Introduction to Finite Dynamical Systems

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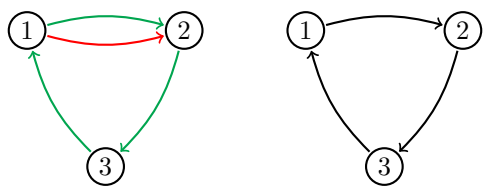
Lecture n° 6, M2 Informatique, October 30, 2019

1 Signed graph

A **signed (directed) graph** is a (directed) graph where each arc is positive or negative. More formally, a signed graph is a couple $D = (V, E)$ where V is a finite set of vertices, and $E \subseteq V^2 \times \{-1, 1\}$. If $(u, v, s) \in E$ then we say that D has an arc from u to v of sign s. Note that D can have both a positive and a negative arc from one vertex to another, and we say that D is **simple** if there is at most one arc from one vertex to another. We denote by $|D|$ the **underlying (unsigned) graph** of D: the vertex set is V and there is an arc from u to v if D has a positive or negative arc from u to v. Below, positive arcs are green, while negative arcs are red.

Example 1. *Here is a example of signed graph with its underlying unsigned graph.*



D $\qquad$ $|D|$

A cycle in D is a simple subgraph C of D such that $|C|$ is a cycle. The **sign** of a cycle is the product of the signs of its arcs. Thus a cycle is positive if and only if it has an even number of negative arcs. The signed graph D above has both a positive and a negative cycle.

If G is a graph and D is a signed graph with $|D| = G$, then D is a **signed version** of G. It is obvious that every graph G has a signed version with only positive cycle, for example the signed version with only positive arcs. But not every G has a signed version with only negative cycles, see Exercice 1. Such graphs G are call **even graphs** and have been studied in e.g. [4, 2].

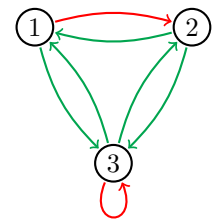
2 Signed interaction graph

Let $f : \{0, 1\}^n \to \{0, 1\}^n$ be a Boolean network with n components. The **signed interaction graph** of f, denoted $D(f)$, is the signed graph with vertex set $[n]$ and with a positive (resp. negative) arc from j to i if there exists $x \in \{0, 1\}^n$ with $x_j = 0$ such that $f_i(\bar{x}^j) - f_i(x)$ is positive (resp. negative). Recall that $\bar{x}^j$ is the state obtained from x by flipping the jth component. Note that $|D(f)| = G(f)$, that is, the interaction graph $G(f)$ is the underlying graph of the signed interaction graph $D(f)$.

Example 2. *Here is an example of Boolean network with its signed interaction graph.*

x	$f(x)$
000	000
001	110
010	101
011	110
100	001
101	100
110	101
111	100

$$\begin{cases} f_1(x) &= x_2 \vee x_3 \\ f_2(x) &= \overline{x_1} \wedge x_3 \\ f_3(x) &= \overline{x_3} \wedge (x_1 \vee x_2) \end{cases}$$



In many applications (mostly in biology) the signed interaction graph $D(f)$ of the system is known (or well approximated) while f itself is unknown. So the basic question is:

What can be said on the dynamics of f according to $D(f)$?

This is a difficult question since many different different networks f can have the same signed interaction graph (see the next section for example). However, $D(f)$ provides much more information on f than $G(f)$, and we can thus hope to have stronger partial answers in the signed case than in the unsigned case.

Given a signed graph D with vertex set $[n]$, we denote by $F(D)$ the set of Boolean networks with an signed interaction graph equal to D.

3 Monotone networks

We equip $\{0,1\}^n$ with the partial order $\leq$ defined by:

$$\forall x, y \in \{0,1\}^n, \qquad x \leq y \quad \Longleftrightarrow \quad x_i \leq y_i \ \forall i \in [n].$$

We say that a Boolean function $h : \{0,1\}^n \to \{0,1\}$ is **monotone** if

$$\forall x, y \in \{0,1\}^n, \qquad x \leq y \Rightarrow h(x) \leq h(y).$$

We say that a Boolean network $f : \{0,1\}^n \to \{0,1\}^n$ is **monotone** if

$$\forall x, y \in \{0,1\}^n, \qquad x \leq y \Rightarrow f(x) \leq f(y).$$

Remark that f is monotone if and only if its n components are monotone. Here is another characterization.

