# Fixing Boolean networks asynchronously

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A **Boolean network (BN)** with $n$ components is a function

$$f : \{0,1\}^n \to \{0,1\}^n$$

$$x = (x_1, \ldots, x_n) \mapsto f(x) = (f_1(x), \ldots, f_n(x)).$$

The **dynamics** is usually described by the successive iterations of $f$

$$x \to f(x) \to f^2(x) \to f^3(x) \to \cdots$$

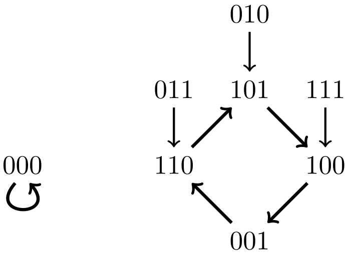
**Fixed points** correspond to stable states.

**Example** with $n = 3$

$$\begin{cases} f_1(x) &= x_2 \vee x_3 \\ f_2(x) &= \overline{x_1} \wedge \overline{x_3} \\ f_3(x) &= \overline{x_3} \wedge (x_1 \vee x_2) \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 000 |
| 001 | 110 |
| 010 | 101 |
| 011 | 110 |
| 100 | 001 |
| 101 | 100 |
| 110 | 101 |
| 111 | 100 |

Dynamics



The **interaction graph** of $f$ is the digraph $G(f)$ on $[n] := \{1, \dots, n\}$ s.t.

$$j \to i \text{ is an arc } \iff f_i \text{ depends on } x_j.$$

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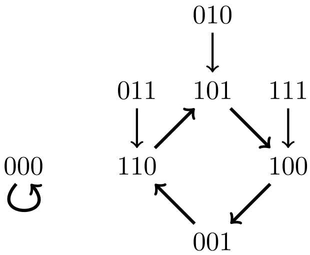
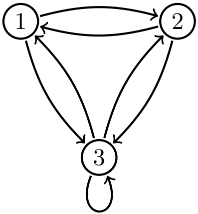



Interaction graph



**Many applications**, in particular:

- Neural networks [McCulloch & Pitts 1943]
- **Gene networks** [Kauffman 1969, Thomas 1973]
- Network Coding [Riis 2007]

**Synchronous** dynamics: all components are updated at each step:

$$x \to f(x) \to f^2(x) \to f^3(x) \to \cdots$$

**Asynchronous**: one component is updated at each step.
$\hookrightarrow$ Update component $i$ at state $x$ means reach the state

$$x \stackrel{i}{\longrightarrow} f^i(x) := (x_1, \ldots, x_{i-1}, f_i(x), x_{i+1}, \ldots, x_n).$$

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The **asynchronous graph $\Gamma(f)$** describes all the possible trajectories: the vertex set is $\{0,1\}^n$ and $x \to f^i(x)$ for all $x \in \{0,1\}^n$ and $i \in [n]$.

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It can be regarded as a **Finite Deterministic Automta** where

1. the alphabet is $\Sigma := [n]$;
2. the set of states is $Q := \{0,1\}^n$;
3. the transition function $\delta : Q \times \Sigma \to Q$ is $\delta(x, i) := f^i(x)$.

## Example

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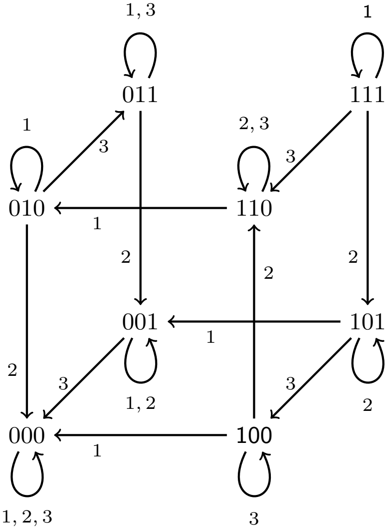

**Notation:** If $w = i_1 i_2 \ldots i_k \in [n]^*$ then $\boldsymbol{f^w(x)}$ is the state obtained from $x$ by updating successively the components $i_1, i_2, \ldots, i_k$, that is,

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**Definition 1.** A word $w \in [n]^*$ **fixes** $f$ if

$$\forall x \in \{0,1\}^n, \qquad f^w(x) \text{ is a fixed point of } f.$$

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**Definition 2.** A word $w$ **fixes a family** $\mathcal{F}$ of BNs if it fixes each $f \in \mathcal{F}$. The **fixing length** $\lambda(\mathcal{F})$ is the min length of a word fixing $\mathcal{F}$.

