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# Approximated Type-2 Fuzzy Set Operations

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*Abstract*— **Type-2 fuzzy sets, an elaboration over type-1 fuzzy sets, are an interesting method for handling uncertainty in rules and parameters in fuzzy systems. However, their adoption has not been as wide as one could have expected. In this paper we provide a simple introduction to type-2 fuzzy sets; then we propose a novel method for calculating operations on type-2 fuzzy sets with normal type-1 membership values, for which we redefine set ordering. Finally, based on the max ordering of fuzzy set and highest degree of separation, we propose an approximation for performing the operations, which ensures that the calculation is accurate for the most important parts of the membership values.**

## I. Introduction

IT is a rule of Nature that everything should evolve toward perfection or vanish. It seems that this rule is also valid for mathematical theories. Without any doubt, the concept of *set* is one of the most important and fundamental in the field of Mathematics. Classical set theory was originally proposed by Georg Cantor at the end of the 19<sup>th</sup> century. Cantor's set theory is to date still used and taught even since the initial years of elementary schools throughout the world. However, one of the most significant developments – one might call it a mutation – that has occurred to set theory was the introduction of fuzzy set theory by Lotfi A. Zadeh in 1965 [13]. Fuzzy set theory, unlike classical set theory, which uses binary-valued characteristic functions, defines a set by means of a membership function, whose values can range in the [0, 1] interval of the real numbers, whose 0 extreme corresponds to no membership at all, whereas 1 corresponds to full membership. In other words, fuzzy set theory adds an infinite number of gray levels to the black and white world of classical set theory.

One of the difficulties with classical set theory, that fuzzy set theory tries to overcome, is to determine whether an element is a member of a given set or not. Fuzzy set theory "provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership"[13]. Thus fuzzy set theory tries to alleviate the problem by associating a degree of membership to the element. However, that leads to the question how membership degrees should be defined. "In fuzzy systems all entities that come into play are defined in terms of fuzzy sets, that is, of their membership functions. The determination of membership functions is then to be correctly viewed as a problem of design" [11]. Therefore, the problem is reduced to determining a membership degree, which is a *crisp* (i.e., exact) number, for all those cases in which, the extent to which, a given element belongs in a set would be uncertain. To solve this paradox [7], in 1975, Zadeh enhanced his theory by introducing type-2 fuzzy sets. Roughly speaking, a type-2 fuzzy set uses fuzzy values as membership degrees.

It is not clear why type-2 fuzzy set theory has not received the attention it deserves from researchers, given that it provides an enhanced framework to understand and replicate the dynamics of the human decision making processes.

Furthermore, type-2 fuzzy set theory is not used as it should by practitioners; perhaps this is due to the intrinsic complexity of the two dimensional space of type-1 fuzzy set theory being extended to three dimensions. However, most probably the reason lies in the aforementioned fact that little effort has been spent by researchers in developing its theory, as it is demonstrated by the fact that (I) there does not exist yet a comprehensive and exhaustive set of definitions for the theory and (II) proposed operations are not so efficient and comprehensible as they should be in order to be adopted in real-world applications. To be sure, type-2 fuzzy sets provide more power and functionality; consequently, it is reasonable that practitioners should expect to pay for that additional power. The fact is, current proposals are probably too expensive in terms of complexity than they could be. Of course, in this paper we do not claim we have found the lowest bounds of complexity; instead, we propose a few results that try to push toward those bounds.

## II. Preliminaries and Notations

A fuzzy set $A$ in universe of discourse $U$ is characterized by a membership function $\mu_A : U \to [0,1]$, and would be denoted as $A = \sum_{u\in U} \frac{\mu_A(u)}{u}$ or $A = \int_{u\in U} \frac{\mu_A(u)}{u}$ when $U$ is discrete or continuous respectively. The *support* of fuzzy set $A$ is defined to be $\{u \in U \mid \mu_A(u) > 0\}$ while its *height* is $Sup_{u\in U}\, \mu_A(u)$. If the *height* of fuzzy set $A$ is 1, then $A$ is a *normal* fuzzy set, otherwise it is called *subnormal*. The *core* of normal fuzzy set $A$ is defined as $\{u \in U \mid \mu_A(u) = 1\}$. Any fuzzy set $A$ can be represented as a (infinite) union of all of its α-level sets. The α-level set of a given fuzzy set $A$ is defined as $A_\alpha = \{u \in U \mid \mu_A(u) \geq \alpha\}$ where $0 \leq \alpha \leq 1$, hence fuzzy set $A$ would be defined as $A = \bigcup_{0\leq\alpha\leq 1} \alpha A_\alpha$.

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Fuzzy set $A$ is said to be *empty* if $Sup_{u \in U} \mu_A(u) = 0$, that is, $Height(A) = 0$. On the other hand, fuzzy set $A$ is said to be *less than or equal to* – or fuzzy subset of – fuzzy set $B$, if $\mu_A(u) \leq \mu_B(u), \forall u \in U$.

The complement of a fuzzy set $A$, when the universe of discourse is discrete, is defined to be $\overline{A} = \sum_{u \in U} \frac{1-\mu_A(u)}{u}$. The union and intersection of two fuzzy sets $A$ and $B$, in discrete universe of discourse is define as $A \bigcup B = \sum_{u \in U} \frac{\mu_A(u) \vee \mu_B(u)}{u}$ and $A \bigcap B = \sum_{u \in U} \frac{\mu_A(u) \wedge \mu_B(u)}{u}$ respectively, such that $\vee$ must satisfy t-conorm conditions -we mainly use *Max* as $\vee$ - and $\wedge$ must satisfy t-norm conditions -we mainly use *Min* as $\wedge$. Fuzzy sets $A$ and $B$ are said to be disjoint if their intersection is empty. Fuzzy set $A$ is said to be convex if all of its α-level sets are convex sets or, equivalently, $\mu_A(\lambda u_1 + (1-\lambda)u_2) \geq Min[\mu_A(u_1), \mu_A(u_2)]$, $\forall u_1, u_2 \in U, \lambda \in [0,1]$.

