# Introduction to Finite Dynamical Systems 

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Recall that, given a signed graph $D$ with vertex set $[n]$, we denote by $F(D)$ the set of Boolean networks $f:\{0,1\}^{n} \rightarrow\{0,1\}^{n}$ with $D(f)=D$. Our aim is to find what can be said on the dynamical properties of Boolean networks in $F(D)$. As a starting point, following what we have regarded in the unsigned case, we study the number of fixed points, and introduce the following two quantities:
$\max (D)=$ maximum number of fixed point in a Boolean network $f \in F(D)$
$\min (D)=$ minimum number of fixed point in a Boolean network $f \in F(D)$.

If $D$ is acyclic, we know that $\min (D)=\max (D)=1$. What are the simplest non-acyclic signed graphs? These are clearly a signed graphs $D$ such that $|D|$ is a cycle. In that case, $|F(D)|=1$ and the unique Boolean network $f$ in $F(D)$ has two fixed points if $D$ is a positive cycle, and no fixed point if $D$ is a negative cycle (Exercice 1). Thus isolated positive and negative cycles have clearly distinct behaviors, and there distinction is naturally justified. So concerning fixed points, the situation for positive and negative cycles is clear. What is the next step? What other simple signed graph families could we study to progress on $\max (D)$ and $\min (D)$ ? One of us propose to study signed cliques. That's an interesting proposition.

## 1 Maximum and minimum number of fixed points for signed cliques

To start, its natural to study the full-positive (resp. full-negative) clique $K_{n}^{+}$(resp. $K_{n}^{-}$) on $n$ vertices, obtained by adding a positive (resp. negative) sign on each arc of $K_{n}$. We then have the following results.
Theorem 1 (Gadouleau Richard Riis [2]). For every $n \geq 1$, we have

$$
\begin{gathered}
\frac{\binom{n}{\lfloor n / 2\rfloor}}{n} \leq \max \left(K_{n}^{+}\right) \leq(2-o(1)) \frac{\binom{n}{\lfloor n / 2\rfloor}}{n} \quad \max \left(K_{n}^{-}\right)=\binom{n}{\lfloor n / 2\rfloor} \\
\min \left(K_{n}^{+}\right)=2 \quad \min \left(K_{n}^{-}\right)=\left\{\begin{array}{l}
0 \text { if } n \geq 4 \\
1 \text { otherwise. }
\end{array}\right.
\end{gathered}
$$

Results concerning the minimum number of fixed points are exact and not difficult. In particular, $\min \left(K_{n}^{+}\right) \leq 2$ is an easy exercice (Exercice 2) and $\min \left(K_{n}^{+}\right) \geq 2$ follow from a more general result given below. The analyse of te maximum number of fixed points is more interesting. The lower bound on $\max \left(K_{n}^{+}\right)$involves a classical result in Coding Theory (the Graham-Sloane bound, see [2]) and the upper bound is an easy consequence of a recent result is Set Theory [3].

To prove $\max \left(K_{n}^{-}\right) \geq\binom{ n}{\lfloor n / 2\rfloor}$, it is sufficient to consider the minority network on $K_{n}^{-}$(Exercice 3). The converse inequality follows from the following simple result, which uses Sperner's theorem given in the last lecture.

Lemma 1. If $D$ is a signed graph with $n$ vertices and only negative arcs, then

$$
\max (D) \leq\binom{ n}{\lfloor n / 2\rfloor}
$$

Proof. Let $x$ and $y$ be distinct fixed points of $f \in F(D)$. Suppose, for a contradiction, that $x \leq y$. Then $f(x) \geq f(y)$ since all the arcs of $D$ are negative. Thus $x \geq y$ since $x$ and $y$ are fixed points, and thus $x=y$, a contradiction. Thus the set of fixed points of $f$ is an antichain, and by Sperner's theorem, the number of fixed points of $f$ is at most $\binom{n}{\lfloor n / 2\rfloor}$.

