M2 Complex Systems - Complex Networks

Lecture 5 - Community detection algorithms

Girvan-Newman, Louvain, Leiden

Automn 2021 - ENS Lyon

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Communities in complex networks

What is a community ?

"Moral" definition

- A group of nodes that share something...
  - People with a common interest
  - Web pages with similar content
  - Proteins realising a common function

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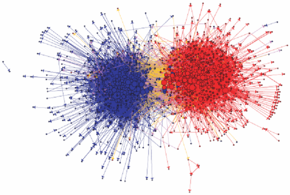
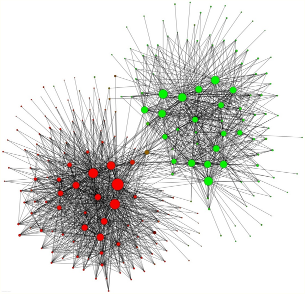
- ... that makes them be in relationship in the network !



Political blogs in US



Languages in Belgium

Communities in complex networks

What is a community ?

Structural definition

- A highly connected group of nodes

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Communities in complex networks

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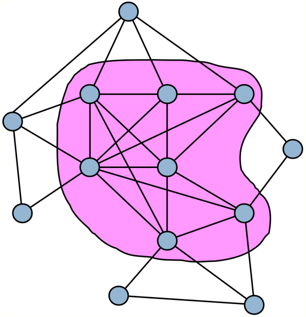
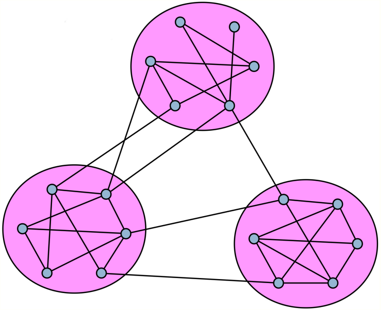
Structural definition

- A highly connected group of nodes
  - Density inside the community much higher than global density of the network
  - Only few edges toward the rest of the network



Types of structural communities

- Partition of the nodes into dense parts sparsely connected between them
  - High density inside communities
  - Few edges between communities



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  A node can belong to several communities
  - more realistic
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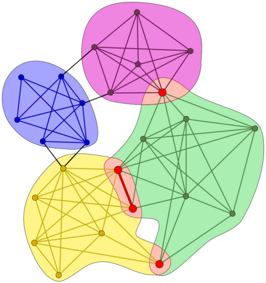

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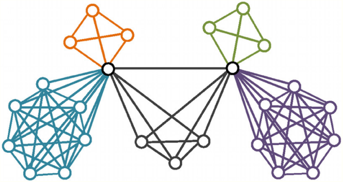
- Partition of the links
  - a link belong to exactly one community
  - a node can have links in different communities



Partition of the nodes

Various approaches, among them :

- random walks
- spectral methods
- hierarchical clustering
- divisive methods
- Louvain, Leiden

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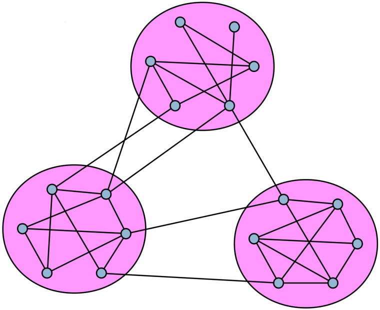
Divisive approach : Girvan & Newman 2002

The idea :

1. identify inter-community links
2. remove them



How to identify inter-community links ?

- Betweenness centrality of links
  - $C_B(e) = \sum_{s \neq t} \frac{\sigma_{st}(e)}{\sigma_{st}}$ where
    - $\sigma_{st} = \#$ shortest paths from s to t
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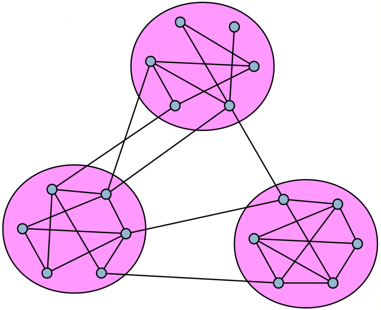