Proposition 1. *A Boolean network is monotone if and only if its signed interaction graph has only positive arcs.*

Proof. Let $f : \{0,1\}^n \to \{0,1\}^n$ and suppose that $D(f)$ has a negative arc from j to i. Then there is $x \in \{0,1\}^n$ with $x_j = 0$ such that $f_i(\bar{x}^j) < f_i(x)$. Since we have $x \leq \bar{x}^j$ and since $f(x) \leq f(\bar{x}^j)$ is false, we deduce that f is not monotone.

Let $f : \{0,1\}^n \to \{0,1\}^n$ and suppose that $D(f)$ has only positive arcs. Let $x, y \in \{0,1\}^n$ with $x \leq y$. We prove, by induction on $d(x,y)$ that $f(x) \leq f(y)$. This is obvious if $d(x,y) = 0$. Suppose that $d(x,y) > 0$. Then there is $j \in [n]$ with $x_j < y_j$, and we have $x \leq \bar{x}^j \leq y$. If $f_i(\bar{x}^j) < f_i(x)$ for some $i \in [n]$ then $D(f)$ has a negative arc form j to i, a contradiction. Thus $f(x) \leq f(\bar{x}^j)$. Since $d(\bar{x}^j, y) = d(x,y) - 1$, by induction hypothesis, $f(\bar{x}^j) \leq f(y)$. We deduce that $f(x) \leq f(y)$. This completes the induction step. $\square$

Proposition 2. *Every monotone Boolean network has at least one fixed point.*

Proof. Let $f : \{0,1\}^n \to \{0,1\}^n$ be monotone. Let $\mathbf{0}$ be the all zero configuration. We have $\mathbf{0} \leq x$ for every $x \in \{0,1\}^n$. In particular, $\mathbf{0} \leq f(\mathbf{0})$. By monotonicity, we obtain

$$\mathbf{0} \leq f(\mathbf{0}) \Rightarrow f(\mathbf{0}) \leq f^2(\mathbf{0}) \Rightarrow f^2(\mathbf{0}) \leq f^3(\mathbf{0}) \Rightarrow \cdots$$

thus

$$\mathbf{0} \leq f(\mathbf{0}) \leq f^2(\mathbf{0}) \leq f^3(\mathbf{0}) \leq \cdots \leq f^n(\mathbf{0}) \leq f^{n+1}(\mathbf{0}).$$

If f has no fixed point, then each inequality is a strict inequality, and we deduce that, for $0 \leq k \leq n+1$, the number of ones in $f^k(\mathbf{0})$ is at least k. But then the number of ones in $f^{n+1}(\mathbf{0})$ is at least $n+1$, which is obviously false. Thus f has a fixed point. $\square$

An **antichain** of $\{0,1\}^n$ is a subset A of $\{0,1\}^n$ such that there is no distinct $x, y \in A$ with $x \leq y$. In other words, distinct members of A are incomparable. Let $\binom{[n]}{k}$ be the set of $x \in \{0,1\}^n$ with exactly k ones. Thus the size of $\binom{[n]}{k}$ is $\binom{n}{k}$. Clearly, for each k, $\binom{[n]}{k}$ is an antichain, and thus $\{0,1\}^n$ can be partitioned in $n+1$ antichains. The dual notion is that of **chain**. A chain is a subset $C \subseteq \{0,1\}^n$ such that any two members of C is comparable. Thus there is an enumeration $x^1, \ldots, x^k$ of the elements of C such that $x^1 < x^2 < \cdots < x^k$. It is clear that the maximum size of a chain is $n+1$. The number of chain of size $n+1$ is $n!$ (Exercice 4).

We denote by $\mathcal{A}(n)$ the set of antichains of $\{0,1\}^n$. The size of $\mathcal{A}(n)$ is the nth **Dedekind number** and is known only for $0 \leq n \leq 8$. We have however the following approximation result.

Theorem 1 (Kleitman 1969 [1])**.**

$$\log_2 |\mathcal{A}(n)| = (1 - o(1)) \binom{n}{\lfloor n/2 \rfloor}.$$

where $o(1) \to 0$ as $n \to \infty$.