**Example: 1231 is fixing** (and no shorter word is fixing, thus $\lambda(f) = 4$).

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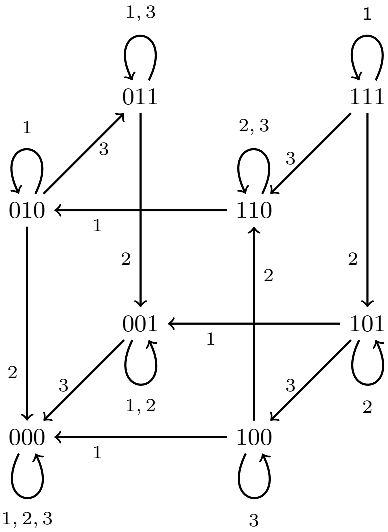

## Remarks

1. $f$ is fixable only if $f$ has a fixed point.
2. If $f$ has a unique fixed point then:

$$w \text{ fixes } f \iff w \text{ is synchronizing.}$$

3. A family $\mathcal{F}$ is fixable if and only if each $f \in \mathcal{F}$ is fixable.

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**Theorem 1 [Bollobás, Gotsman and Shamir 1993]**

There is a positive fraction $\phi(n)$ of fixable BNs with $n$ components:

$$\lim_{n \to \infty} \phi(n) = 1 - \frac{1}{e} \geq 0.64.$$

## Example of fixable families

1. $\boldsymbol{F_M(n)}$: **Monotone BNs** $(2^{\Theta(\sqrt{n}2^n)})$:

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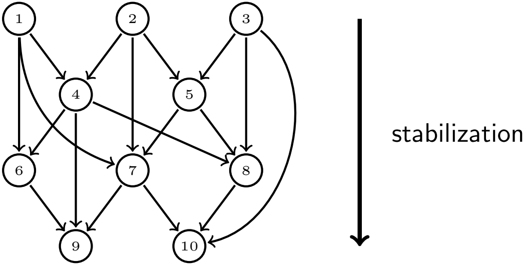
4. $\boldsymbol{F_P(n)}$: **Monotone BNs whose interaction graph is a Path** $(2n!)$.

**Theorem [Aracena, Gadouleau, R., Salinas 2018+]**

| Networks | $\mathcal{F}$ | $\max_{f\in\mathcal{F}}\lambda(f)$ | $\lambda(\mathcal{F})$ |
|---|---|---|---|
| Acyclic | $F_A(n)$ | $n$ | $\Theta(n^2)$ |
| Path | $F_P(n)$ | $n$ | $\Theta(n^2)$ |
| Increasing | $F_I(n)$ | $\Theta(n^2)$ | $\Theta(n^2)$ |
| **Monotone** | $\boldsymbol{F_M(n)}$ | $\boldsymbol{\Omega(n^2)}$ | $\boldsymbol{O(n^3)}$ |

# Acyclic networks

$G(f)$

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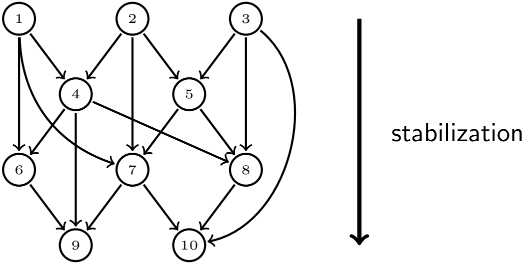


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**Proposition.** Let $f \in F_A(n)$ and $w \in [n]^*$.