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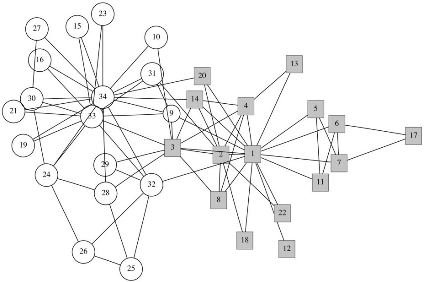

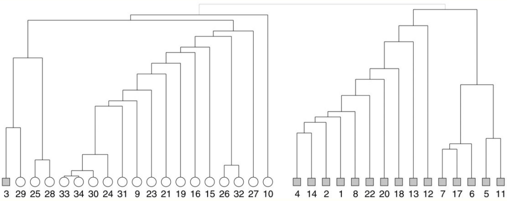

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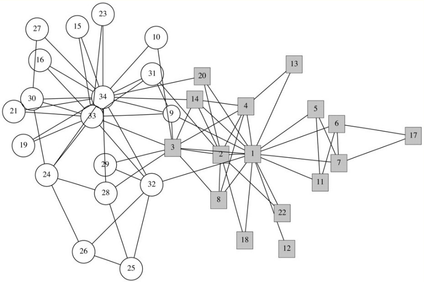

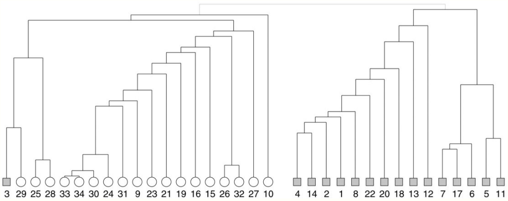

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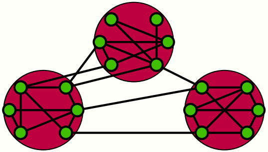
- Complexity
  - betweenness for all links : $O(nm)$
  - connected components : $O(m)$
  - m iterations
  - Overall : $O(nm^2)$

The Louvain algorithm

- Idea : optimize a quality function for node partitions



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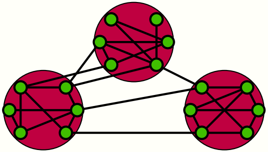

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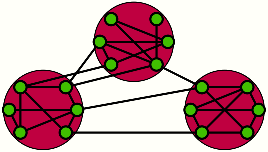

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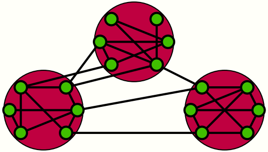
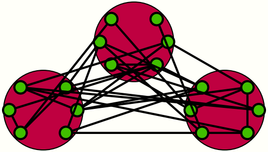
- Correction : compare to a randomized version of the network

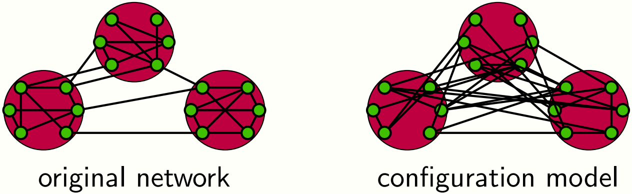


original network



configuration model

Modularity



- Proportion of edges inside communities

A the adjacency matrix of G
k_i the degree of node i
c_i the community of node i
δ is the Kronecker symbol : $\delta(c_i, c_j) = 1$ iff $c_i = c_j$

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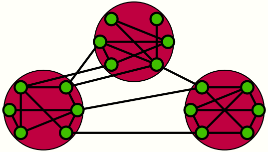

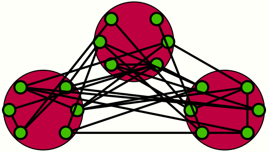

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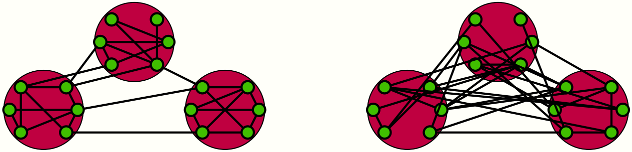

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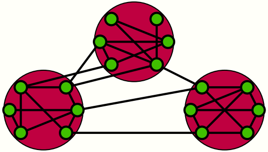

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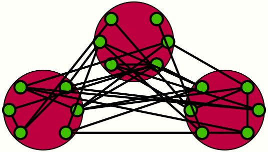