The *highest degree of separation* of two convex fuzzy sets $A$ and $B$ is $1 - Sup_{u \in U} \mu_{A \cap B}(u)$ [13].

One of the most important results in the field of fuzzy set theory is the extension principle. The extension principle allows one to *fuzzify* any mathematical theory. In brief, let $f$ be a mapping from $U$ to $V$ and $A$ be a fuzzy subset of U defined as $A = \sum_{u \in U} \frac{\mu_A(u)}{u}$. The extension principle states that $f(A) = \sum_{u \in U} \frac{\underset{v \in f^{-1}(u)}{Sup} \mu_A(v)}{f(u)}$, which is a fuzzy subset in $V$.

## III. Type-2 Fuzzy Set

As mentioned above, a paradox in fuzzy set theory is determined by the requirement to assign an exact membership degree to each element of the universe of discourse. The problem of defining binary membership criteria for classical set is reduced to the problem of defining the membership degree of each element in the universe to a given fuzzy set as a crisp number in [0, 1]. Instead of requiring the assignment of a value in {0, 1} to each element of the universe, fuzzy set theory requires the assignment of a value in [0, 1]. As an example, fuzzy set of numbers around 10 would be defined as:

$$A = \{\frac{0.1}{1}, \frac{0.2}{2}, \frac{0.3}{3}, \frac{0.4}{4}, \frac{0.5}{5}, \frac{0.6}{6}, \frac{0.7}{7}, \frac{0.8}{8}, \frac{0.9}{9}, \frac{1}{10}, \frac{0.9}{11}, \frac{0.8}{12}, \frac{0.7}{13}, \frac{0.6}{14}, \frac{0.5}{15}, \frac{0.4}{16}, \frac{0.3}{17}, \frac{0.2}{18}, \frac{0.1}{19}\}.$$

While this extension offers a more powerful framework to handle uncertainty, the problem of "sharply defined criteria" for assigning a value in [0,1] to each element of the universe remains. It would be more natural to express a membership degree for each element in a given fuzzy set by means of linguistic values. However, these linguistic values are special, because they have to denote the degree of membership of any element in a given fuzzy set.

*Definition*[14]: A fuzzy set is of type-n, n= 2, 3, 4, …. if its membership function ranges over fuzzy sets of type n-1. The membership function of type-1 fuzzy sets ranges over [0,1].

By the way, Mendel has implied in [7] and [3] that type-2 fuzzy sets seem to be enough to handle uncertainties in rules or system parameters in most fuzzy systems of practical interest.

To introduce the concept of type-2 fuzzy sets simply, let us express the above example, namely the set of numbers around 10, by using linguistic values as shown in Fig. 1.

| Not at All | Almost No | Very Low | Low | Almost Low | Medium | Almost High | High | Very High | Very Very High | Complete | Very Very High | Very High | High | Almost High | Medium | Almost Low | Low | Very Low | Almost No | Not at All |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

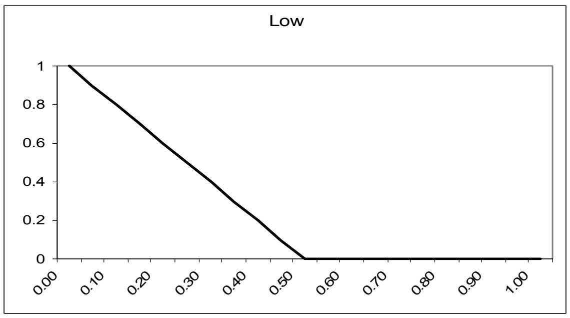
Fig.1. Fuzzy set of numbers around 10, using linguistic values as membership values.

In agreement with Zadeh [14], the linguistic variable used here is defined by the five-tuple $(\mathrm{X}, T(\mathrm{X}), U, G, M)$, where $\mathrm{X}$ is a *membership value*, $T(\mathrm{X})$ ={*Not at all, Almost No, Low, Medium, High, Complete, Very Low, Very High,* . . . }, U = [0,1] is the universe of discourse since the variable is to denote the degree of membership, and G and M denote definite syntactic and semantic rules respectively. Let the primary terms be defined as shown in Fig. 2.

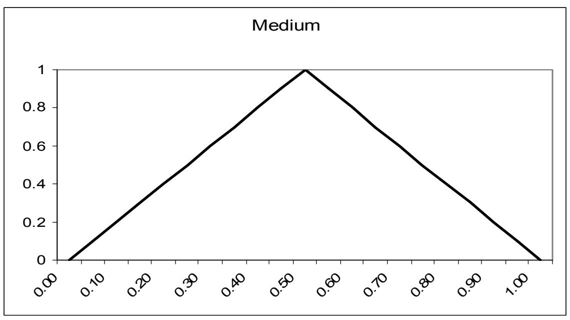
Now, if we replace each linguistic value by its *meaning* [14], we will obtain the three-dimensional shape of the above fuzzy set shown in Fig. 3, which is analogous to the table mentioned. Notice that, once again, it is understood that $\forall u \in U$ not explicitly mentioned, the membership degree is *Not At All*.

This fuzzy set, whose membership values are other fuzzy sets, is called type-2 fuzzy set [14]. More precisely, $\mu^{II} : U \to \mathrm{X}$, where $\mathrm{X}$ is a linguistic variable defined as above. In fact the terms used as the linguistic values are to indicate the membership degree of a given element in the type-2 fuzzy set. As it is apparent, according to the meaning of the terms used, the membership degree of a given element is not limited to a single value, but to a set of values with different strengths. That allows us to handle uncertainties in a cleaner fashion. In other words, the type-2 fuzzy set framework provides a mechanism to assign different membership degrees to a given element with different, so to speak, strengths. The set of membership degrees with their corresponding strengths for any given element is, as a matter of fact, a type-1 fuzzy set.

