One-relation languages and ω-code generators

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23 Juin 2010

ω-words

Let A be a finite alphabet and L be a language over A:

- ω-word : infinite sequence of letters in A;
- ω-language : set of ω-words;
- ω-power of L:

$$L^\omega = \{u_0 u_1 \cdots \mid \ u_i \in L \setminus \epsilon\}$$

Example

$L = a^* b$
$L^\omega = (a^* b)^\omega$: set of ω-words which contain an infinite number of b.

ω-generators

Set of ω-generators of L^ω:

$$[L^\omega] = \{G \subseteq A^+ \mid G^\omega = L^\omega\}$$

Example

$L^\omega = (a^* b)^\omega$: set of words which contain an infinite number of b.
$G = a^* b a^* \in [L^\omega]$.

Codes

- L code: any word in L^+ has a unique factorization in words in L;
  $\{a, ab, ba\}$ is not a code

$$aba = a(ba) = (ab)a$$

- L ω-code: any ω-word in L^ω has a unique factorization in words in L;
  $\{a, ab, b^2\}$ is not an ω-code

$$abbb\cdots = (a)b^2b^2\cdots = (ab)b^2b^2\ldots$$

- L prefix code: no element in L is a proper prefix of another element in L.

Prefix code $\Rightarrow$ ω-code $\Rightarrow$ code.

An open problem

"Does a rational ω-language have an ω-code generator ?"

The problem is solved for prefix code generators [Litovsky, 91].

Note that an ω-code is minimal w.r.t. inclusion into the set of ω-generators.

Finite vs infinite case

$L \neq \emptyset$ and L rational.

	monoid L^*	ω-power L^ω
maximal generators	L^*	finite
minimal generators	$(L^* \setminus \epsilon) \setminus (L^* \setminus \epsilon)^2$	**?**

- L^* is generated by a code iff its minimal set generator is a code.
- For ω-power, the case is very different.

Example

$L = \{a^2, a^3, b\}$ is a minimal generator of L^ω.
L^* is the greatest generator of L^ω.
$\{a^2, a^3 b, b\}$ is an ω-code generator of L^ω.

Remarks

1) If L is a two-element language, then L^ω always has an ω-code generator.

- If $L = \{u, v\}$ is not an ω-code, then u, v are powers of a same word. Thus $\{u, v\}^\omega = u^\omega$.

2) There exists a three-element language L such that L^ω has no ω-code generator.

Example

$\{a, ab, baba\}^\omega$ has no ω-code generator.

Non-trivial Relations

$L = \{u_1, u_2, \ldots, u_n\}$ a finite language over A;
$\Sigma = \{x_1, x_2, \ldots, x_n\}$ an alphabet.
A relabeling of L is a morphism $\tilde{}: \Sigma^\infty \to L^\infty$ defined by

$$x_i \mapsto u_i$$

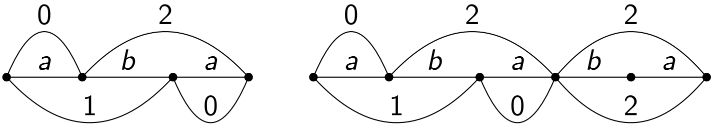
- The set of *relations* is satisfied by L:

$$\cong \quad = \quad \{(x, y) \in \Sigma^\infty \times \Sigma^\infty \mid \tilde{x} = \tilde{y}\}$$

- $x \cong y$ is called minimal relation if $x' \ncong y'$ for all $x' \in \mathit{Pref}(x)$ and $y' \in \mathit{Pref}(y)$.

Example

For $L = \{a, ab, ba\}$ and $\Sigma = \{0, 1, 2\}$.



The set of minimal relations of L is

$$\begin{cases} 02^n \cong 1^n 0 & \text{for all } n > 0 \\ 02^\omega \cong 1^\omega \end{cases}$$

Example

For $L = \{a, ab, ba\}$ and $\Sigma = \{0, 1, 2\}$. The set of minimal relations of L is recognized by a finite automaton:

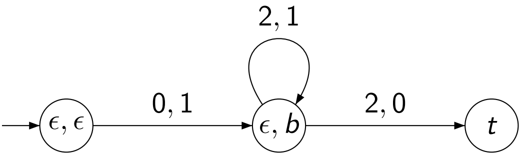


Figure: Automaton recognizing the language $(0, 1)(2, 1)^\infty(2, 0)$.

Thus, the set of minimal relations of L:

$$\begin{cases} 02^n \cong 1^n 0 & \text{for all } n > 0 \\ 02^\omega \cong 1^\omega \end{cases}$$

Minimal relations

Theorem (Lothaire, 2002)

The set of minimal relations for a finite language is a rational relation.