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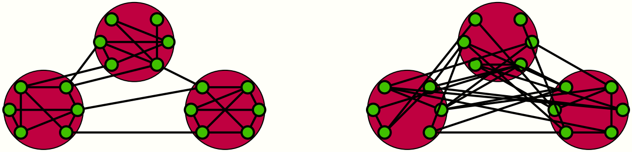

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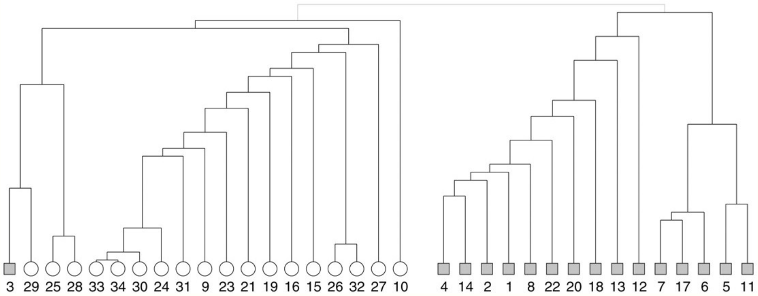
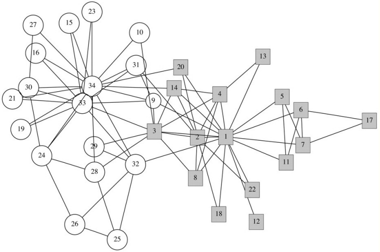
  - **NP-hard** to maximize modularity

Utility of modularity

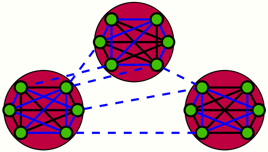
- Come back to the dendogram produced by Girvan-Newman





Other quality functions

- Distance to cluster graphs



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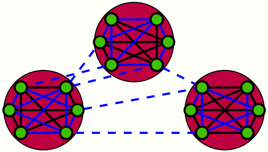

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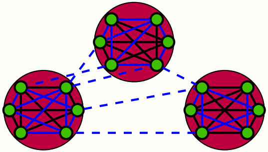

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  - CPM($\mathcal{P}$)=$\sum_{c}[e_c - \gamma\binom{n_c}{2}]$
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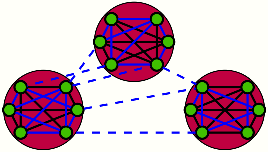

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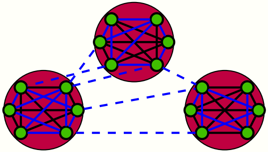

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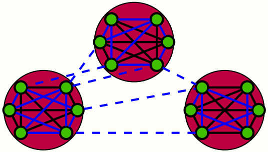

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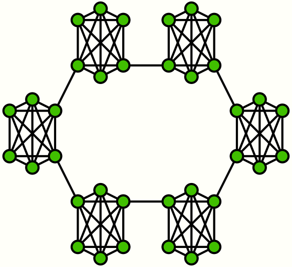

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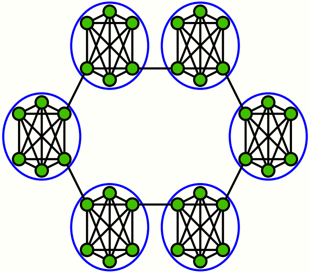

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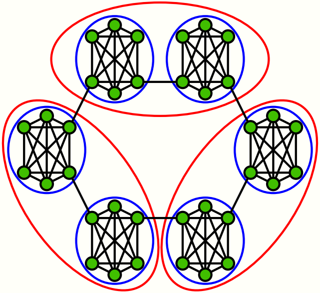

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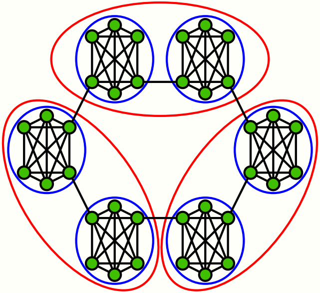

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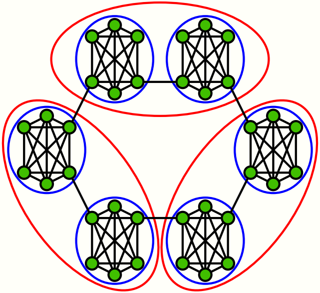

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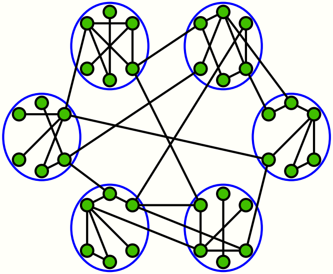
- Which one is "morally" the best community partition ?
- Which one has higher modularity ?

