Decompositions of a small tensor

Tensors are arrays of values with 1, 2, 3, ..., N indexes. Tensors with 1 index are named 1^{st} order tensors and we commonly call them vectors. Tensors with 2 indexes are named 2^{nd} order tensors and we commonly call them matrices. When the number of indexes is larger than 2, say N, no common names are available and we simply call them N^{th} order tensors.

With the recent advances in data acquisition, data can now be acquired under many different contexts, using different acquisition modalities and from many different individuals. As a consequence, N^{th} order tensors (with N>2) are now available in many data analysis problems. This motivated and motivates research on how to extend matrix decomposition techniques used in data analysis, such as the singular value decomposition (SVD) or non negative matrix factorization (NMF), to such higher order data arrays.

In this project, the students will be introduced to such extensions by evaluating analytically two tensor decompositions, the canonical polyadic (CP) decomposition [1,2] and the higher order singular value decomposition (HOSVD) [3] of a 3^{rd} order tensor with size 2 x 2 x 2 and elements :

$$x_{i,j,k} \in \mathbb{R} \quad \text{with} \quad i,j,k \in \{1,2\} \times \{1,2\} \times \{1,2\}$$

The students will first evaluate both tensor decompositions for the simpler case, where the tensor is symmetric. Then, later, they will tackle the more general non symmetric case. From their results for this simple special case, the students may highlight some peculiar facts about tensors with N>2, such as:

1) The tensor CP decomposition for N>2 can be essentially unique even without orthogonality constraints such as those imposed in the SVD factors. This is in contrast with unconstrained matrix decompositions which are never essentially unique.

2) CP decomposition rank, also called the tensor rank, can be larger than any of the tensor dimensions. Notice that for matrices the rank is always smaller than the smallest dimension.

3) The fiber related ranks obtained in the HOSVD for N>2 can be different, whereas in the matrix case, the column rank is always equal to the row rank.

At the end of the project, the students will compare their analytical results with decompositions given by numerical methods in a few examples. The comparison will be carried out either in python or in scilab.

References:

[1] Kolda, T. G., & Bader, B. W. (2009). *Tensor decompositions and applications*. SIAM review, 51(3), 455-500.

[2] Comon, P., Luciani, X., & De Almeida, A. L. (2009). *Tensor decompositions, alternating least squares and other tales*. Journal of chemometrics, 23(7-8), 393-405.

[3] De Lathauwer, L., De Moor, B., Vandewalle, J. (2000). *A Multilinear Singular Value Decomposition*. SIAM Journal on Matrix Analysis and Applications. 21(4): 1253–1278.

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