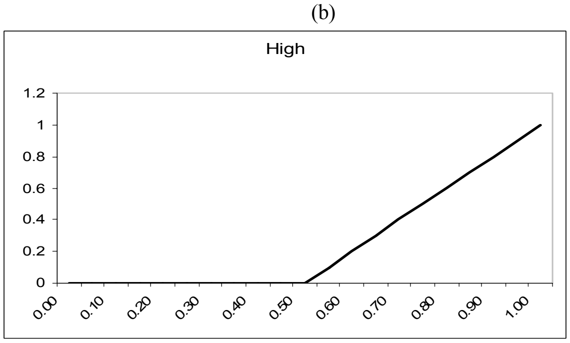
Before delving into the details of type-2 fuzzy sets, let us mention the important fact that, using type-2 fuzzy sets, the problem of directly assigning crisp values to any element as a membership degree is alleviated and replaced by the assignment of linguistic values as membership degrees. As a consequence, the problem of defining an exact membership degree, in type-2 fuzzy set is really reduced to defining the *meaning* of basic term sets of the linguistic variable *membership value;* a relevant discussion can be found in [2], which argues that traditional membership functions are not enough to model human expertise, whilst type-2 fuzzy membership functions would be up to that task [9]. However this freedom has its own expenses that will be discussed below.




(a)




(b)



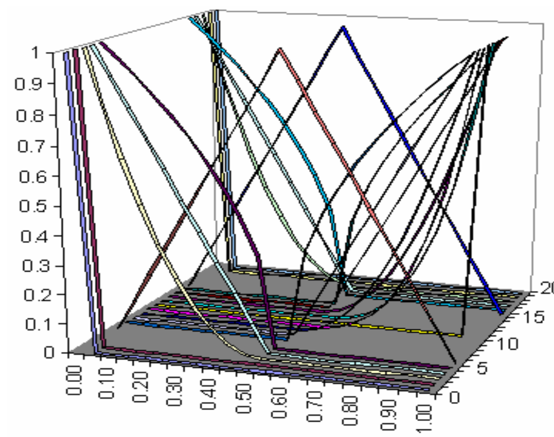

(c)

Fig.2. Meaning of the primary terms, (a) *Low*, (b) *Medium*, (c) *High*.

## IV. DEFINITIONS AND OPERATIONS

A discrete type-2 fuzzy set $A$ denoted by $\widetilde{A}$ can be denoted extensively as $\widetilde{A} = \sum_{u \in U} \left[ \sum_{i \in J_u} \frac{S_i^{(u)}}{\mu_i^{(u)}} \Big/ u \right]$, where $J_u \subseteq [0,1]$. In this notation $\mu_i^{(u)}$ is the i$^{\text{th}}$ membership value of $u$ with strength $S_i^{(u)}$. $\sum_{i \in J_u} \mu_i^{(u)}$ constitutes the primary membership and $\sum_{i \in J_u} S_i^{(u)}$ constitutes the secondary membership degree of element $u$. In other words, $S_i^{(u)}$ is the strength of the i$^{\text{th}}$ membership value of $u$, i.e. $\mu_i^{(u)}$. Clearly, $\left[ \sum_{i \in J_u} \frac{S_i^{(u)}}{\mu_i^{(u)}} \right]$ for any given $u \in U$ is a special type-1 fuzzy set that describes the imprecise membership value of $u$ in $\widetilde{A}$ and is usually referred to as a *vertical slice.*

The set $\{ \sum_{u \in U} \sum_{i \in J_u} \mu_i^{(u)} \Big/ u \quad , \quad S_i^{(u)} > 0 \}$ is called *Footprint of Uncertainty* (FoU). The FoU identifies the region of membership degrees but it does not say anything about the strength of each membership degree. Before continuing, we observe that type-1 fuzzy sets are a special case of type-2 fuzzy sets, where, for all $u \in U$, the set of primary membership degrees, namely $\sum_{i \in J_u} \mu_i^{(u)}$, is a singleton with the strength of unity.




| | Not at All | Almost No | Very Low | Low | Almost Low | Medium | Almost High | High | Very High | Very Very High | Complete | Very Very High | Very High | High | Almost High | Medium | Almost Low | Low | Very Low | Almost No | Not at All |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0.00 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0.05 | 0 | 0 | 0.8 | 0.9 | 0.9 | 0.1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0.9 | 0.9 | 0.8 | 0 | 0 |
| 0.10 | 0 | 0 | 0.6 | 0.8 | 0.9 | 0.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.2 | 0.9 | 0.8 | 0.6 | 0 | 0 |
| 0.15 | 0 | 0 | 0.5 | 0.7 | 0.8 | 0.3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.3 | 0.8 | 0.7 | 0.5 | 0 | 0 |
| 0.20 | 0 | 0 | 0.4 | 0.6 | 0.8 | 0.4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.4 | 0.8 | 0.6 | 0.4 | 0 | 0 |
| 0.25 | 0 | 0 | 0.3 | 0.5 | 0.7 | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.5 | 0.7 | 0.5 | 0.3 | 0 | 0 |
| 0.30 | 0 | 0 | 0.2 | 0.4 | 0.6 | 0.6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.6 | 0.6 | 0.4 | 0.2 | 0 | 0 |
| 0.35 | 0 | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.7 | 0.5 | 0.3 | 0.1 | 0 | 0 |
| 0.40 | 0 | 0 | 0 | 0.2 | 0.4 | 0.8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.8 | 0.4 | 0.2 | 0 | 0 | 0 |
| 0.45 | 0 | 0 | 0 | 0.1 | 0.3 | 0.9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.9 | 0.3 | 0.1 | 0 | 0 | 0 |
| 0.50 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0.55 | 0 | 0 | 0 | 0 | 0 | 0.9 | 0.3 | 0.1 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0.3 | 0.9 | 0 | 0 | 0 | 0 | 0 |
| 0.60 | 0 | 0 | 0 | 0 | 0 | 0.8 | 0.4 | 0.2 | 0 | 0 | 0 | 0 | 0 | 0.2 | 0.4 | 0.8 | 0 | 0 | 0 | 0 | 0 |
| 0.65 | 0 | 0 | 0 | 0 | 0 | 0.7 | 0.5 | 0.3 | 0.1 | 0 | 0 | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0 | 0 | 0 | 0 | 0 |
| 0.70 | 0 | 0 | 0 | 0 | 0 | 0.6 | 0.6 | 0.4 | 0.2 | 0.1 | 0 | 0.1 | 0.2 | 0.4 | 0.6 | 0.6 | 0 | 0 | 0 | 0 | 0 |
| 0.75 | 0 | 0 | 0 | 0 | 0 | 0.5 | 0.7 | 0.5 | 0.3 | 0.1 | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| 0.80 | 0 | 0 | 0 | 0 | 0 | 0.4 | 0.8 | 0.6 | 0.4 | 0.2 | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 0.4 | 0 | 0 | 0 | 0 | 0 |
| 0.85 | 0 | 0 | 0 | 0 | 0 | 0.3 | 0.8 | 0.7 | 0.5 | 0.3 | 0 | 0.3 | 0.5 | 0.7 | 0.8 | 0.3 | 0 | 0 | 0 | 0 | 0 |
| 0.90 | 0 | 0 | 0 | 0 | 0 | 0.2 | 0.9 | 0.8 | 0.6 | 0.5 | 0 | 0.5 | 0.6 | 0.8 | 0.9 | 0.2 | 0 | 0 | 0 | 0 | 0 |
| 0.95 | 0 | 0 | 0 | 0 | 0 | 0.1 | 0.9 | 0.9 | 0.8 | 0.7 | 0 | 0.7 | 0.8 | 0.9 | 0.9 | 0.1 | 0 | 0 | 0 | 0 | 0 |
| 1.00 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Fig. 3. Three dimensional shape and its related table of Type 2 fuzzy set of numbers around 10.