Louvain algorithm

- **Given a partition**, make a pass through all the vertices :
  - consider each vertex x once in an arbitrary order
  - move x to the community that gives the largest increase in modularity

G (n=30,m=46)



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Obs. : non-neighbouring community is never the best

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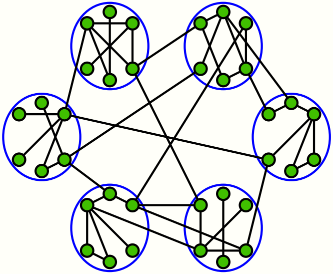

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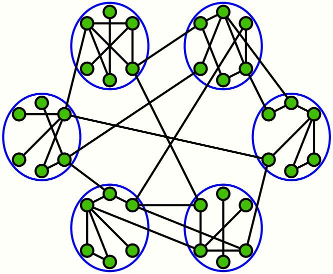
Obs. : non-neighbouring community is never the best

G (n=30,m=46)



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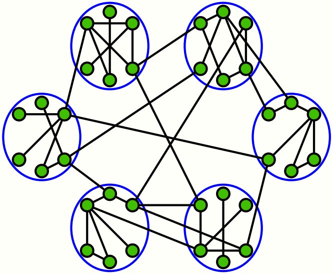

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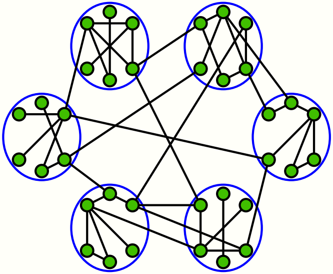

$$\Delta Q(C,i) = \left[\frac{e_C+k_{i,C}}{2m} - \left(\frac{a_C+k_i}{2m}\right)^2\right] - \left[\frac{e_C}{2m} - \left(\frac{a_C}{2m}\right)^2 - \left(\frac{k_i}{2m}\right)^2\right]$$

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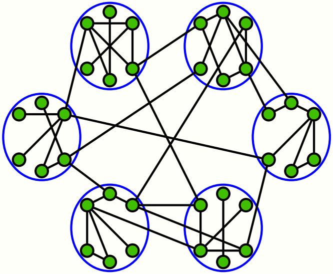
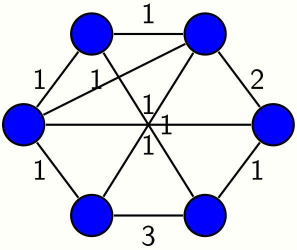


$G/\mathcal{P}$



Louvain algorithm

augmented$\leftarrow$ true;
while *augmented* **do**
  $\mathcal{P}_0 \leftarrow \{\{x\} \mid x \in V(G)\}$; $\mathcal{P} \leftarrow \mathcal{P}_0$; $Q \leftarrow 0$;
  while *augmented* **do**
    augmented$\leftarrow$ faux;
    for *i de 1 a n* **do**
      $Q_{ori} \leftarrow Q$;
      i moves to $c_{iso} = \{i\}$; $Q \leftarrow Q - \Delta Q_{out}(i)$;
      $Q_{max} \leftarrow Q$; $c_{max} \leftarrow c_{iso}$;
      for $c \in \mathcal{P}$ **do**
        if $Q + \Delta Q_{in}(c) > Q_{max}$ **then**
          $Q_{max} \leftarrow Q + \Delta Q_{in}(i, c)$;
          $c_{max} \leftarrow c$;
        end
      end
      If $Q_{max} = Q_{ori}$ **then** $c_{max} \leftarrow c_{ori}$ **else** augmented$\leftarrow$ true;
      i moves to c_{max} ; $Q \leftarrow Q_{max}$;
    end
  end
  If $\mathcal{P} \neq \mathcal{P}_0$ **then** augmented $\leftarrow$ true ; $G \leftarrow G/\mathcal{P}$;
end
return $\{Expand(P) \mid P \in \mathcal{P}\}$;

Leiden algorithm

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  - Contract only the obtained sub-communities
  - At the next step start from the partition defined by the whole communities