1. If $w$ is a topological sort of $G(f)$, then $w$ fixes $f$, thus $\lambda(f) = n$.
2. If $w$ contains a topological sort of $G(f)$ then $w$ fixes $f$.
3. If $w$ contains all the permutations of $[n]$, then it fixes $F_A(n)$.

An **$n$-complete** word is a word $w \in [n]^*$ that contains (as subsequences) all the permutations of $[n]$.

$$\lambda(n) := \text{minimum length of an } n\text{-complete word.}$$

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**Corollary.** Every $n$-complete word fixes $F_A(n)$, thus

$$\lambda(F_A(n)) \leq \lambda(n).$$

## What is the magnitude order of $\lambda(n)$?

For an **upper-bound**, let

$$w := \underbrace{123\dots n}_{1}\underbrace{123\dots n}_{2}\cdots\underbrace{123\dots n}_{n}$$

Let $\pi = i_1 i_2 \dots i_n$ be a permutation of $[n]$. Then

$$w := \underbrace{123\dots n}_{\text{contains } i_1}\underbrace{123\dots n}_{\text{contains } i_2}\cdots\underbrace{123\dots n}_{\text{contains } i_n}$$

Hence $w$ contains $\pi$. Thus $w$ is $n$-complete: $\lambda(n) \leq |w| = n^2$.

## What is the magnitude order of $\lambda(n)$?

For a **better upper-bound**, let

$$w := \underbrace{123\dots n}_{1}\,\underbrace{n(n-1)\dots 321}_{2}\,\underbrace{123\dots n}_{3}\cdots\underbrace{123\dots n}_{n}$$

Then $w$ is $n$-complete, and

$$w' := \underbrace{123\dots n}_{1}\,\underbrace{(n-1)\dots 321}_{2}\,\underbrace{23\dots n}_{3}\cdots\underbrace{23\dots n}_{n}$$

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**Theorem**

| | | |
|---|---|---|
| $\lambda(n) \leq n^2 - 2n + 4$ | for all $n \geq 1$ | [Adleman 1974] |
| $\lambda(n) \leq n^2 - 2n + 3$ | for all $n \geq 10$ | [Zlinescu 2011] |
| $\lambda(n) \leq \lceil n^2 - \frac{7}{3}n + \frac{19}{3} \rceil$ | for all $n \geq 7$ | [Radomirovic 2012] |

## What is the magnitude order of $\lambda(n)$?

For a **lower-bound**, note that if $w$ is $n$-complete then

$$n! \leq |\{\text{subsequences of length } n \text{ contained in } w\}| \leq \binom{|w|}{n} \leq \frac{|w|^n}{n!}$$

Hence,

$$|w|^n \geq (n!)^2 \geq \left(\frac{n}{e}\right)^{2n} \quad \text{thus} \quad |w| \geq \left(\frac{n}{e}\right)^2.$$

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**Theorem [Kleitman, Kwiatkowski 1976]**

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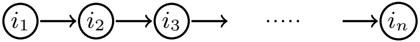


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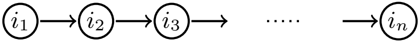


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**Proposition 2.** A word fixes $F_P(n)$ if and only if it is $n$-complete, thus

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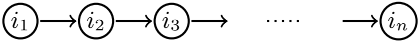


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Since $F_P(n) \subseteq F_A(n)$ we deduce that $\lambda(n) \leq \lambda(F_A(n))$ and thus

**Theorem.** $\lambda(F_P(n)) = \lambda(F_A(n)) = \lambda(n) = n^2 - o(n^2)$.

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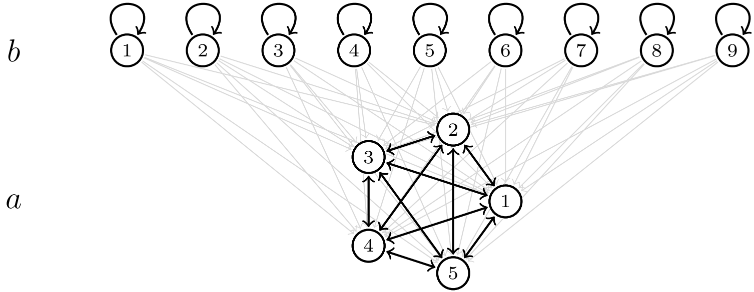
# A monotone network hard to fixe

