a contradiction. □

Theorem 6 is an obvious consequence of Lemma 4.

Example 1 *$n = 4$ and F is defined by:*

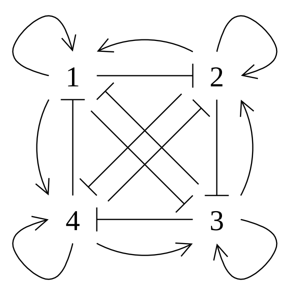
$$\begin{aligned} f_1(x) &= x_1\, x_2\, \overline{x_3}\, \overline{x_4} \\ f_2(x) &= x_2\, x_3\, \overline{x_4}\, \overline{x_1} \\ f_3(x) &= x_3\, x_4\, \overline{x_1}\, \overline{x_2} \\ f_4(x) &= x_4\, x_1\, \overline{x_2}\, \overline{x_3} \end{aligned}$$

Equivalently, F can be defined by the following table:

x	$F(x)$
0000	0000
0001	0000
0010	0000
0011	0010
0100	0000
0101	0000
0110	0100
0111	0000
1000	0000
1001	0001
1010	0000
1011	0000
1100	1000
1101	0000
1110	0000
1111	0000

F has a unique fixed point (0000). The global interaction graph $G(F)$ is the following (T-end arrows correspond to negative arcs, the other arrows correspond to positive arcs):

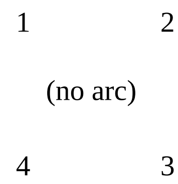
$G(F)$



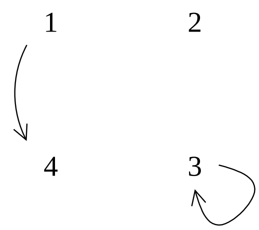
$G(F)$ has 8 positive circuits (4 of length 1, 2 of length 2, and 2 of length 4), and it has 16 negative circuits (4 of length 2, 8 of length 3, and 4 of length 4). So Theorems 1 and 3 cannot be applied to deduce that F has a fixed point. The local interaction graphs are the

following:

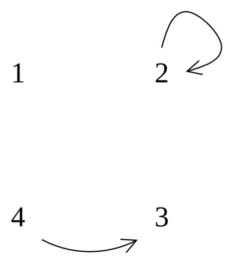
$G_F(0000)$



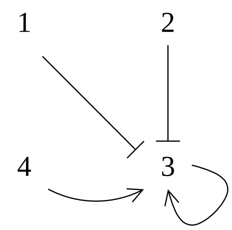
$G_F(0001)$



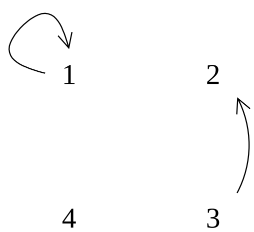
$G_F(0010)$



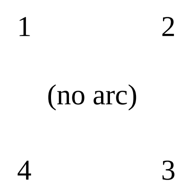
$G_F(0011)$



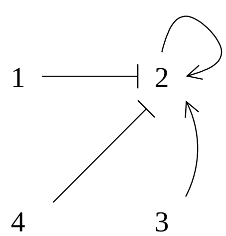
$G_F(0100)$



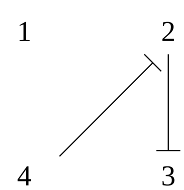
$G_F(0101)$



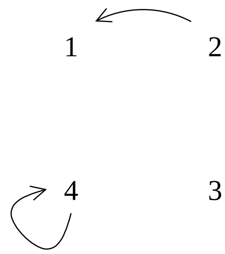
$G_F(0110)$



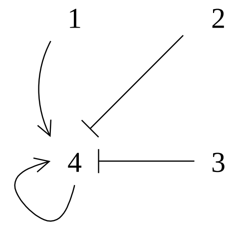
$G_F(0111)$



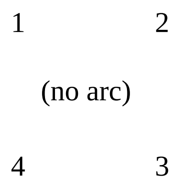
$G_F(1000)$



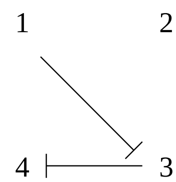
$G_F(1001)$



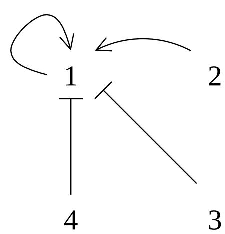
$G_F(1010)$



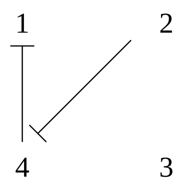
$G_F(1011)$



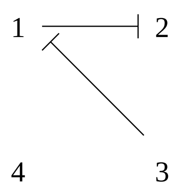
$G_F(1100)$



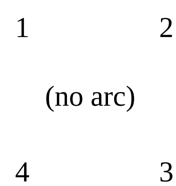
$G_F(1101)$



$G_F(1110)$



$G_F(1111)$



The maximal out-degree of each local interaction graph is at most one, so F is non-expansive, and all the local interaction graphs are without negative circuit. So F satisfies *the conditions of Theorem 6 (and F has indeed a fixed point). Since some local interaction graphs contain a positive circuit (of length one), Theorem 4 cannot be applied to deduce that F has a fixed point.*

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