**Representation Principle:** Let us define α-level sets of a type-2 fuzzy set $\widetilde{A}=\sum_{u\in U}\left[\sum_{i\in J_u}\frac{S_i^{(u)}}{\mu_i^{(u)}}\Big/u\right]$ as

$\widetilde{A}_\alpha=\{(u,\mu_i^{(u)}),\forall u\in U, i\in J_u, S_i^{(u)}\geq\alpha\}=$ $\bigcup_{u\in U}\{(u,\mu_i^{(u)})\mid i\in J_u, S_i^{(u)}\geq\alpha\}$ ,where $0\leq\alpha\leq 1$.

Consequently, $\widetilde{A}=\bigcup_{\alpha\in[0,1]}\alpha\widetilde{A}_\alpha$, where the strength of $\mu_i^{(u)}$ will be the largest α such that $(u,\mu_i^{(u)})\in A_\alpha$, more precisely $S_i^{(u)}=\underset{(u,\mu_i^{(u)})\in\widetilde{A}_\alpha}{Sup}\ \alpha$.

Type-2 fuzzy set $\widetilde{A}$ is empty if $\widetilde{A}_\alpha=\{(u,0),\forall u\in U\}$ for all $0<\alpha\leq 1$. Notice that for an empty type-2 fuzzy set, for each $u\in U$, the primary membership set, namely $\sum_{i\in J_u}\mu_i^{(u)}$, is a singleton which is equal to *{0}* with strength $S_0^{(u)}=1$.

The complement of type-2 fuzzy set $\widetilde{A}$ is defined to be

$$\overline{\widetilde{A}}=\sum_{u\in U}\left[\sum_{i\in J_u}\frac{S_i^{(u)}}{(1-\mu_i^{(u)})}\Big/u\right],\text{ where } J_u\subseteq[0,1].$$

Under the assumption that all type-1 fuzzy sets are normal – that is, the membership degree of any element in a given type-2 fuzzy set is a normal type-1 fuzzy sets, although we make no assumption on their convexity – then type-2 fuzzy set $\widetilde{A}$, defined as $\widetilde{A}=\sum_{u\in U}\left[\sum_{i\in J_u}\frac{S_i^{(u)}}{\mu_i^{(u)}}\Big/u\right]$, is less than or equal to (i.e., is a type-2 fuzzy subset of) type-2 fuzzy set $\widetilde{B}$, defined as $\widetilde{B}=\sum_{w\in U}\left[\sum_{k\in `J_w}\frac{`S_k^{(w)}}{`\mu_k^{(w)}}\Big/w\right]$, if $\widetilde{A}_\alpha\leq\widetilde{B}_\alpha$ for all $0\leq\alpha\leq 1$, where $\widetilde{A}_\alpha\leq\widetilde{B}_\alpha$ if and only if $\{(u,`\mu_k^{(u)})\}_\alpha\geq\{(u,\mu_i^{(u)})\}_\alpha$, that is, $\forall u\in U$,

$(\exists(u,`\mu_k^{(u)})\in\widetilde{B}_\alpha\mid\forall(u,\mu_i^{(u)})\in\widetilde{A}_\alpha,(u,`\mu_k^{(u)})\geq(u,\mu_i^{(u)})$

such that $(u,`\mu_k^{(u)})\geq(u,\mu_i^{(u)})$ if and only if $`\mu_k^{(u)}\geq\mu_i^{(u)}$ – which also implies our definition of set ordering. Type-2 fuzzy set $\widetilde{A}$ is equal to type-2 fuzzy set $\widetilde{B}$ if $\widetilde{A}_\alpha=\widetilde{B}_\alpha$ for all $0\leq\alpha\leq 1$.

**Theorem 1:** The intersection of type-2 fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ is a type-2 fuzzy set $\widetilde{C}=\widetilde{A}\cap\widetilde{B}$ such that $\widetilde{C}_\alpha=Sup(\widetilde{C}'_\alpha\mid\widetilde{C}'_\alpha\leq\widetilde{A}_\alpha\ \ and\ \ \widetilde{C}'_\alpha\leq\widetilde{B}_\alpha)$.