## A monotone BN $f$ with $n = a + b$, which is hard to fixe:



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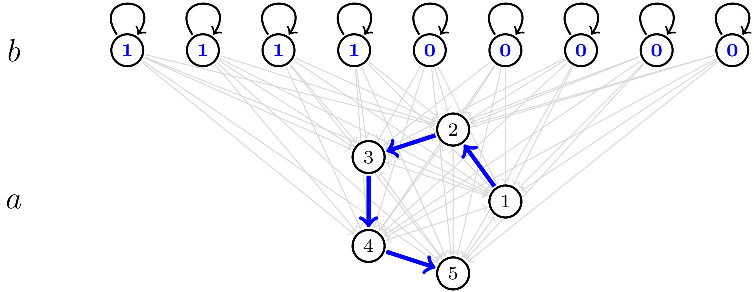


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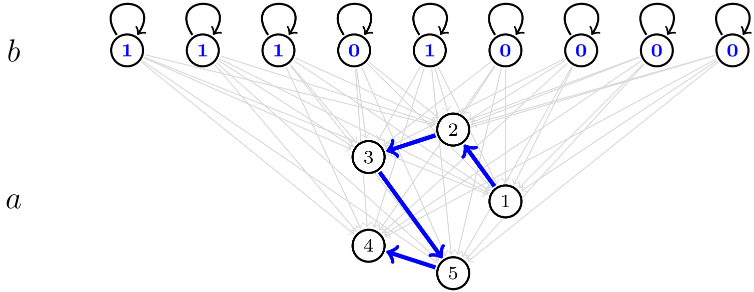


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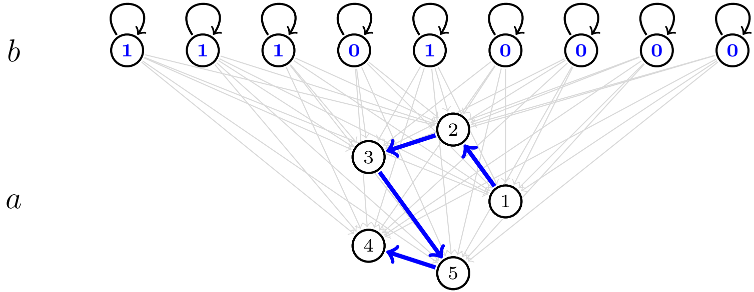


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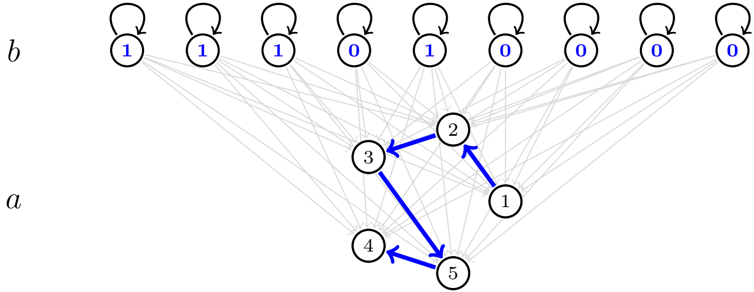


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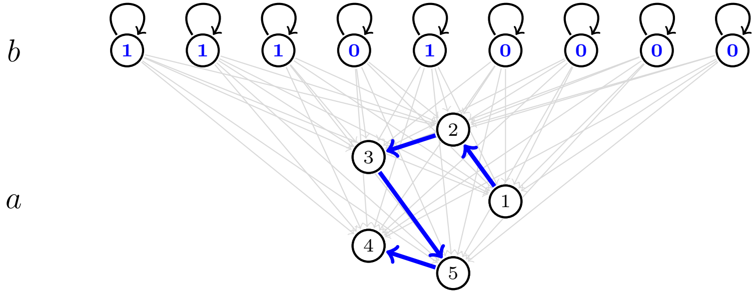