Conjecture 1. If $D$ is a simple signed digraph without cycle of length one, then

$$
\max (D) \leq\binom{ n}{\lfloor n / 2\rfloor}
$$

What about other signed cliques? There are no partial answer that are not directed consequences of results about the full-positive and full-negative clique. We have the following conjecture.

Conjecture 2. Let $D$ be a simple signed digraph with $|D|=K_{n}$.

$$
\max \left(K_{n}^{+}\right) \leq \max (D) \leq \max \left(K_{n}^{-}\right) .
$$

## 2 Absence of positive and negative cycles

Since positive and negative cycles, when isolated, play distinct roles, it is interesting to see what happens when we forbid the presence of positive or negative cycles. For that we introduce some definitions and preliminary result. Given $D$ with vertex set $[n]$ and $x \in\{0,1\}^{n}$ we denote by $D(x)$ the spanning subgraph of $D$ obtained by removing all the positive arcs from $i$ to $j$ with $x_{i} \neq x_{j}$ and all the negative arcs from $i$ to $j$ with $x_{i}=x_{j}$. The basic property concerning $D(x)$ is the following. The proof is an easy exercice (Exercice 4).

Proposition 1. Let $D$ be a signed graph with vertex set $[n]$ and $x \in\{0,1\}^{n}$. Then $D(x)$ has only positive cycles.

A basic result is signed graph theory is the following. A walk of length $\ell \geq 1$ in $D$ is a sequence of vertices $i_{0}, \ldots, i_{\ell}$ such that $D$ has an arc $a_{k}$ from $i_{k}$ to $i_{k+1}$ for all $0 \leq k<\ell$. If $D$ is simple, $a_{k}$ is the unique arc from $i_{k}$ to $i_{k+1}$, and the sign of $D$ is defined, without possible ambiguity, as the product of the sign of the arcs $a_{k}$. Hence, a walk can be regarded as a path where repetition of vertices are allowed. A closed walk is a walk $i_{0}, \ldots, i_{\ell}$ with $i_{0}=i_{\ell}$.

Theorem 2 (Harary 1954). Let $D$ be a strongly connected signed graph with vertex set $[n]$. If $D$ has only positive cycles, then $D(x)=D$ for some $x \in\{0,1\}^{n}$.

Proof. Suppose that $D$ has only positive cycles (this implies that $D$ is simple). Let $T$ be a spanning out-tree of $D$ rooted in a vertex $v$. For all vertex $i$, let $T_{j}$ be the path from $v$ to $j$ contained in $T$ ( $T_{v}$ is of length zero and positive). Let $x \in\{0,1\}^{n}$ be defined by: for all $i \in[n], x_{i}=0$ is $T_{i}$ is positive and $x_{i}=1$ otherwise. We claim that $D(x)=D$. Suppose, for a contradiction, that $D(x) \neq D$. Thus $D$ has a positive arc from $i$ to $j$ with $x_{i} \neq x_{j}$ or a negative arc from $i$ to $j$ with $x_{i}=x_{j}$.

Suppose first that $D$ has a positive arc from $i$ to $j$ with $x_{i} \neq x_{j}$. Then $T_{i}$ and $T_{j}$ have opposite sign. If $i_{1}, \ldots, i_{l}$ are the vertices of $T_{i}$ in the order, then $i_{1}, \ldots, i_{l}, j$ is a walk with the same sign as
$T_{i}$, since sign the arc from $i=i_{\ell}$ to $j$ is positive. We deduce that $D$ has a positive walk $W^{+}$from $v$ to $j$ and a negative walk $W^{-}$from $v$ to $j$. Let $P$ be a path from $j$ to $v$. If $P$ is positive, then the concatenation of $W^{-}$and $P$ gives a negative walk from $v$ to $v$ and if $P$ is negative, then the concatenation of $W^{+}$and $P$ gives a negative walk from $v$ to $v$. So in both case, $D$ has a negative closed walk and we deduce that $D$ has a negative cycle (Exercice 5), a contradiction.