*Proof*: Let us rewrite $\widetilde{A}$ and $\widetilde{B}$ as the union of their α-level sets: $\widetilde{A}=\bigcup_{\alpha\in[0,1]}\alpha\widetilde{A}_\alpha$ and $\widetilde{B}=\bigcup_{\alpha\in[0,1]}\alpha\widetilde{B}_\alpha$, where $\widetilde{A}_\alpha=\bigcup_{u\in U}\{(u,\mu_i^{(u)})\mid i\in J_u,S_i^{(u)}\geq\alpha\}$ and $\widetilde{B}_\alpha=\bigcup_{w\in U}\{(w,`\mu_k^{(w)})\mid k\in J_w,`S_k^{(w)}\geq\alpha\}$. We define $\widetilde{A}_\alpha(u)=\{(u,\mu_i^{(u)})\mid i\in J_u,S_i^{(u)}\geq\alpha\}$, $\widetilde{B}_\alpha(w)=\{(w,`\mu_k^{(w)})\mid k\in J_w,`S_k^{(w)}\geq\alpha\}$, $\widetilde{A}(u)=\bigcup_{0<\alpha\leq 1}\alpha\widetilde{A}_\alpha(u)$, and $\widetilde{B}(w)=\bigcup_{0<\alpha\leq 1}\alpha\widetilde{B}_\alpha(w)$. Similarly, let us denote $\widetilde{C}(u)=\bigcup_{0<\alpha\leq 1}\alpha\widetilde{C}_\alpha(u)=\bigcup_{u\in U}\{(u,\mu_t''^{(u)})\mid t\in J''_u,S_t''^{(u)}\geq\alpha\}$.

According to the *extension principle*,

$\widetilde{C}=\widetilde{A}\cap\widetilde{B}=\bigcup_{u\in U}\{\widetilde{A}(u)\wedge\widetilde{B}(u)\}$, which reduces to finding $\widetilde{C}(u)=\widetilde{A}(u)\wedge\widetilde{B}(u)$ – we refer to $\wedge$ as *Min* – such that $\widetilde{C}(u)=Sup\{\widetilde{C}'(u)\mid\widetilde{C}'(u)\leq\widetilde{A}(u)\ \ and\ \ \widetilde{C}'(u)\leq\widetilde{B}(u)\}$.

Regarding the above definition, $\widetilde{C}\leq\widetilde{A}$ if and only if $\forall u\in U,\widetilde{C}(u)\leq\widetilde{A}(u)$, and $\widetilde{C}(u)\leq\widetilde{A}(u)$ if $\forall 0<\alpha\leq 1,\widetilde{C}_\alpha(u)\leq\widetilde{A}_\alpha(u)$. Consequently

$\widetilde{C}_\alpha(u)=\{(u,\mu_t''^{(u)})\mid S_t''^{(u)}\geq\alpha\ ,(u,\mu_t''^{(u)})\in\widetilde{A}_\alpha(u)\ \ or$ $(u,\mu_t''^{(u)})\in\widetilde{B}_\alpha(u),\exists(u,\mu_i^{(u)})\in\widetilde{A}_\alpha(u),\exists(u,`\mu_k^{(u)})\in\widetilde{B}_\alpha(u)\ ,$ $(u,\mu_t''^{(u)})\leq(u,\mu_i^{(u)})\ \ and\ \ (u,\mu_t''^{(u)})\leq(u,`\mu_k^{(u)})\ \}$, that is,

$\widetilde{C}_\alpha(u)=Sup\{\widetilde{C}'_\alpha(u)\mid\widetilde{C}'_\alpha(u)\leq\widetilde{A}_\alpha(u)\ \ and\ \ \widetilde{C}'_\alpha(u)\leq\widetilde{B}_\alpha(u)\}$.

**Theorem 2:** The union of type-2 fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ is a type-2 fuzzy set $\widetilde{C}=\widetilde{A}\cup\widetilde{B}$ such that $\overline{\widetilde{C}}_\alpha=Sup\ (\overline{\widetilde{C}'_\alpha}\mid\overline{\widetilde{C}'_\alpha}\leq\overline{\widetilde{A}}_\alpha\ \ and\ \ \overline{\widetilde{C}'_\alpha}\leq\overline{\widetilde{B}}_\alpha)$.

The proof is straightforward by using the above theorem and the definition of complement of type-2 fuzzy sets.

## V. Approximate Type-2 Fuzzy Set Operations

Based on the ordering of type-1 fuzzy sets, mostly *max order set* [10] and its extension [5], and the concept of highest degree of separation in [13], it is possible to deal with type-2 operations very simply. In what follows, we assume that all membership values are convex type-1 fuzzy set.

Let us remind that when *A* and *B* are type-1 fuzzy sets, in order to calculate their intersection, for any $u\in U$, we have to find $Min\{\mu_A(u),\mu_B(u)\}$ where $\mu_A(u)$ and $\mu_B(u)$ are both crisp numbers, on which a total order is defined.

However, if we are dealing with type-2 fuzzy sets, their members have imprecise membership degrees modeled as type-1 fuzzy set, and a total ordering among the type-1 fuzzy sets must be clearly defined to be able to compute the minimum two between type-1 fuzzy sets. We now define such an ordering unambiguously, which results in a simplification of type-2 fuzzy set operations. The simplification is in fact obtained by means of an approximation; however, the loss of precision induced by the approximation is negligible in the important regions of the membership degrees.

The negation operator is not affected by the simplification. Therefore, we now discuss the calculation of the intersection operation; the calculation of union is dual to that of intersection and can be obtained from it.

Let $\widetilde{A}(u)$ and $\widetilde{B}(u)$ represent a vertical slice of type-2 fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ respectively. Accordingly, $\widetilde{A}(u)$ and $\widetilde{B}(u)$ represent type-1 fuzzy sets as imprecise membership degrees of element $u$ in type-2 fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ respectively.

If $\widetilde{A}(u)$ and $\widetilde{B}(u)$ are disjoint fuzzy sets, clearly their minimum is the one which lies closer to 0. For example in the Fig. 4, $\widetilde{A}(u)$ is clearly less than $\widetilde{B}(u)$. Their highest degree of separation is 1.

