Simple dynamics on graphs

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Let $A = \{0, 1, \ldots, q\}$ be a **finite alphabet**.

A **finite dynamical system** with n components is a function

$$\begin{aligned} f : A^n &\to A^n \\ x = (x_1, \ldots, x_n) &\mapsto f(x) = (f_1(x), \ldots, f_n(x)) \end{aligned}$$

The **dynamics** is described by the successive iterations of f

$$x \to f(x) \to f^2(x) \to f^3(x) \to \cdots$$

The **interaction graph** of f, denoted $\mathbf{IG}(f)$, is the **signed directed graph** with vertices $\{1, \ldots, n\}$ such that:

- there is a **positive arc** $j \to i$ if there exists $x \in A^n$ such that

$$f_i(x_1, \ldots, x_j, \ldots, x_n) < f_i(x_1, \ldots, x_j + 1, \ldots, x_n)$$

- there is a **negative arc** $j \to i$ if there exists $x \in A^n$ such that

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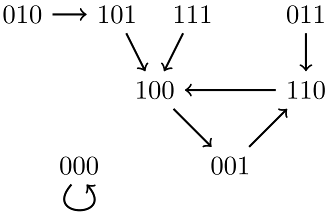
We can have both $j \to i$ and $j \to i$. The interaction from j to i is then **non-monotone**. We indicate this with the colored arc

$$j \to i$$

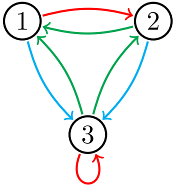
Example: with $f : \{0,1\}^3 \to \{0,1\}^3$ defined by

$$
\begin{aligned}
f_1(x) &= x_2 \text{ OR } x_3 \\
f_2(x) &= \text{NOT}(x_1) \text{ AND } x_3 \\
f_3(x) &= \text{NOT}(x_3) \text{ AND } (x_1 \text{ XOR } x_2)
\end{aligned}
$$

Dynamics



Interaction graph



What can be said on f according to its interaction graph ?

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Theorem [Robert 80]

If the interaction graph of f is acyclic, then f^n is constant.

$f^k = \mathrm{cst} \iff$ f has a unique fixed point and, starting from any initial configuration, the system reaches this fixed point in at most k iterations.

$\iff$ **f converges in k steps**.

Robert's result shows that:

"simple" interaction graph (i.e. acyclic)

$\Downarrow$

"simple" dynamics (i.e convergence)

Does the converse holds ?

"complex" interaction graph

$\Downarrow$ **?**

"complex" dynamics

Notation: Given a signed digraph G with n vertices and $q \geq 2$

$$F(G,q) := \{ f : A^n \to A^n \text{ such that } |A| = q \text{ and } \mathrm{IG}(f) = G \}.$$

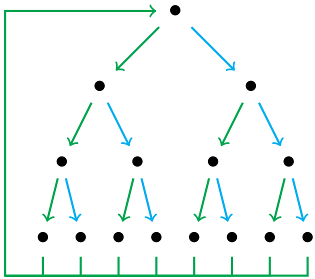
Theorem [Gadouleau R 05]

Let G be any signed digraph with n vertices.

- *If $q \geq 4$ there exists $f \in F(G,q)$ such that $f^2 = \mathrm{cst}$.*
- *If $q = 3$ there exists $f \in F(G,q)$ such that $f^{\lfloor \log_2 n \rfloor + 2} = \mathrm{cst}$.*

In the case $q = 3$ the convergence time $\lfloor \log_2 n \rfloor + 2$ is optimal.

Example: If G is as follows



- there exists $f \in F(G, 3)$ such that $f^{\lfloor \log_2 n \rfloor + 2} = \mathrm{cst}$.
- there is no $f \in F(G, 3)$ such that $f^{\lfloor \log_2 n \rfloor + 1} = \mathrm{cst}$.

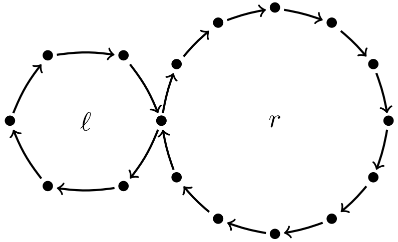
The boolean case $q = 2$ is much more difficult.

There is not necessarily a boolean convergent system $f \in F(G, 2)$.
Ex: G is strongly connected and all its cycles have the same sign.

It is very hard to understand which are the signed digraphs G such that $F(G, 2)$ contains a convergent system.

This lead us to consider the **unsigned case**.

Example: Let G be the digraph obtained from a cycle of length ℓ and a cycle of length $r \geq \ell$ by identifying one vertex.



- $F(G, 2)$ has a convergent system if and only if ℓ divides r.
- If $f \in F(G, 2)$ converges then $f^{2r-1} = \mathrm{cst}$ and $f^{2r-2} \neq \mathrm{cst}$.

Theorem [Gadouleau R 05]

1) If G has a strongly connected spanning subgraph $H \neq G$ such that the gcd of the lengths of the cycles of H is one, then there exists $f \in F(G,2)$ such that

$$f^{n^2-2n+2} = \mathrm{cst}.$$

2) If G is strongly connected and has a loop (an arc $i \to i$) then there exists $f \in F(G,2)$ such that

$$f^{2n-1} = \mathrm{cst}.$$

3) If G is symmetric ($i \to j$ iff $j \to i$), has no loop and $n \geq 3$, then there exists $f \in F(G,2)$ such that

$$f^{3} = \mathrm{cst}.$$

Conclusion

In the **non-boolean case**, every signed digraph admits a very simple dynamics: a system that converges toward a unique fixed point in logarithmic time.

In the **boolean case**, we have only provide some sufficient conditions for the existence of a convergent system.

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In the **boolean case**, we have only provide some sufficient conditions for the existence of a convergent system.

Question 1: *Given a digraph G, what is the complexity of deciding if G admits a boolean system that converges ?*

Question 2: *Is there exists a constant c such that, for every digraph G with n vertices, if G admits a boolean system that converges, then G admits a boolean system that converges in at most cn steps ?*