Un algorithme pour décider de la surjectivité sur les configurations finies des automates cellulaires en dimension 1

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Objective

- Implication diagram between global properties of 1D CA

G bijective $\Leftrightarrow$ G reversible $\Leftrightarrow$ G injective $\Leftrightarrow$ $G|_{\mathcal{P}}$ bijective $\Leftrightarrow$ $G|_{\mathcal{P}}$ injective

$\Downarrow$

$G|_{\mathcal{F}}$ bijective $\Leftrightarrow$ $G|_{\mathcal{F}}$ surjective

$\Downarrow$

$G|_{\mathcal{F}}$ injective $\Leftrightarrow$ $GoE = \emptyset$ $\Leftrightarrow$ G surjective $\Leftrightarrow$ $G|_{\mathcal{P}}$ surjective

- Decision algorithm for surjectivity on finite configurations for 1D CA

1D CA

Formally, a (one-dimensional) cellular automaton (1D CA) is a structure

$$\mathcal{A} = \langle r, \Sigma, [-r, r], \delta \rangle$$

- $r \in \mathbb{N}$ — *radius*
- Σ — (finite) set of *states*
- $[-r, r]$ — *neighborhood*
- $\delta \colon \Sigma^{2r+1} \to \Sigma$ — *local function*

A 1D CA is *elementary* (ECA) $\to \Sigma = \{0, 1\}$ and $r = 1$.

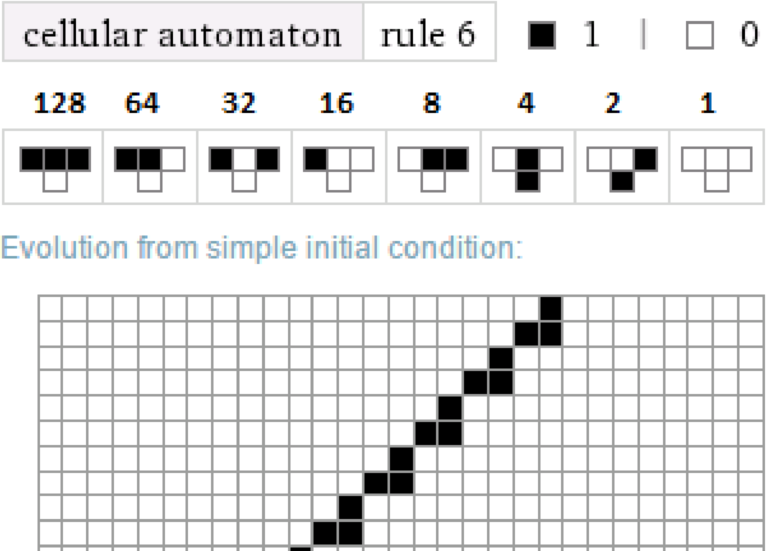
Wolfram number

The *Wolfram number* of a table $\delta : \{0,1\}^{2r+1} \to \{0,1\}$ is given by

$$\sum_{x \in \{0,1\}^{2r+1}} 2^{dec(x)} \cdot \delta(x)$$

where $dec(x)$ is the function which assigns the decimal representation to the binary number x.

Example : rule 6



Global rule

The global rule $G\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ describes the overall behavior of a CA from time step t to $t+1$ for any $t \in \mathbb{N}$:

$$\forall c \in \{0,1\}^{\mathbb{Z}}, \forall i \in \mathbb{Z},\ G(c)_i = \delta(c_{[i-r,i+r]}) \ .$$

Finite and periodic configurations

A configuration c is x-finite for $x \in \Sigma$ if $|\{i \in \mathbb{Z}, c_i \neq x\}| < \infty$.

Let $\mathcal{F}$ be the set of all x-finite configurations for a fixed $x \in \Sigma$.

A configuration c is spatially periodic if $\exists p \in \mathbb{N}$ *st.* $\sigma^p(c) = c$ where σ is the shift map.

Let $\mathcal{P}$ be the set of all spatially periodic configurations.