Suppose first that $D$ has a negative arc from $i$ to $j$ with $x_{i}=x_{j}$. Then $T_{i}$ and $T_{j}$ have the same sign. If $i_{1}, \ldots, i_{l}$ are the vertices of $T_{i}$ in the order, then $i_{1}, \ldots, i_{l}, j$ is a walk which has not the same sign as $T_{i}$, since sign the arc from $i=i_{\ell}$ to $j$ is negative. We deduce that $D$ has a positive walk $W^{+}$from $v$ to $j$ and a negative walk $W^{-}$from $v$ to $j$, and as above, we deduce that $D$ has a negative cycle, a contradiction.

Remark 1. The proof gives an $O\left(n^{2}\right)$-time algorithm to find a negative cycle in $D$ if it exists. By the theorem above, which is a real tour de force, it is also polynomial to decide if $D$ has a positive cycle.

Theorem 3 (Robertson Seymour Thomas [4]). There is a polynomial time algorithm that decides if a given signed digraph $D$ has a positive cycle.

Given a signed graph $D$ and a vertex $i$, we denote by $N^{s}(i)$ be the set of vertices $j$ such that $D$ has an arc from $j$ to $i$ of sign $s$.

Lemma 2. Let $D$ be a sign graph with vertex set $[n]$, let $x \in\{0,1\}^{n}$ and let $i$ be a vertex of $D$ which is not a source. Then the following holds:

- If $x_{j}=1$ for all $j \in N^{+}(i)$ and $x_{j}=0$ for all $j \in N^{-}(i)$, then $f_{i}(x)=1$.
- If $x_{j}=0$ for all $j \in N^{+}(i)$ and $x_{j}=1$ for all $j \in N^{-}(i)$, then $f_{i}(x)=0$.

Proof. To prove the first assertion, let $x$ with $x_{j}=1$ for $j \in N^{+}(i)$ and $x_{j}=0$ for all $j \in N^{-}(i)$. Suppose, for a contradiction, that $f_{i}(x)=0$. Let $y \in\{0,1\}^{n}$. We prove by induction on the Hamming distance $d(x, y)$ that $f_{i}(y)=0$ for all $y \in\{0,1\}^{n}$. If $d(x, y)=0$ there is nothing to prove. So suppose that $d(x, y)>0$. Let $j \in \Delta(x, y)$. We have two cases:

1. Suppose $x_{j}<y_{j}$. By induction, $f_{i}\left(y+e_{j}\right)=0$. If $f_{i}(y)=1$, then $D$ has a positive arc from $j$ to $i$, thus $x_{j}=1$ and we obtain a contradiction. Thus $f_{i}(y)=0$.
2. Suppose $x_{j}>y_{j}$. By induction, $f_{i}\left(y+e_{j}\right)=0$. If $f_{i}(y)=1$, then $D$ has a negative arc from $j$ to $i$, thus $x_{j}=0$ and we obtain a contradiction. Thus $f_{i}(y)=0$.

So $f_{i}(y)=0$ in both cases, and this proves the induction step. Hence $f_{i}(y)=0$ for all $y \in\{0,1\}^{n}$. So $f_{i}$ is a constant function and we deduce that $i$ is a source of $D$, a contradiction. This proves the first assertion, and the second has a similar proof.

Theorem 4 (Aracena 2008 [1]). Let $D$ be a strong connected signed graph with vertex set $[n]$.