Let $\mathcal{M}(n)$ be the set of monotone Boolean functions $h : \{0,1\}^n \to \{0,1\}$.

Proposition 3. *For all $n \geq 1$,*

$$|\mathcal{M}(n)| = |\mathcal{A}(n)|$$

Proof. For each antichain A of $\{0,1\}^n$ and we define $h^A : \{0,1\}^n \to \{0,1\}$ by

$$h^A(x) = \begin{cases} 1 & \text{if } x \geq a \text{ for some } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then h^A is monotone. Indeed, suppose that $x \leq y$ and $h^A(x) = 1$. Then $x \geq a$ for some $a \in A$ and since $y \geq x$ we have $y \geq a$ and thus $h^A(y) = 1$. So h^A is monotone. Let A, B be distinct antichains of $\{0,1\}^n$ and let us prove that $h^A \neq h^B$. Suppose that there is $a \in A \setminus B$. Then $h^A(a) = 1$ and if $h^B(a) = 0$ we are done. So suppose that $h^B(a) = 1$. Then $a \geq b$ for some $b \in B$. If $h^A(b) = 1$, then $b \geq a'$ for some $a' \in A$. Then $a \geq b \geq a'$. Since $a \notin B$, $a \neq a'$. So A contains distinct comparable elements, a contradiction. Thus $h^A(b) = 0 \neq h^B(b)$ and we are done. If $B \setminus A$ is not empty the proof is similar. Hence, $A \mapsto f^A$ is an injection from $\mathcal{A}(n)$ to $\mathcal{M}(n)$, so $|\mathcal{M}(n)| \geq |\mathcal{A}(n)|$.

For each monotone $h : \{0,1\}^n \to \{0,1\}$ let $A(h)$ be the set of minimal elements of $h^{-1}(1)$, that is, the set of a with $h(a) = 1$ such that there is no x distinct from a with $x \leq a$ and $h(x) = 1$. It is clear that $A(h)$ is an antichain. Let $h, h' : \{0,1\}^n \to \{0,1\}$ be monotone and distinct, and let us

prove that $A(h) \neq A(h')$. Since $h \neq h'$, there is x such that $h(x) \neq h'(x)$ and, without loss, we can suppose that $h(x) = 1$ and $h'(x) = 0$. Then $x \geq a$ for some $a \in A(h)$. If $a \in A(h')$ then $h'(a) = 1$ and since $a \leq x$ we have $h'(x) = 1$ by monotonicity, a contradiction. Thus $h \mapsto A(h)$ is an injection from $\mathcal{M}(n)$ to $\mathcal{A}(n)$, so $|\mathcal{M}(n)| \leq |\mathcal{A}(n)|$. $\square$

Let $\mathcal{M}'(n)$ be the set of monotone Boolean functions $h : \{0,1\}^n \to \{0,1\}$ that depends on its n components. Following the second lecture, we easily deduce that

$$|\mathcal{M}'(n)| = |\mathcal{M}(n)| - \sum_{i=0}^{n-1} \binom{n}{i} |\mathcal{M}'(i)|.$$

From that and the result of Kleitman, we deduce that $|\mathcal{M}'(n)|$ is doubly exponential with n. So if D is a signed graph with only positive arcs, then $|F(D)|$ is doubly exponential with the maximum in-degree of $|D|$. Actually, it is not difficult to see that this is true for every signed graph D.

Lemma 1 (Sperner's theorem 1928 [3])**.** *If A is an antichain of $\{0,1\}^n$ then $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