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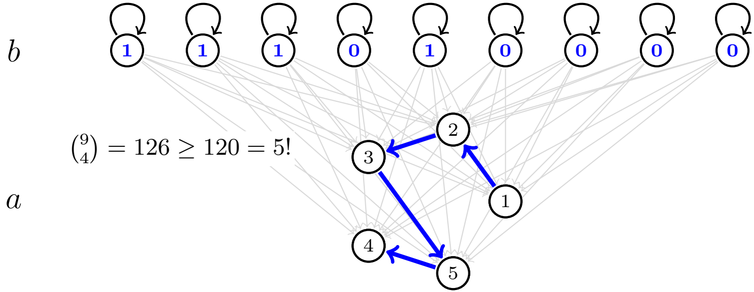


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The construction is based on the fact that a word containing the $n!$ permutations of $S_n$ must be of quadratic length.

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In the construction $f$, we encode $S_a$ with $b = O(a \log a)$ inputs.

But we can encode $\Pi_a$ with $b = o(a)$ only, and get

$$\boldsymbol{\lambda(f) \geq \frac{a^2}{3} \sim \frac{n^2}{3}}.$$

**Theorem [Aracena, Gadouleau, R., Salinas 2018+]**

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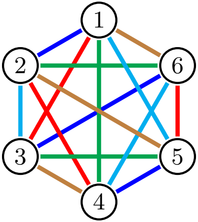
**Theorem.** There exists $\Pi_n \subseteq S_n$ of size $2^{o(n)}$ such that any word containing all the permutations in $\Pi_n$ is of length $\geq n^2/3$.

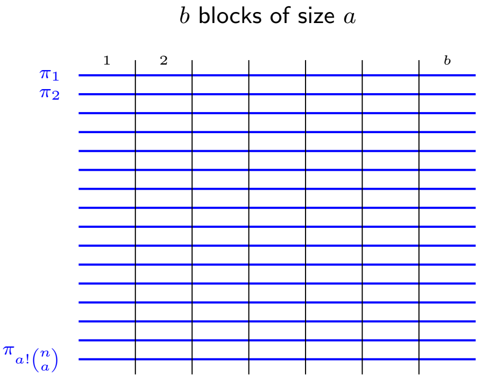
**Baranyai' theorem [1975]**

If $n = ab$, there exists a collection of $\frac{1}{b}\binom{n}{a}$ partitions of $[n]$ into $a$-sets, such that each $a$-subset of $[n]$ appears in exactly one partition.

For $a = 2$ this this equivalent to that, for $n$ even, there is a partition of the edges of $K_n$ into perfect matchings.



For $n = ab$, there exists a collection $\Pi_n \subseteq S_n$ of size $a!\binom{n}{a} \leq n^a$ with the following properties:



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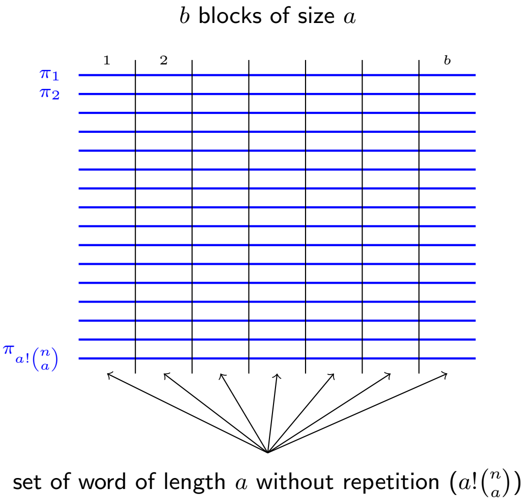