Remark

Two different languages can have the same set of minimal relations.

Example

- $L_0 = \{a, ab, ba\}$;
- $L_1 = \{aba, ab, ba\}$;
- or all languages in the class

$$\mathcal{L} = \{(uv)^n u, uv, vu \mid u, v \in \Sigma^+, uv \neq vu, \text{ and } n \geq 0\}.$$

Their ω-powers have an ω-code generator $\widetilde{C}$, where $C = 0 \cup 1^*2$.

Zero-relation languages = Codes

Theorem (Julia and al., 1996)

Let L be a code such that L^* is the greatest generator of L^ω. Then L^ω has an ω-code generator if and only if L is an ω-code generator.

What about one-relation languages ?

Properties of set $\cong$

P1) $\cong$ is a congruence;

P2) If $x^\omega \cong w$ then $xw \cong w$;

P3) If $xw \cong w$ then $x^\omega \cong w$.

Some relations are derivated from the others:

Example

$$
\begin{aligned}
02 \cong 10 &\Rightarrow 022 \cong 102 \cong 110 \\
&\Rightarrow 02^n \cong 1^n 0.
\end{aligned}
$$

$$
\begin{aligned}
10 \cong 02 &\Rightarrow 102^\omega \cong 022^\omega = 02^\omega \\
&\Rightarrow 1^\omega \cong 02^\omega
\end{aligned}
$$

One-relation languages

Definition

We say that L is a *one-relation language* if there is a pair $(x, y) \in \Sigma^+ \times \Sigma^+,\ x \neq y$ such that $\cong$ is the smallest subset of $\Sigma^\infty \times \Sigma^\infty$ containing (x, y) and satisfying the properties P1–P3. Then $x \cong y$ is called *basic relation*.

Example

Let $L = \{a, ab, ba\}$, and $\Sigma = \{0, 1, 2\}$. L is a one-relation language, where $02 \cong 10$ is the basic relation.

Example

Let $L = \{a, ab, ba, ac, ca\}$, and $\Sigma = \{0, 1, 2, 3, 4\}$. L is not one-relation language; i.e., we have $02 \cong 10$ and $04 \cong 30$.

Overlaps of basic relation

$$OVL(x,y) = (Pref(x) \cap Suf(y)) \cup (Suf(x) \cap Pref(y)).$$

The set of overlapping words of the basic relation generate other relations.

Example

- Let $L = \{a, a^2b, ba^2\}$ and $\Sigma = \{0,1,2\}$.
- The basic relation $002 \cong 100$, and $OVL(002, 100) = \{0, 00\}$.

$$100 \cong 002 \Rightarrow 1002^\omega \cong 0022^\omega = 002^\omega$$
$$\Rightarrow 1^\omega \cong 002^\omega$$

$$100 \cong 002 \Rightarrow 100(02)^\omega \cong 002(02)^\omega = 0(02)^\omega$$
$$\Rightarrow (10)^\omega \cong 0(02)^\omega$$

Set of overlapping words is empty

Proposition

Let L be a finite language such that L^* is the greatest generator of L^ω. If L is one-relation language with the basic relation $x \cong y$ and $OVL(x, y) = \emptyset$, then L^ω has no ω-code generator.

Set of overlapping words is singleton

Proposition

Let L be a finite language such that L^* is the greatest generator of L^ω. If L is a one-relation language with the basic relation $xz \cong yx$ and $OVL(xz, yx) = \{x\}$, then L^ω has an ω-code generator if and only if

$$x = 0 \text{ and } y = 1 \quad (0z \cong 10)$$

or

$$x = 0, z = 2, \text{ and } y \in 10^+ \quad (02 \cong 10^n).$$

Set of overlapping words is singleton (cont.)

Proposition

Let L be a finite language such that L^* is the greatest generator of L^ω. If L is a one-relation language with the basic relation $xz \cong yx$ and $OVL(xz, yx) = \{x\}$, then L^ω has no finite ω-code generator.

Conclusions: some natural extensions

Example

- $L = \{a, a^2b, ba^2\}$ and $\Sigma = \{0, 1, 2\}$.
- Basic relation $002 \cong 100$ and $OVL(002, 100) = \{0, 00\}$.
- ω-code generator of L^ω:

$$a \cup \{a^2b, a^2ba\}^* ba^2.$$

Example

- $L = \{a, ab, ba, ac, ca\}$, and $\Sigma = \{0, 1, 2, 3, 4\}$;
- Basic relations: $02 \cong 10$ and $04 \cong 30$;
- ω-code generator of L^ω:

$$a \cup \{ab, ac\}^* \{ba, ca\}.$$