G bijective $\Leftrightarrow$ G reversible $\Leftrightarrow$ G injective
$\Leftrightarrow$ $G|_{\mathcal{P}}$ bijective $\Leftrightarrow$ $G|_{\mathcal{P}}$ injective

$\Downarrow$

$G|_{\mathcal{F}}$ bijective
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$\Downarrow$

$G|_{\mathcal{F}}$ injective $\Leftrightarrow$ $GoE = \emptyset$ $\Leftrightarrow$ G surjective
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We differentiate surjectivity according to $x \in \Sigma$:

- $G|_{\mathcal{F}_x}$ is the restriction of G to x-finite configurations.
- $G|_{\mathcal{F}_\star} := \bigcup_{x \in \Sigma} G|_{\mathcal{F}_x}$.
- $G|_{\mathcal{F}^2} := G^2|_{\mathcal{F}}$ with $\delta(0 \ldots 0) = 1$ and $\delta(1 \ldots 1) = 0$.

Definition

In the case of binary CA, $G|_{\mathcal{F}}$ is *surjective* if $G|_{\mathcal{F}_x}$ is surjective for some $x \in \Sigma$.

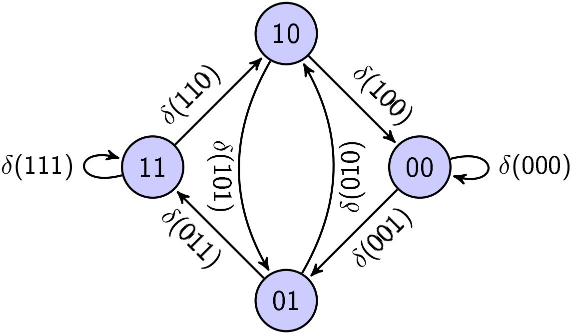
$G|_{\mathcal{F}_x}$ is surjective for some $x \in \Sigma$: each finite configuration has a pre-image (finite).

Definition

Given two binary strings $x, y \in \{0,1\}^{2r+1}$, the *fusion* $x \odot y$ of x and y is the string $x_0 y$ if $x_i = y_{i-1}$ for $i \in [1, 2r]$, $\perp$ otherwise.

The *De Bruijn* graph B_δ associated with a CA $\langle r, \Sigma, [-r, r], \delta \rangle$ is a structure $\langle V, E, l \rangle$,

- $V = \{0,1\}^{2r}$
- $E = \{(x, y) \in V \times V \mid x \odot y \neq \perp\}$
- $l \colon E \to \{0,1\}$ is a labelling function such that $l(x, y) = \delta(x \odot y)$.

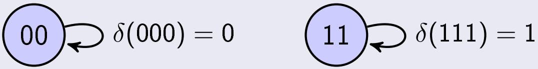


Remark

Given a bi-infinite path $P = \ldots p_0, \ldots, p_n \ldots$ over B_δ, the string $\ldots \odot p_0 \odot p_1 \odot \ldots \odot p_n \odot \ldots$ is a configuration for the CA δ and $l(P) = \ldots l(p_0, p_1) \cdot \ldots \cdot l(p_{n-1}, p_n) \ldots$ is its image.

Remark

$$G({}^{\omega}0^{\omega}) = {}^{\omega}0^{\omega} \qquad G({}^{\omega}1^{\omega}) = {}^{\omega}1^{\omega}$$



Proposition 1

For $u, v \in \mathcal{F}$, if $G(x^{\omega} v x^{\omega}) = x^{\omega} u x^{\omega}$, then the configuration has to start and end in vertex $i = \{x\}^{2r}$ with $\delta(i \odot i) = x$ on the associated De Bruijn graph.

Basic idea

Modify the initial and final sets of vertices of B in a new graph B^T in which we trim configurations by x *st.*

$$u \in L(B^T) \Leftrightarrow \exists k, k' \geq 0 : x^k u x^{k'} \in L(B)$$

000 11 00 reconnu par L(B) $\Leftrightarrow$ 11 reconnu par $L(B^T)$

Construction of B^T from B

Let a De Bruijn graph $B = (\Sigma, I, F)$

- set of states Σ
- set of initial states I
- set of final states F

Let $i = \{x\}^{2r}$
$I = F = \{i\}$ if $\delta(i \odot i) = x$, $\{\perp\}$ otherwise.