Proof. Let A be an antichain of $\{0,1\}^n$. For each $x \in \{0,1\}^n$ we denote by C_x the chaines of $\{0,1\}^n$ of size $n+1$ containing x. Let $w(x)$ be number of ones in x. Then there are $w(x)!$ chain from $\mathbf{0}$ to x of size $w(x)+1$, and $(n-w(x))!$ chains from x to $\mathbf{1}$ of size $n-w(x)+1$. Since any chain of size $n+1$ containing x is the union of a chain from $\mathbf{0}$ to x of size $w(x)+1$ and a chain from x to $\mathbf{1}$ of size $n-w(x)+1$, we deduce that

$$|C_x| = w(x)!(n-w(x))!$$

Suppose that $C \in C_x \cap C_y$ for distinct $x, y \in A$. Then C contains both x and y, thus x and y are comparable, a contradiction. Thus

$$\forall x, y \in A, \qquad x \neq y \Rightarrow C_x \cap C_y = \emptyset.$$

Since there are $n!$ chains of size $n+1$, we deduce that

$$|\cup_{x \in A} C_x| = \sum_{x \in A} |C_x| = \sum_{x \in A} w(x)!(n-w(x))! \leq n!.$$

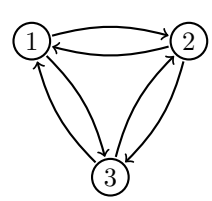
Thus

$$\frac{|A|}{\binom{n}{\lfloor n/2 \rfloor}} = \sum_{x \in A} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{x \in A} \frac{1}{\binom{n}{w(x)}} = \sum_{x \in A} \frac{w(x)!(n-w(x))!}{n!} \leq 1.$$

$\square$

4 Exercises

1. *Prove that the following graph G is even.*

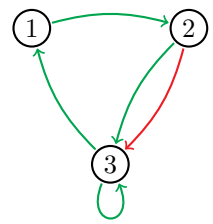


Answer. Let D be a signed version of G, and suppose, for a contradiction, that all the cycles of D are negative. Then D is simple. Let s_{ij} be the sign of the arc from i to j. We have $s_{ij} = -s_{ji}$ since otherwise, the cycle of length two between i and j is positive. Hence

$$s_{12}s_{23}s_{31} = (-s_{21} - s_{32} - s_{13}) = -(s_{13}s_{32}s_{21})$$

Thus $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 2 \to 1$ are cycles of opposite signs, thus exactly one is positive, a contradiction.

2. *Find a Boolean network with the following signed interaction graph.*



Answer.

$$\begin{cases} f_1(x) &= x_3 \\ f_2(x) &= x_1 \\ f_3(x) &= (\overline{x_2} \wedge x_1) \vee (x_2 \wedge x_3) \end{cases}$$

3. *Prove that* $\log_2 |\mathcal{A}(n)| \geq \binom{n}{\lfloor n/2 \rfloor}$.

Answer. Since $A = \binom{[n]}{\lfloor n/2 \rfloor}$ is an antichain if size $\binom{n}{\lfloor n/2 \rfloor}$, and since every subset of A is an antichain, there is at least $2^{|A|}$ antichains in $\{0,1\}^n$.

4. *Prove that* $\{0,1\}^n$ *has* $n!$ *chains of size* $n+1$.

Answer. Let $C = x^0 < x^1 \cdots < x^n$ be a chain of size $n+1$. For each $k \in [n]$ there is a unique component i_k that differs between x^{k-1} and x^k, and $i_1, \ldots, i_n$ is a permutation of $[n]$, denoted $\pi(C)$. It is clear that if C and C' are distinct chains of size $n+1$ then $\pi(C) \neq \pi(C')$. Thus $\{0,1\}^n$ has at most $n!$ chains of size $n+1$. Conversely, given a permutation $\pi = i_1, \ldots, i_n$ of $[n]$ we define the chain $C = x^0 < x^1 < \cdots < x^n$, where $x^0 = \mathbf{0}$ (all zero state) and where the components at one in x^k are $i_1, \ldots, i_k$. Then if π and π' are distinct permutations, then $C(\pi)$ and $C(\pi')$ are distinct, and thus $\{0,1\}^n$ has at least $n!$ chains of size $n+1$.

References

[1] Daniel Kleitman. On dedekind's problem: the number of monotone boolean functions. *Proceedings of the American Mathematical Society*, 21(3):677–682, 1969.

[2] Paul Seymour and Carsten Thomassen. Characterization of even directed graphs. *Journal of Combinatorial Theory, Series B*, 42(1):36–45, 1987.

[3] E. Sperner. Ein satz über untermengen einer endlichen menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928.

[4] Carsten Thomassen. Even cycles in directed graphs. *European Journal of Combinatorics*, 6(1):85–89, 1985.