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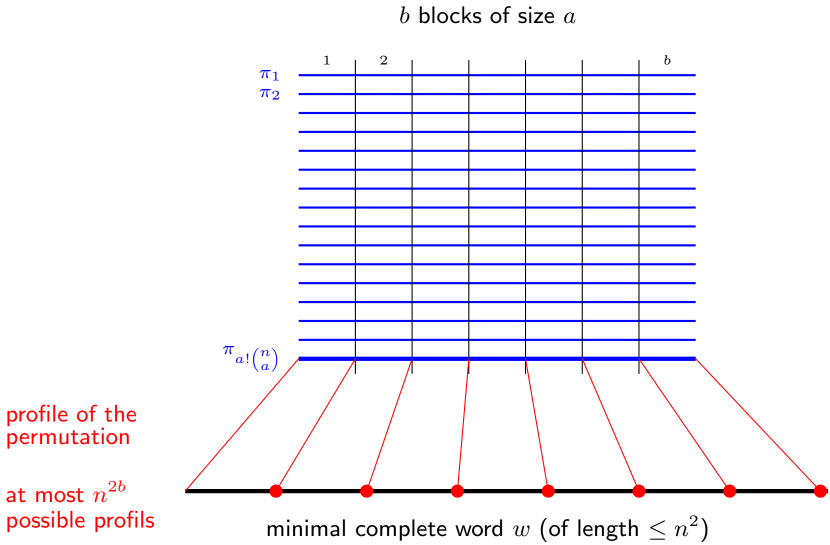

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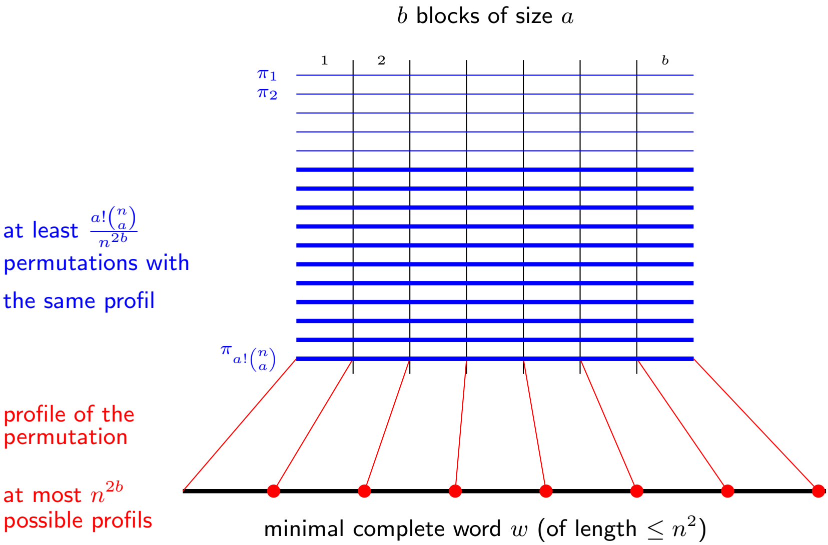

By a counting argument

$$|w| \geq \left(n^{-\frac{2b}{a}}\right) \frac{n(n-a)}{e}$$

Taking $a = n^{\frac{1}{2}+\epsilon}$ and $b = n^{\frac{1}{2}-\epsilon}$ we get

$$|w| \sim \frac{n^2}{e} \qquad \text{and} \qquad |\Pi_n| \leq n^{n^{\frac{1}{2}+\epsilon}} = 2^{o(n)}.$$

**Theorem [Aracena, Gadouleau, R., Salinas 2018+]**

| Networks | $\mathcal{F}$ | $\max_{f \in \mathcal{F}} \lambda(f)$ | $\lambda(\mathcal{F})$ |
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## Fixing all increasing networks

Let $f$ be any BN with $n$ components and $x \in \{0,1\}^n$.

1. $f$ is **increasing from** $x$ if $f^u(x) \leq f^{uv}(x)$ for all $u, v \in [n]^*$.
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**Proposition.** $\lambda(F_I(n)) = \lambda(n) = n^2 - o(n^2)$.

