Introduction to Finite Dynamical Systems

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1 The value of $\min(G)$

This is a simple quantity: by Robert's theorem we known that $\min(G) = 1$ is G is acyclic and, actually, $\min(G) = 0$ otherwise. For the second case, the proof is a construction which uses the following notion: a **Feedback Vertex Set** (FVS) in a graph G is a subset $I \subseteq V(G)$ of vertices that every cycle of G has a vertex in I, that is, the graph $G \setminus I$ obtained from G by removing I (and the attached arcs) is acyclic. The set of in-neighbor of a vertex i in G is denoted $N(i)$.

Theorem 1 (Aracena, Salinas 2013)**.** *For every graph G we have*

$$\min(G) = \begin{cases} 1 & \textit{if } G \textit{ is acyclic} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. As mentioned above, by Robert's theorem we known that $\min(G) = 1$ is G is acyclic. So suppose that G has at least one cycle. Let I be a minimal feedback vertex set of G. We define $f \in F(G)$ by

$$f_i(x) = \bigwedge_{j \in N(i)} \overline{x_j} \quad \forall i \in I, \qquad f_i(x) = \bigvee_{j \in N(i)} x_j \quad \forall i \notin I.$$

We prove that f has no fixed point, and thus $\min(G) = 0$. Suppose, for a contradiction, that x is a fixed point of f. We consider two cases.

Suppose first that $x_i = 0$ for some $i \in I$. Since I is a minimal FVS, there is a cycle C in G that intersects I only in i (Exercise 1). Let $i_1, \dots, i_\ell$ an enumeration of the vertices of C in the order, starting from $i_1 = i$. We have $x_{i_1} = 0$. Suppose that $x_{i_k} = 0$, with $1 \leq k < \ell$. Then $i_{k+1} \notin I$ and we deduce that $f_{i_{k+1}}(x) = 0$ and thus $x_{i_{k+1}} = 0$. Consequently, $x_{i_k} = 0$ for all $1 \leq k \leq \ell$. In particular, $x_{i_\ell} = 0$, and since $i_1 \in I$, we deduce that $f_{i_1}(x) = 1 \neq x_{i_1}$, a contradiction.

Suppose now that $x_i = 1$ for all $i \in I$. Let J be the set of vertices i with $x_i = 0$. If J is empty then it is clear that $f_i(x) = 0$ for all $i \in I$, a contradiction. Thus J is not empty. Let $j \in J$. Since J and I are disjoint, $j \notin I$ and since $f_j(x) = x_j = 0$, there is at least one in-neighbor k of j such that $x_k = 0$. Thus $k \in J$. We have proved that $G[J]$ (the subgraph induced by J) has minimum in-degree at least one. Thus $G[J]$ has a cycle C, and since J is disjoint from I we get a contradiction: C does not intersect I.

This proves that f has no fixed points. $\square$

2 A lower bound one $\max(G)$

The **packing number** of G is the maximum size of a collection of vertex-disjoint cycles. For instance $\nu(K_n) = \lfloor n/2 \rfloor$. Note also that $\nu(G) = 0$ if and only if G is acyclic. By Robert's theorem, $\max(G) = 1$ if G is acyclic. The following lower bound generalizes this.

Theorem 2 (Aracena, Richard, Salinas 2017)**.** *For every graph G we have*

$$\max(G) \geq \nu(G) + 1.$$

Conjecture 1. *For every $k \geq 0$, there is G with $\nu(G) = k$ and $\max(G) = k + 1$. (for $k = 0, 1, 2$ this is easy, see Exercice 2.)*