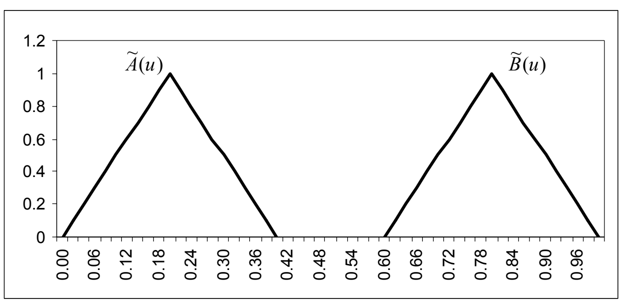



Fig. 4. Simple example, where highest degree of separation is 1.

When the highest degree of separation is less than 1, let us define $\widetilde{A}'(u) = \widetilde{A}(u) - (\widetilde{A}(u) \cap \widetilde{B}(u))$ and $\widetilde{B}'(u) = \widetilde{B}(u) - (\widetilde{A}(u) \cap \widetilde{B}(u))$. By construction, $\widetilde{A}'(u)$ and $\widetilde{B}'(u)$ are disjoint. However, more precisely, $\widetilde{A}'(u)$ and $\widetilde{B}'(u)$ in general can be rewritten as $\widetilde{A}'(u) = \bigcup_i \widetilde{A}'_i(u) \mid \widetilde{A}'_l(u) \cap \widetilde{A}'_m(u) = \phi, l \neq m$ and $\widetilde{B}'(u) = \bigcup_i \widetilde{B}'_i(u) \mid \widetilde{B}'_l(u) \cap \widetilde{B}'_m(u) = \phi, l \neq m$ respectively.

We define $\widetilde{C}(u) = Minn\{\widetilde{A}'(u), \widetilde{B}'(u)\} + (\widetilde{A}'(u) \cap \widetilde{B}'(u))$, as the approximation of the minimum of $\widetilde{A}(u)$ and $\widetilde{B}(u)$, where $Minn\{\widetilde{A}'(u), \widetilde{B}'(u)\} = Min\{\underset{i}{Min}\{\widetilde{A}'_i(u)\}, \underset{j}{Min}\{\widetilde{B}'_j(u)\}\}$, such that $Min\{A, \phi\} = A$ and $Min\{\phi, \phi\} = \phi$. Figure 5 depicts a classical example that, this method produces exact answer.

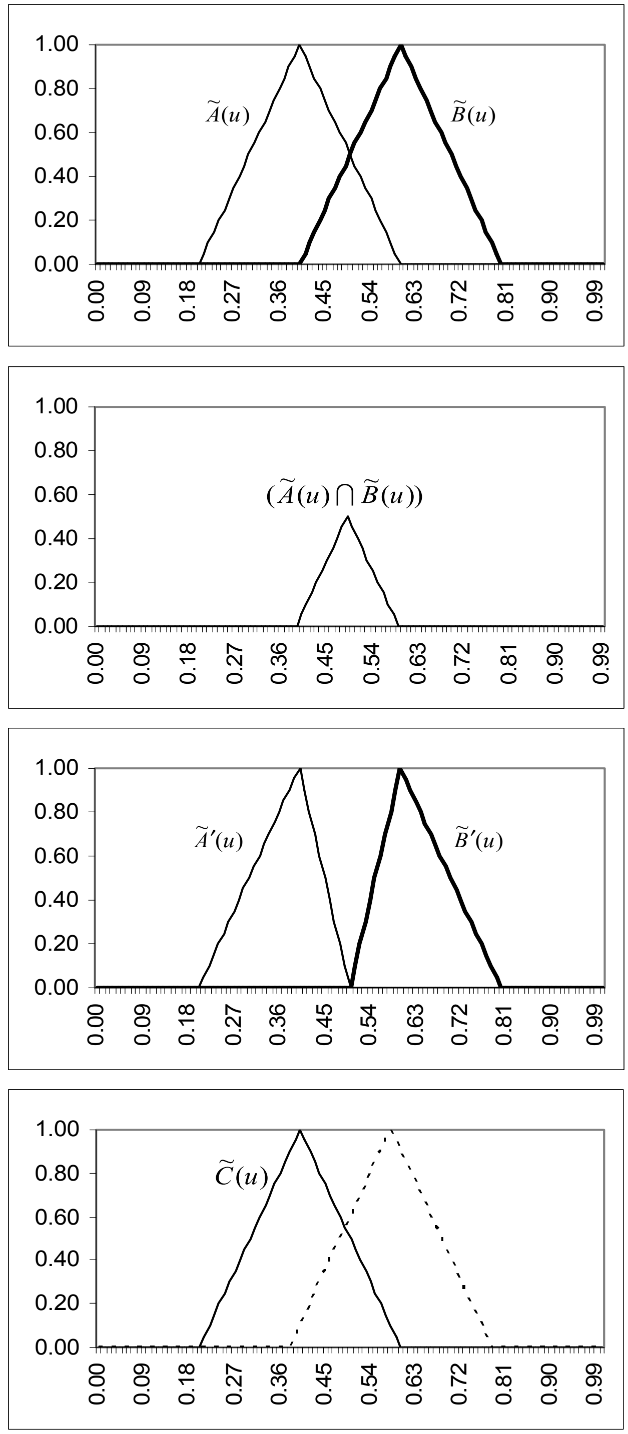



Fig. 5. Classical example, where highest degree of separation is less than 1 and the approximate method produces exact answer.

As another example, consider two convex type-1 fuzzy sets $\widetilde{A}'(u)$ and $\widetilde{B}'(u)$, as shown in the Fig. 6. Here, the result of intersection given by the simplified method, as it was the case with the above example, is also exact. Be noticed that highest degree of separation, here is 0.