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## A cubic word fixing all the monotone networks

**Lemma.** If $f$ is monotone and $w$ is $n$-complete, then

$x \leq f(x) \;\Rightarrow\; f$ is increasing from $x \;\Rightarrow\; f^w(x)$ if a fixed point of $f$

$x \geq f(x) \;\Rightarrow\; f$ is decreasing from $x \;\Rightarrow\; f^w(x)$ if a fixed point of $f$

Let $\omega^n$ be an $n$-complete word of length $\lambda(n)$ for each $n \geq 1$

**Theorem.** The word $W^n := \omega^1\omega^2 \dots \omega^n$ fixes $F_M(n)$ and $|W^n| \leq n^3$.

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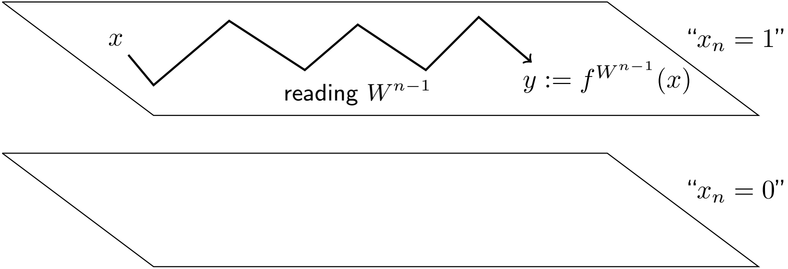

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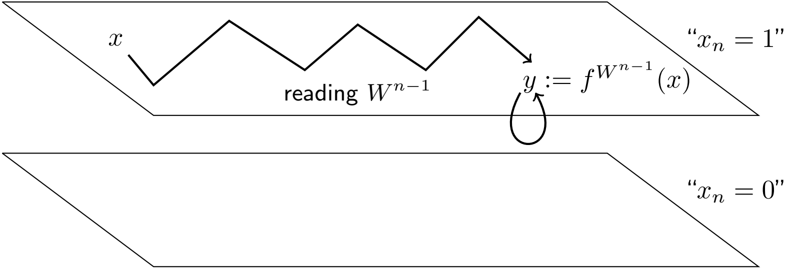


1. If $y$ is a FP then $f^{\omega^n}(y)$ is a FP.

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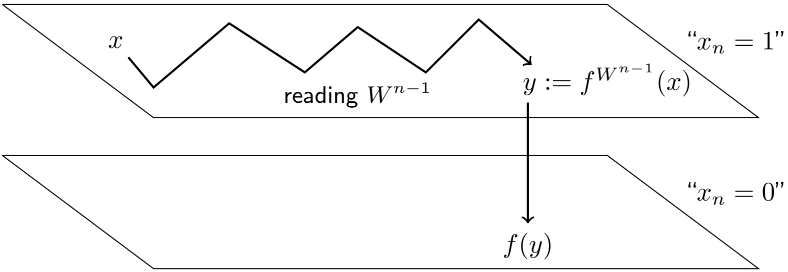


1. If $y$ is a FP then $f^{\omega^n}(y)$ is a FP.
2. If not $f(y) = y + e_n$ thus $y \leq f(y)$ or $y \geq f(y)$, thus $f^{\omega^n}(y)$ is FP.

**Theorem [Aracena, Gadouleau, R., Salinas 2018+]**

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# Conclusion

It is interesting to regard the asynchronous dynamics as a DFA.

Emphasis on the notion of **fixing words**.

$\hookrightarrow$ Some results in the monotone case, with various technics

$\hookrightarrow$ $n$-complete words, Baranyai's theorem etc.

**Question.** Is it as hard to fixe one $f \in F_M(n)$ as $F_M(n)$?

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**Question.** Is it as hard to fixe one $f \in F_M(n)$ as $F_M(n)$?

What about classical notions in DFA?

A word $w$ is a **synchronizing word** of a BN $f$ if $f^w$ is constant.

**Černý's conjecture for Boolean networks**

If a BN $f$ has a synchronizing word, then it has one of length $\leq 2^{2n}$.