We will just prove that $\max(G) \geq 2$ if G has a cycle (the general proof is slightly more technical). So suppose that G has a cycle. Let $f \in F(G)$ defined by $f_i(x) = \bigwedge_{j \in N(i)} x_j$ for all i. Then the all one state $x = (11\dots1)$ is obviously a fixed point. Let I be the set of vertices reachable from a cycle in G. It is not empty since G has at least one cycle C and then $V(C) \subseteq I$. It is clear that

$$\delta^-(G[I]) \geq 1.$$

Let x defined by $x_i = 0$ if $i \in I$ and $x_i = 1$ otherwise. If $x_i = 0$ then there is some $j \in N(i)$ with $x_j = 0$ since $\delta^-(G[I]) \geq 1$. Thus $f_i(x) = 0 = x_i$. If $x_i = 1$ then $i \notin I$, thus there is no $j \in N(i) \cap I$ and thus $x_j = 1$ for all $j \in N(i)$. We deduce that $f_i(x) = 1 = x_i$. Thus x is a fixed point, and it is not the all one state since $I \neq \emptyset$. Thus f has at least two fixed points, and we deduce that $\max(G) \geq 2$.

The packing number and the minimum in-degree are connected each other.

Theorem 3 (Alon 1996)**.** *For every graph G,*

$$\nu(G) \geq \left\lfloor \frac{\delta^-(G)}{64} \right\rfloor.$$

Conjecture 2 (Bermond, Thomassen 1981)**.** *For every graph G,*

$$\nu(G) \geq \left\lceil \frac{\delta^-(G)}{2} \right\rceil.$$

If true, the bound is optimal (see Exercice 3).

3 An upper-bound on $\max(G)$

If G is acyclic, then $\max(G) = 1$. From that we may think that if G is not so far from being acyclic, then $\max(G)$ is not se big. But to prove something like that we need a kind of distance from acyclicity. The right notion is the **transversal number** of G, defined as the minimum size of a FVS of G. It is denoted $\tau(G)$. Note that $\tau(G) = 0$ if and only if G is acyclic. Note also that $\nu(G) \leq \tau(G)$ (Exercise 4).

Theorem 4 (Feedback bound; Riis 2007 and Aracena 2008)**.** *For every graph G, we have*

$$\max(G) \leq 2^{\tau(G)}.$$

Proof. Let I be a FVS of size $\tau(G)$. Let $f \in F(G)$ and let x, y be fixed points of f (not necessarily distinct). We prove that

$$x_I = y_I \Rightarrow x = y. \qquad (*)$$

Suppose that $x_I = y_I$. Let n be the number of vertices in G and $m = n - \tau(G)$. Let $i_1, \dots, i_m$ be a topological sort of $G \setminus I$, that is, there is no arc from i_ℓ to i_k for all $1 \leq k \leq \ell \leq m$. Let $I_0 = I$ and $I_k = I \cup \{i_1, \dots, i_k\}$ for $1 \leq k \leq n - \tau(G)$. We prove, by indiction on k from 0 to m that $x_{I_k} = y_{I_k}$.

For $k = 0$ this is true by hypothesis. Let $1 < k \leq m$. We have $x_{I_{k-1}} = y_{I_{k-1}}$ by induction. Since $N(i_k) \subseteq I_{k-1}$, we deduce that $f_{i_k}(x) = f_{i_k}(y)$ and thus $x_{i_k} = y_{i_k}$ since x and y are fixed points. This proves that $x_{I_k} = y_{I_k}$ and completes the induction step. In particular $x_{I_m} = y_{I_m}$, that is, $x = y$ since $I_m = V(G)$. This proves $(*)$.

Let X be the set of fixed points of f. Suppose, for a contradiction, that $|X| > 2^{|I|}$. Then, by the projection lemma (Exercice 5), we have $x_I = y_I$ for distinct $x, y \in X$. But by $(*)$ if $x_I = y_I$ then $x = y$ and we get the desired contradiction. Thus $|X| \leq 2^{|I|}$. $\square$

We have seen that $\nu(G) \leq \tau(G)$. Thus a large packing number forces a large transversal number. Conversely, the following very difficult theorem says that, conversely, a large transversal number forces a large packing number.