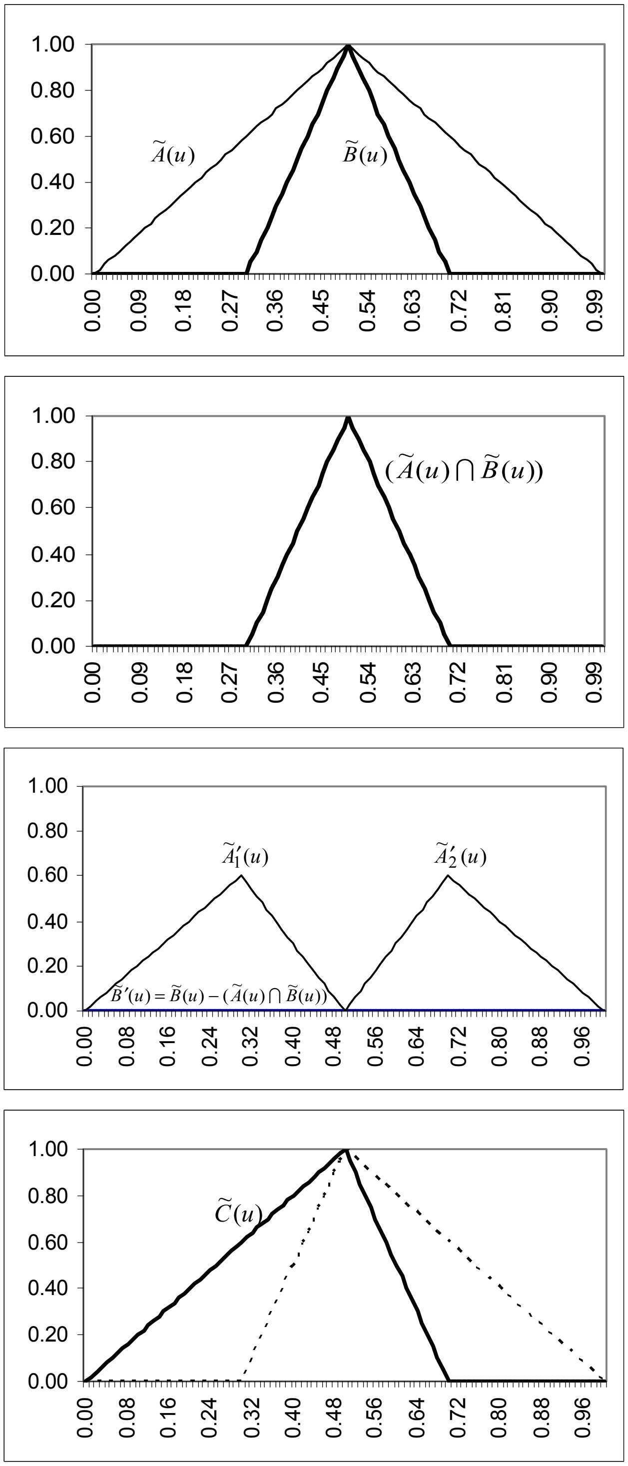



Fig. 6. An example, where highest degree of separation is 0 and the approximate method produces exact answer.

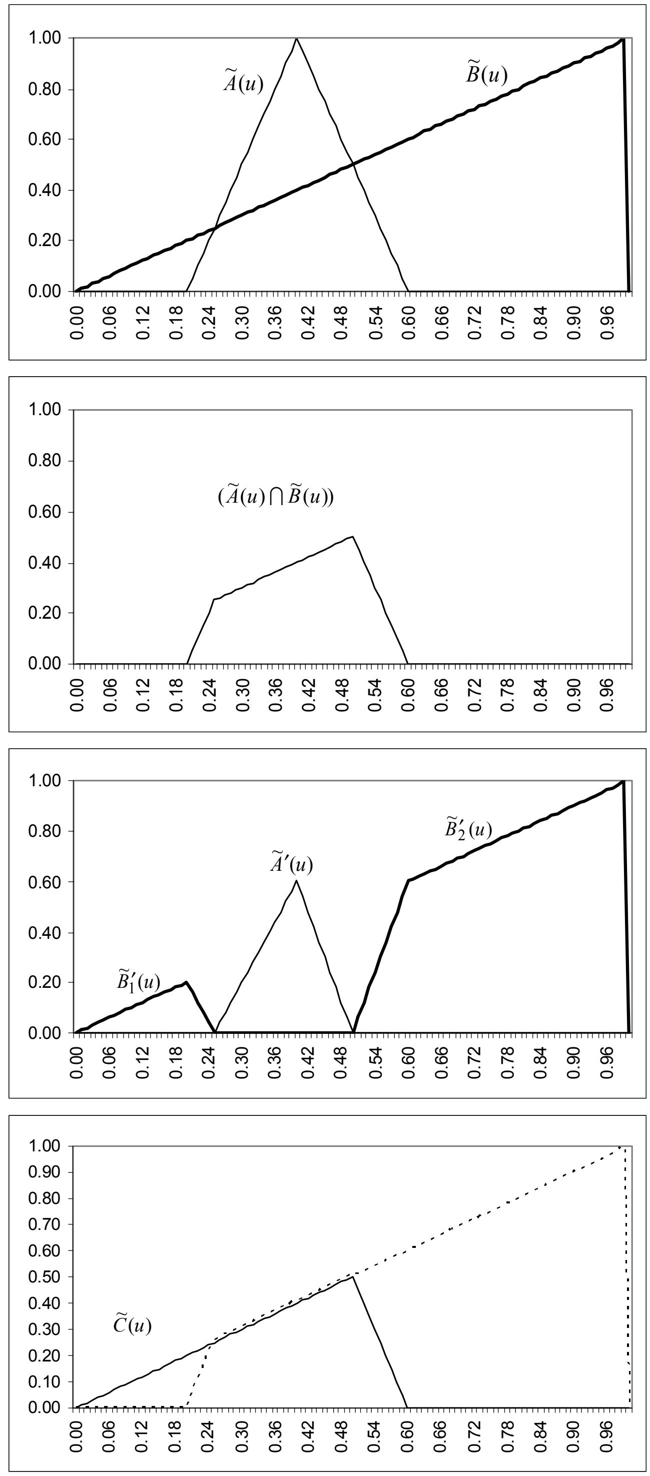



Fig. 7. An example, where the answer is approximated.