1. If $D$ has only negative cycles, $\min (D)=0$.
2. If $D$ has only positive cycles, $\max (D) \geq 2$.

Proof. For the first point, suppose that $D$ has only negative cycles but $\min (D)>0$. Thus there is $f \in F(D)$ with a fixed point, say $x$. Let $i \in[n]$. If $x_{i}=0$ then $f_{i}(x)=x_{i}=0$, thus $f_{i}(x) \neq 1$ and we deduce from the lemma that there is $j \in N^{+}(i)$ with $x_{j}=0=x_{i}$ or $j \in N^{-}(i)$ with $x_{j}=1 \neq x_{i}$. Thus, in both case, the arc from $j$ to $i$ is in $D(x)$. If $x_{i}=1$ we prove similarly that there is an arc
from $j$ to $i$ in $D(x)$. Thus the minimum in-degree of $D(x)$ is at least one. We deduce that $D(x)$ has a cycle, which is positive by the previous proposition, a contradiction. Thus $\min (D)=0$.

For the second point, suppose that $D$ has only positive negatives and let $f \in F(D)$. By Harary's theorem, there is $x$ such that $D(x)=D$. Using the previous lemma, we deduce that:

1. if $x_{i}=1$ then $x_{j}=1$ for all $j \in N^{+}(i)$ and $x_{j}=0$ for all $j \in N^{-}(i)$, so $f_{i}(x)=1$;
2. if $x_{i}=0$ then $x_{j}=0$ for all $j \in N^{+}(i)$ and $x_{j}=1$ for all $j \in N^{-}(i)$, so $f_{i}(x)=0$.

Hence, $f_{i}(x)=x_{i}$ in both cases. We deduce that $x$ is a fixed point. Let $\bar{x}$ be the state opposite to $x$, that is obtained by changing all the components. Then we have $D(\bar{x})=D$ and we prove similarly that $\bar{x}$ is a fixed point. Hence $f$ has at least two fixed points and we deduce that $\max (D) \geq 2$.

## 3 Exercises

1. Let $D$ be a simple signed graphe with $|D|=C_{n}$. Prove that $\min (D)=\max (D)=0$ if $D$ is negative and $\min (D)=\max (D)=2$ if $D$ is positive.

Answer. For $i \in[n]$, and let $s_{i}=0$ is the sign of the arc from $i-1$ to $i$ is positive, and $s_{i}=1$ if it is negative (where the soustraction is modulo $n$ ). Then $F(D)$ contains a unique Boolean network $f$, since $f_{i}(x)=x_{i-1}+s_{i}$ for all $x \in\{0,1\}^{n}$ (sum modulo two). Suppose that $x=f(x)$. Then

$$
\begin{aligned}
& x_{1}=f_{1}(x)=x_{n}+s_{1} \\
& x_{2}=f_{2}(x)=x_{1}+s_{2}=x_{n}+s_{1}+s_{2} \\
& x_{3}=f_{3}(x)=x_{2}+s_{3}=x_{n}+s_{1}+s_{2}+s_{3} \\
& \vdots \\
& x_{n}=f_{n}(x)=x_{n-1}+s_{n}=x_{n}+s_{1}+s_{2}+\cdots+s_{n-1}+s_{n} .
\end{aligned}
$$

We deduce that $s_{1}+s_{2}+\cdots+s_{n-1}+s_{n}=0$, thus there is an even number of ones in the sum, that is, $D$ has an even number of negative arcs, and thus it is positive. We deduce that if $D$ is negative then $f$ has no fixed points and thus $\min (D)=\max (D)=0$. Conversely, for $a \in\{0,1\}^{n}$, let $x^{a} \in\{0,1\}^{n}$ be defined recursively by $x_{1}^{a}=a$ and $x_{i}^{a}=x_{i-1}^{a}+s_{i}$ for $1<i \leq n$. Then, we obviously have $f_{i}\left(x^{a}\right)=x_{i}^{a}$ for $1<i \leq n$, and

$$
x_{n}^{a}=x_{n-1}^{a}+s_{n}=x_{n-2}^{a}+s_{n-1}+s_{n}=\cdots x_{1}^{a}+s_{2}+s_{3}+\cdots+s_{n-1}+s_{n} .
$$

Thus $f_{1}\left(x^{a}\right)=x_{n}^{a}+s_{1}=x_{1}+s_{1}+s_{2}+\cdots+s_{n}$. If $D$ is positive, $s_{1}+s_{2}+\cdots+s_{n}=0$ thus $f_{1}\left(x^{a}\right)=x_{1}^{a}$, that is $x^{a}$ is a fixed point of $f$. Thus $x^{0}$ and $x^{1}$ are fixed points of $f$, and it is easy to see that $f$ has no other fixed point. Thus $f$ has exactly two fixed points and we deduce that $\min (D)=\max (D)=2$.
2. Prove that $\min \left(K_{n}^{+}\right) \leq 2$.