Theorem 5 (Reed, Robertson, Seymour, Thomas 1996)**.** *There is a function $h : \mathbb{N} \to \mathbb{N}$ such that, for every graph G,*

$$\tau(G) \leq h(\nu(G))$$

The function h of the proof is astronomic, but a folklore conjecture asserts that $h(k)$ is only in $O(k \log k)$. This is based on the best lower bound we known: $h(k) = \Omega(k \log k)$ [Alon, Seymour 1993]. The only exact value is $h(1) = 3$, and this is a deep result [MacCuaig 1991] (see Exercice 6).

4 Exercises

1. *Prove that if I is a minimal FVS of G then for every $i \in I$ there is a cycle C in G that intersects I only in i.*

   Answer. Let I be a minimal and $i \in I$. Let $I' = I \setminus \{i\}$. Then $G \setminus I'$ has a cycle C since I' is strictly included in I, which is a minimal FVS. Thus $V(C) \cap I' = \emptyset$ and, by definition, $V(C) \cap I \neq \emptyset$. Thus $V(C) \cap I = \{i\}$.

2. *Prove that for $k = 0, 1, 2$ there is G such that $\nu(G) = k$ and $\max(G) = k + 1$.*

   Answer. For $k = 0$, we have $\nu(K_1) = 0$ and $\max(K_1) = 1$. For $k = 1$, we have $\nu(C_1) = 1$ and $\max(C_1) = 2$. For $k = 2$, let G be the graph with vertex set $\{1, 2\}$ with a loop on each vertex and an arc $1 \to 2$. We have $\nu(G) = 2$. If $\max(G) = 4$ then some $f \in F(G)$ has four fixed points. But then f is the identity on $\{0, 1\}^2$, that is, $f_1(x) = x_1$ and $f_2(x) = x_2$ for all $x \in \{0, 1\}^2$. But the interaction graph of such a f has no arc from 1 to 2, a contradiction. Thus $\max(G) \leq 3$. Let $f \in F(G)$ defined by $f_1(x) = x_1$ and $f_2(x) = x_1 \wedge x_2$. Then 00, 10 and 11 are fixed points, thus $\max(G) \geq 3$. We deduce that $\max(G) = 3$ has desired.

3. *Find for every k a graph G with $\delta^-(G) = k$ and $\nu(G) = \lceil k/2 \rceil$.*

   Answer. We have $\nu(K_n) = \lfloor n/2 \rfloor = \lceil (n-1)/2 \rceil = \lceil \delta^-(K_n)/2 \rceil$.

4. *Prove that $\nu(G) \leq \tau(G)$ for every G.*

   Answer. Let $C_1, \ldots, C_\nu$ be vertex-disjoint cycles in G, with $\nu = \nu(G)$. Such a collection of cycles exists by definition of the packing number. Let I be any FVS of G of size $\tau = \tau(G)$. By definition of a FVS, for each $1 \leq k \leq \nu$ there is a vertex j_k in C_k which belongs to I. Let $J = \{j_1, \ldots, j_\nu\}$. Since the cycles are vertex-disjoint, the j_r are pairwise distinct, thus $|J| = \nu$. Since $J \subseteq I$ we have $\nu \leq |I| = \tau$.

5. *Prove the projection lemma: If* $X \subseteq \{0,1\}^n$, $I \subseteq [n]$ *and* $|X| > 2^{|I|}$, *then* $x_I = y_I$ *for distinct* $x, y \in X$.

   Answer. Let X and I has in the statement. Let $h : X \to \{0,1\}^I$ defined by $h(x) = x_I$. Since $|X| > 2^{|I|} = |\{0,1\}^I|$ the function h is not an injection: there exists distinct $x, y \in X$ such that $h(x) = h(y)$.

6. *Find a graph* G *with* $\nu(G) = 1$ *and* $\tau(G) = 3$.

   Answer.