$I(y, u) \rightarrow$ the vertex reached starting from vertex y after reading word u as a sequence of labels.

Construction of B^T

$$I^T = \{I(q', x^*), q' \in I\}$$

$$F^T = \{q \in \Sigma, \exists k \geq 0, I(q, x^k) \in F\}$$

$$B^T = (\Sigma, I^T, F^T)$$

Proposition 2

$u \in L(B^T) \Leftrightarrow \exists k, k' \geq 0 : x^k u x^{k'} \in L(B).$

Proof

Proof.

[$\Rightarrow$] ($\Leftarrow$ is obvious from the construction of B^T)

For $\{x\}^{2r} \in I$, $u \in L(B^T) \Rightarrow \exists q \in I^T, \exists q' \in F^T : l(q,u) = q'$

$$\exists k \geq 0 : l(\{x\}^{2r}, x^k) = q,$$

$$\exists k' \geq 0 : l(q', x^{k'}) = \{x\}^{2r} \in F \Rightarrow x^k u x^{k'} \in L(B)$$

□

Test surjectivity on $\mathcal{F}$

Proposition 3

$\forall x \in \Sigma,\ G|_{F_x}$ is surjective *iff* $L(B^T) = \Sigma^*$.

Proof

Proof.

From Proposition 2, we know that if $u \in L(B^T)$, then $\exists k, k' \geq 0 : x^k u x^{k'} \in L(B)$. In other words, it means that for a recognized configuration u in B^T, the first symbol of u is equal to the last symbol of u and is not equal to x. In addition, to be surjective, one need that each configuration has a least one pre-image. Thus, all configurations of B^T should have a pre-image, which is equivalent to test if $L(B^T) = \Sigma^*$. □

Test equality to Σ^* .

Basic idea

Determinize B^T and detect failure during construction.

In the deterministic graph, starting from vertex I^T

- there should be an outgoing edge from current vertex labelled by x for any possible x belonging to Σ
- if it contains a vertex $v \in F^T$, then it is final.

Since we construct the deterministic graph and we want to read each symbol, if the current created vertex is not final or has not an outgoing edge labelled by x, $\forall x \in \Sigma$, then $L(B^T) \neq \Sigma^*$.

Complexity

There are:

- 2^{2r} vertices in B
- 2^{2r} vertices in B^T
- $2^{2^{2r}}$ vertices in the deterministic graph of B^T *(in the worst case)*

Complexity

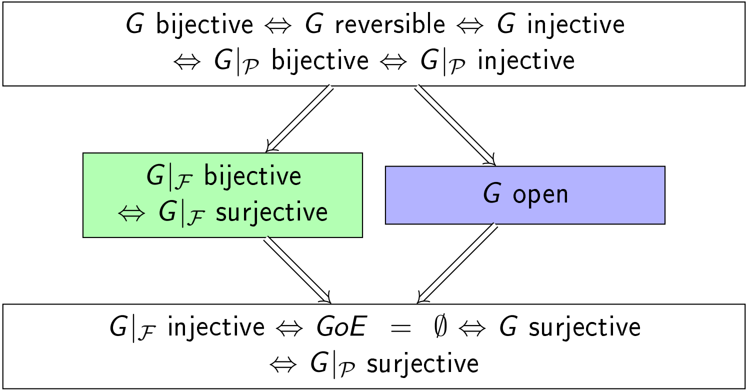
Our algorithm is exponential in the size of the local rule.

Results for ECA

Wolfram number of ECA			Surjectivity type
	166	180	$G\|_{\mathcal{F}_0}$
	154	210	$G\|_{\mathcal{F}_1}$
170	204	240	$G\|_{\mathcal{F}_\star}$
15	51	85	$G\|_{\mathcal{F}^2}$

In order to test if $G|_{\mathcal{F}^2}$ is surjective, increase radius by 1.

Implication diagram



Conclusions

- New decision algorithm for surjectivity on $\mathcal{F}$ for 1D CA
- Completion of the implication diagram of [durand97] in 1D *w.r.t.* the openess property

Thank you for your attention :-)