Answer. Let $f$ be the an-network over $K_{n}^{+}$, that is $f_{i}(x)=\bigwedge_{j \neq i} x_{j}$. Clearly, the all-zero and all-one configurations are fixed points. Suppose that $x=f(x)$, and let $i, j$ distinct components, and suppose that $x_{j} \neq x_{i}$. If $x_{i}=0$ then $f_{j}(x)=0 \neq x_{j}$, a contradiction, and if $x_{j}=0$ then $f_{i}(x)=0 \neq x_{i}$, a contradiction. We deduce that all the components in $x$ have the same value, thus $x$ is the all-zero or the all-one configuration. Hence $f$ has exactly two fixed points, and this implies $\min \left(K_{n}^{+}\right) \leq 2$.
3. Prove that $\max \left(K_{n}^{-}\right) \geq\binom{ n}{\lfloor n / 2\rfloor}$.

Answer. Considering the minority function on $K_{n}^{-}$, that is, let $f \in F\left(K_{n}^{-}\right)$defined by: for all $i \in[n]$ and $x \in\{0,1\}$,

$$
f_{i}(x)=\left\{\begin{array}{l}
0 \text { if } \sum_{j \neq i} x_{j} \geq\lfloor n / 2\rfloor, \\
1 \text { otherwise. }
\end{array}\right.
$$

Let any $x$ containing exactly $\lfloor n / 2\rfloor$ ones. If $x_{i}=0$ then $\sum_{j \neq i} x_{j}=\lfloor n / 2\rfloor$ thus $f_{i}(x)=0=x_{i}$, and if $x_{i}=1$ then $\sum_{j \neq i} x_{j}=\lfloor n / 2\rfloor-1$ thus $f_{i}(x)=1=x_{i}$. So $x$ is a fixed point. Since the number of $x$ with exactly $\lfloor n / 2\rfloor$ ones is $\binom{n}{\lfloor n / 2\rfloor}$, we deduce that $f$ has at least $\binom{n}{\lfloor n / 2\rfloor}$ fixed points.
4. Let $D$ be a signed graph with vertex set $[n]$ and $x \in\{0,1\}^{n}$. Prove that $D(x)$ has only positive cycles.

Answer. Let $i_{0}, \ldots, i_{n}$ be a walk in $D(x)$. Then this walk is positive if and only if $x_{0}=x_{\ell}$. We deduce that all the closed walks of $D(x)$ are positive. In particular, all the cycles of $D(x)$ are positive.
5. Let $D$ be a signed graph. Prove that $D$ has a negative cycle if it has a negative closed walk.

Answer. Let $W=i_{0}, \ldots, i_{\ell}$ be a negative closed walk in $D$ of minimal length. If the vertices $i_{0}, \ldots, i_{\ell-1}$ are all distinct, then $W$ itself form a negative cycle and we are done. Otherwise, there are $0 \leq p<q<\ell$ such that $i_{p}=i_{q}$. Then, $W_{1}=i_{p}, \ldots, i_{q}$ and $W_{2}=$ $i_{0}, \ldots, i_{p}, i_{q+1} \ldots, i_{\ell}$ are closed walk, and the sign of $W$ is the product of the sign of $W_{1}$ and $W_{2}$. Thus exactly one of $W_{1}, W_{2}$ is negative, and since both are shorter than $W$, we obtain a contradiction. Thus every negative closed walk in $D$ of minimal length is a negative cycle.

## References

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