In the example, shown in Fig. 7, where $Support(\widetilde{A}'(u)) \subseteq Support(\widetilde{B}'(u)$ and $Height(\widetilde{A}'(u)) \geq Height(\widetilde{A}'(u) \cap \widetilde{B}'(u))$ the result is approximated but we observe that the most important part, that is the part closer to zero, which is more important in judging on being minimum, is correctly retained.

The simplified method has been tested on a variety of non-convex fuzzy sets and the results obtained were always, if not exact, at least correct in the sense that the more informative portion of the resulting membership function was correctly retained. An example is reported in the Fig. 8.

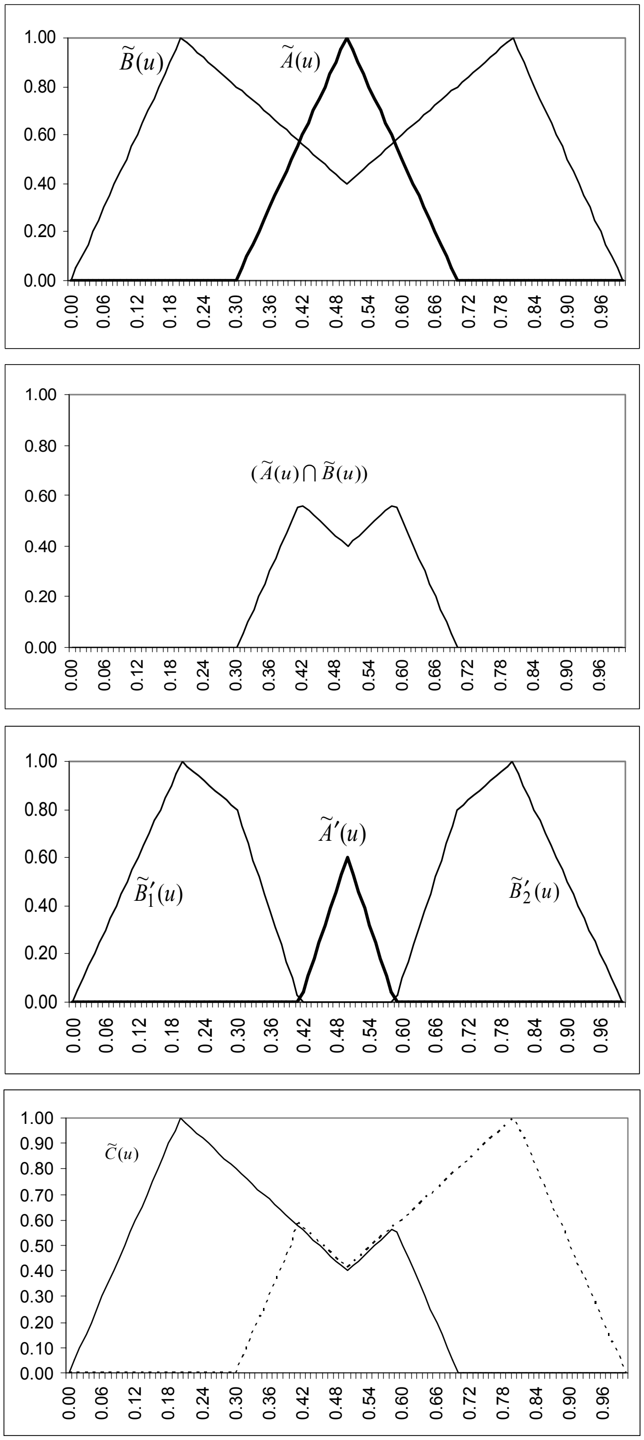



Fig. 8. Applying the method on non-convex fuzzy sets.

In all the above examples, approximate maximum of the two linguistic values, which is to be used in calculating the union of two type-2 fuzzy sets, is shown in dashed line and without label.

## VI. Conclusion

Type-2 fuzzy sets are a significant improvement over type-1 fuzzy sets. However, as mentioned in [8], they have not yet received the attention they deserve mainly because (1) no simple and uniform definition of them exists, (2) they are hard to graph due to their three-dimensional nature, (3) derived formulas are hard to understand, (4) complex to implement and (5) in our belief rarely attractive for researchers. One of the main bottlenecks of type-1 fuzzy sets is the assignment of a membership degree – a crisp number – to elements of the universe of discourse. Such an assignment has to be performed arbitrarily, or as a result of a trial-and-error process; alternatively, techniques like neural networks or evolutionary algorithms are used to tune the parameters of fuzzy systems. However, type-2 fuzzy sets would clearly represent a better choice in case there exists uncertainty about the exact membership degree of each element and we aim at keeping the system as natural as possible [9]. However, in [12] it is shown that when input data in type-1 fuzzy set is noisy, which results in uncertainty in type-1 fuzzy sets, the system is amenable to take advantage of type-2 fuzzy sets.

In this paper, respecting the FoU and the three-dimensional nature of type-2 fuzzy sets, we introduced type-2 fuzzy sets by naturally integrating membership degrees of type-1 fuzzy sets with linguistic variables, which undoubtedly simplifies the concept; on the other hand, for performing operations, we introduced $\alpha$-level sets over type-2 fuzzy sets and extended the set ordering concept, based on the corresponding concept on interval sets mentioned in [10], [6], [5], and [1], which results in operation definitions that make just one restriction on membership values, namely being normal type-1 fuzzy sets, comparable to those proposed in [3], [4], and [8]. On the other hand, to make type-2 fuzzy sets more suited for applications, we introduced an approximate method for calculating the relevant set operations, based mainly on the corresponding operations on type-1 fuzzy sets. Finally, we discussed where the suggested method produces exact results and where it produces approximate results. We argue that the approximation is acceptable because it calculates the most important part of the result accurately.

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